

Transformations

Unit Overview

This unit is about functions and their transformations. The graph of a function can easily be found by looking at how an equation has been changed from the parent function.

Transformations are used in many disciplines including science, business, advertising and art.

Transformations

A **transformation** of a function is an alteration of the function rule that produces a new function. A **translation** of a function shifts the graph horizontally or vertically.

If you have ever played a video game in which a figure moves across the screen, you have experienced translations. Programmers of games first put the images on coordinate planes and then use horizontal and vertical movements to “slide” the image. The original position of the image is called the **pre-image** and the final position is called the final **image**. With equations, the original equation is called the **parent function**.

Follow along with the example below.

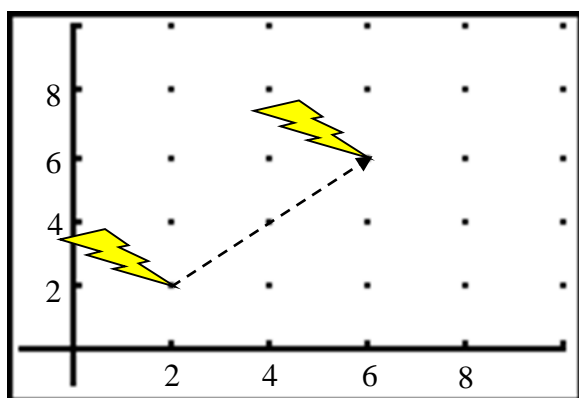


Figure 1

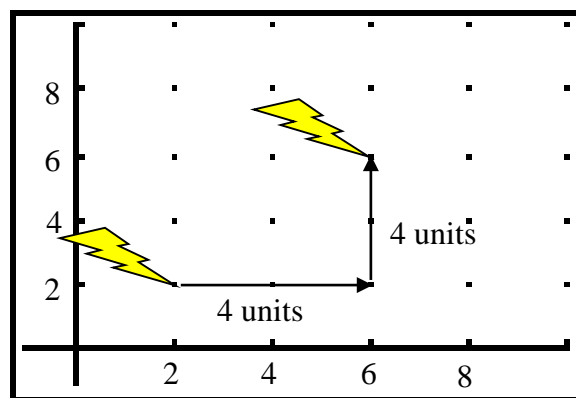


Figure 2

In figure 1, you are shown that the pre-image was translated right and up. In figure 2, it is illustrated how this is done – the translation was 4 units right and 4 units up.

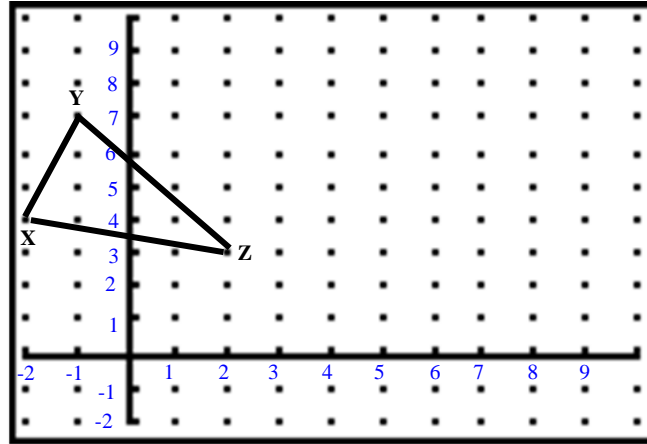
Let's say that the pre-image point was (x, y) , then the image point after a slide of 4 units right and 4 units up would be $(x + 4, y + 4)$.

- Adding a negative number to the x -coordinate results in a slide to the left.
- Adding a negative number to the y -coordinate results in a slide down.

Study the example below.

Example: Draw $\triangle XYZ$ on a coordinate plane with vertices $X(-2, 4)$, $Y(-1, 7)$, and $Z(2, 3)$. Draw the image of $\triangle XYZ$ after a slide of 2 units right and 4 units down.

a.) Draw $\triangle XYZ$.



b.) Determine the coordinates of the vertices after the slide.

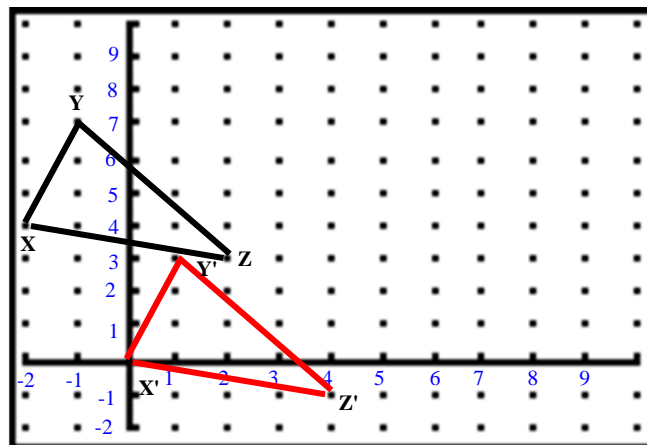
Add 2 to all x -coordinates. **Subtract 4** from all y -coordinates.

$$X((-2+2, 4-4)) \quad Y(-1+2, 7-4) \quad Z(2+2, 3-4)$$

$$X'(0, 0) \quad Y'(1, 3) \quad Z'(4, -1)$$

*Notice that the coordinates of the new image are denoted with X' , Y' , and Z' . This notation is read as X-prime, Y-prime, and Z-prime.

c.) Draw the new image of $\triangle XYZ$.



Stop! Go to Questions #1-6 about this section, then return to continue on to the next section.

Vertical Translations

Translations of a function are very similar to the translations you just studied. First let's take a look at the parent function of a quadratic $f(x) = x^2$ and compare it to the graph of $g(x) = x^2 + 2$.

Example #1: Compare the graph of the parent function $f(x) = x^2$ with the graph of $g(x) = x^2 + 2$.

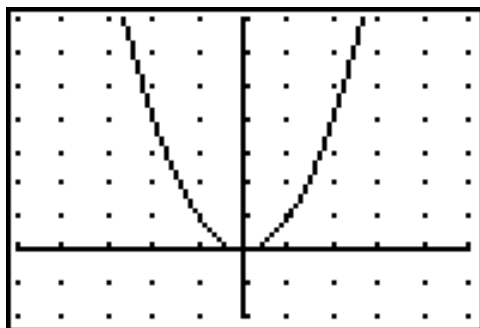


Figure 1

a.) Graph the parent function $f(x) = x^2$.
(Figure 1)

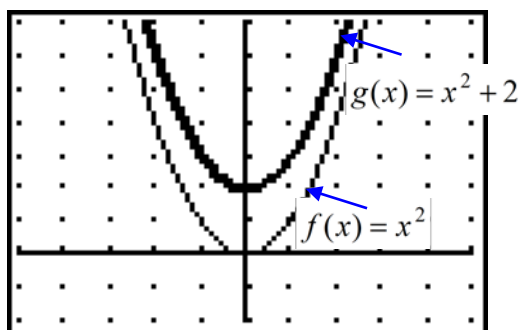


Figure 2

b.) Graph the function $g(x) = x^2 + 2$.
(Figure 2)

c.) Notice that the graph of $g(x)$ is the same as $f(x)$, but is translated **up** 2 units because of the +2 after the quadratic term.

Example #2: Without graphing, describe the translation of the graph $g(x) = x^2 - 4$ from the parent function $f(x) = x^2$.

If you answered the translation was 4 units down, you are correct. The two preceding examples were explanations of vertical translations and in general are defined as:

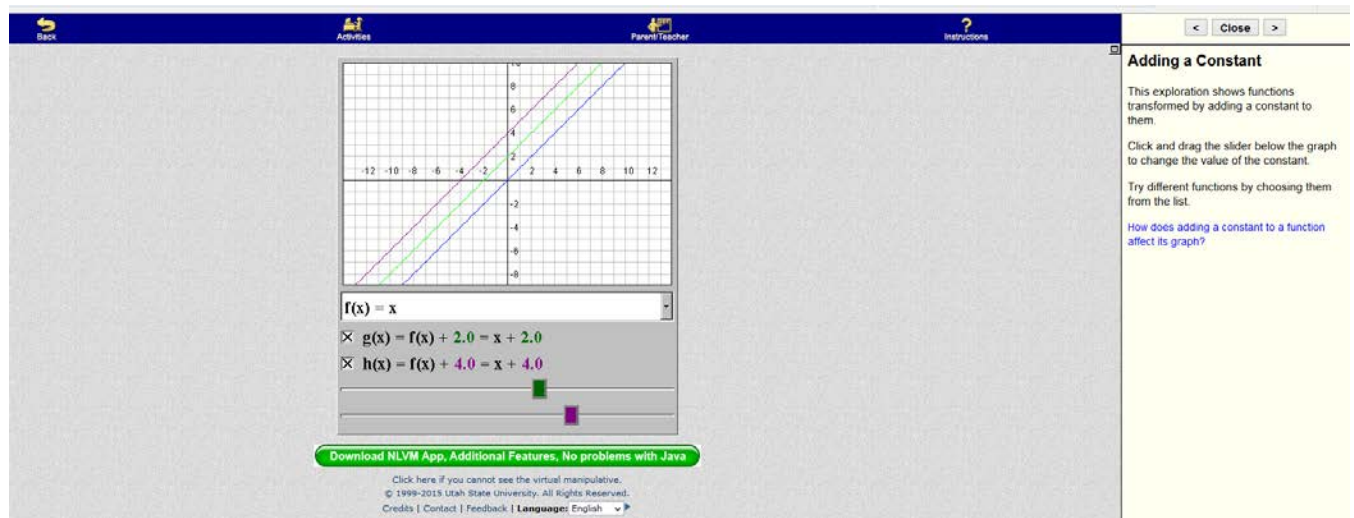
The graph of $f(x) = x^2 + k$ is identical to $f(x) = x^2$ translated vertically $|k|$ units.

When k is positive, the translation is in a positive direction, or up along the y-axis.

When k is negative, the translation is in a negative direction, or down along the y-axis.

Activity: Go to: [Function Transformations - NLVM](#)

It should look like this:



A new window will pop up and you will see graphs. The parent function has a default setting of $f(x) = x$. You can change this by clicking the drop down box next to the equation. This will allow you to see how different equations are affected. This parent function is graphed in black. There are two other lines, one green and one purple. They have a constant added to the function. You can use the sliders underneath the equations to change this constant. If you scroll far enough left, you will get negatives (or $-$ a constant). Play around with these graphs until you are familiar with how the functions change by adding (or subtracting) a constant.

Horizontal Translations

Horizontal translations shift functions left and right and are useful in creating simple patterns and pictures as shown below.



Example #1: Compare the graph of $f(x) = |x|$ which is the parent function of an absolute value function with the function $g(x) = |x - 3|$.

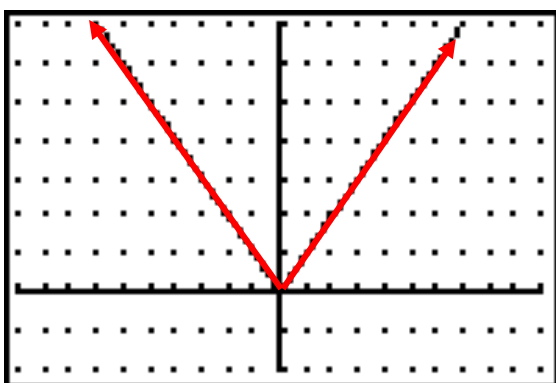


Figure 1

a.) Graph the parent function. (Figure 1)

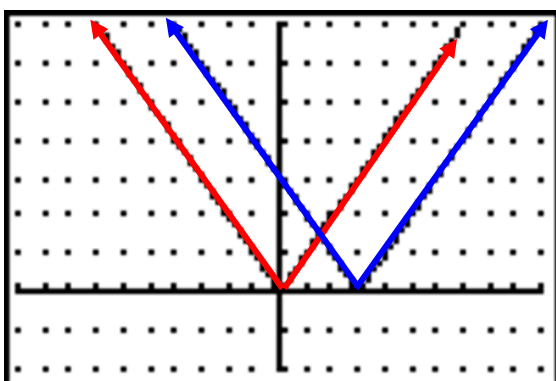


Figure 2

b.) Graph $g(x) = |x - 3|$. (Figure 2)

c.) Notice the translation does not change the shape of the graph but it does move the graph 3 units **right**.

Example #2: Without graphing, describe the translation of the graph $g(x) = |x + 4|$ from the parent function of $f(x) = |x|$.

If you answered the translation is 4 units left, you are correct. The two preceding examples were explanations of **horizontal translations** and in general are defined as:

The graph of $f(x) = |x - k|$ is translated horizontally $|k|$ units from the graph of $g(x) = |x|$.

When k is positive, the translation is in a positive direction, or to the right along the x -axis.

When k is negative, the translation is in a negative direction, or to the left along the x -axis.

*Notice that it may seem opposite of what you might expect!

Now, let's take a look at a function that has both a vertical and horizontal translation.

Example #3: Compare $f(x) = (x - 3)^2 + 2$ with the parent function of $g(x) = x^2$.

- a.) Graph the parent function of $g(x) = x^2$.
- b.) Graph the function $f(x) = (x - 3)^2 + 2$.

You can do this by making a table of values, using a graphing calculator, or using an online resource such as [Math is Fun](#) or [Desmos](#).

- c.) Determine the translation beginning with the horizontal translation (3 units right).
- d.) Determine the vertical translation (2 units up).



*We can establish a rule in determining the horizontal and vertical translations of functions.

If given $f(x) = (x-h)^2 + k$ as a quadratic function, the translation from the parent function will be h units left or right and k units up or down. This general rule is also used for absolute value functions in the form $g(x) = |x-h| + k$.

You can see this by going to the previous activity: [Function Transformations - NLVM](#)

However, when it opens, you want to click the $>$ arrow at the top right next to the close button. This is circled below in red. Click it once to go to the next page, which shows what happens when you have $f(x+k)$. Again, use the drop down arrow to try this with different equations, x for a linear equation, x^2 for a quadratic and so forth. Also, adjust the green slider bar to try adding different values for k .

The screenshot shows the NLVM interface for the 'Adding a Constant' activity. At the top, there are navigation buttons: 'Back', 'Activities', 'Parent/Teacher', and 'Innovations'. The main area features a coordinate plane with a blue line representing the parent function $f(x) = x$. Two other lines are shown: a green line representing $g(x) = f(x) + 2.0 = x + 2.0$ and a purple line representing $h(x) = f(x) + 4.0 = x + 4.0$. Below the graph, there is a dropdown menu currently set to $f(x) = x$, and two equations are listed with checkboxes: $g(x) = f(x) + 2.0 = x + 2.0$ and $h(x) = f(x) + 4.0 = x + 4.0$. A green slider bar is positioned below the equations. At the bottom, there is a green button that says 'Download NLVM App, Additional Features, No problems with Java.' and a small text link: 'Click here if you cannot see the virtual manipulative.' The right sidebar contains the title 'Adding a Constant', a brief description, and instructions: 'Click and drag the slider below the graph to change the value of the constant. Try different functions by choosing them from the list. How does adding a constant to a function affect its graph?'. A red box highlights the right-pointing arrow next to the 'Close' button in the top right corner.

Stop! Go to Questions #7-13 about this section, then return to continue on to the next section.

Vertical Stretches and Compressions

Example #1: What happens when the parent function of $f(x) = x$ is multiplied by 3?
 $3f(x) = 3x$.

You can see how the values are affected by making a graph or a table of values.

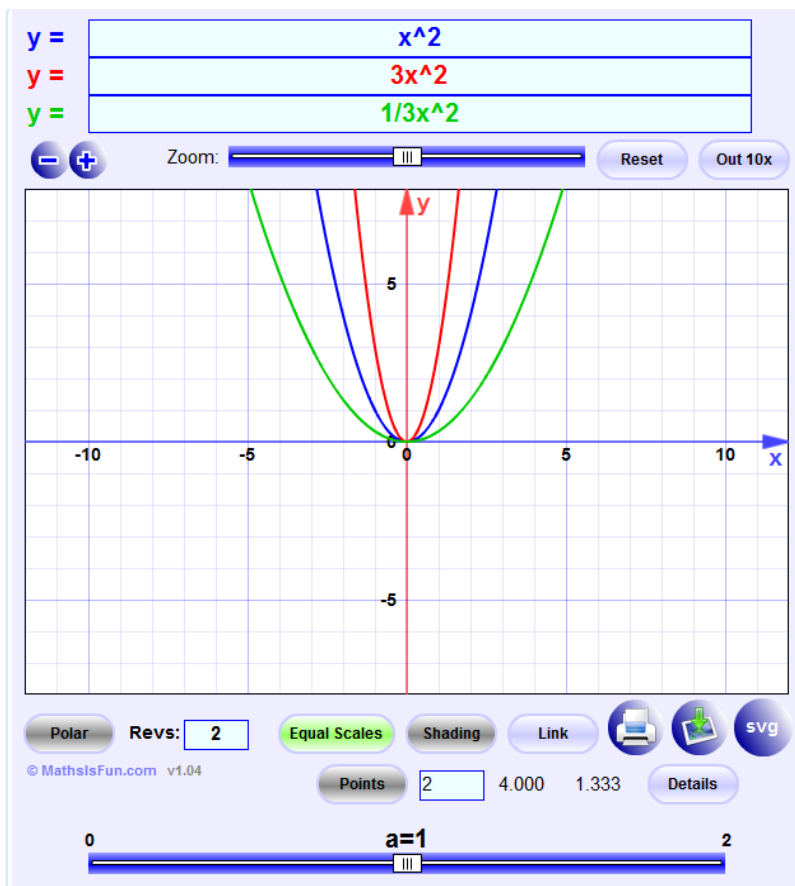
x	$f(x) = x$	$3f(x) = 3x$
-2	-2	-6
-1	-1	-3
0	0	0
1	1	3
2	2	6

Notice that all of the y values have been multiplied by 3. Due to the steeper slope, you have a **vertical stretch**.

You may see this easier with a quadratic equation. Try a parent function of $f(x) = x^2$. Then try $f(x) = 3x^2$.

Try graphing these at: [Math is Fun](#).

Type in $y=x^2$ for the blue equation and $y=3x^2$ for the red equation. The ^ symbol is found above the 6 key and denotes an exponent. You can also do this for the equations in example 4. After you have typed both in, the graphs appear. You should notice that the red graph is much steeper than the blue graph. We call this a **stretch** because the y values are higher for the same x -values. Now, under the red y , you will see a + and – button. Clicking these allows you to delete an equation or add additional equations. Click the + button. You should now see a green $y=$. You should now be able to input a 3rd equation. We are going to enter a new equation of $y=(1/3)x^2$. How does the green graph compare to the others? Is it narrower? Wider? In case you are having trouble, click here to see the graphs.



You should notice that the green graph is wider. We call this a **vertical compression** because the y -values are smaller for the same x -values.

If you would like to see this using the activity we used earlier, go to [Function Transformations - NLVM](#) and make sure to click the forward arrow $>$ as we did before until you reach "Multiplying by a Constant."

What happens if k is negative? What if we had $-3f(x)$? Try it! Click the $+$ button under the green y to add another function. Then type $y = -3x^2$.

You get a mirror image of $3f(x)$. This is called a reflection about the x -axis. You still have a vertical stretch but in the opposite direction.

The graph of $f(x) = x$ is stretched vertically when multiplied by k , a number > 1 .

The graph of $f(x) = x$ is compressed vertically when multiplied by k , a number between 0 and 1, also written as $0 < k < 1$.

This is represented by $kf(x)$ in the table at the end of the lesson.

When k is negative, the graph is also reflected about the x -axis as well as being stretched or compressed.

Example #5: How does the graph of $f(3x)$ compare to the parent function $f(x) = x^3$? Notice that this is different from the previous problems because now, instead of multiplying the whole function by 3, the input is $3x$.

To find the the new function, substitute $3x$ for x . $f(3x) = (3x)^3$.

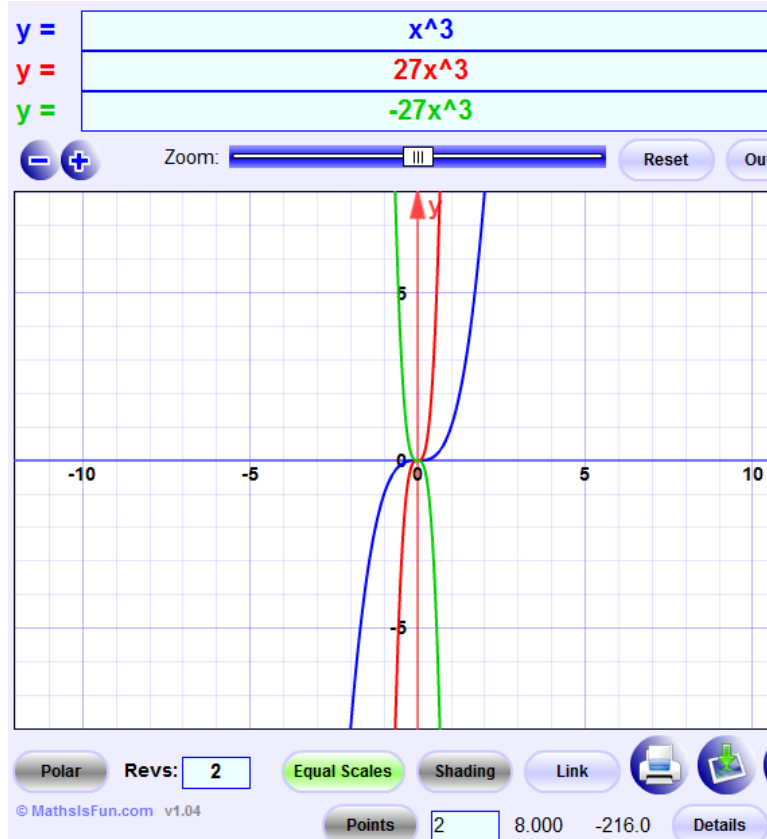
$$(3x)^3 = 27x^3$$

Graph these 2 equations to see how they compare. Type x^3 for the blue equation and $27x^3$ for the red equation. Notice how the x -values change. All x -values are multiplied by $1/k$ (or divided by k) while the y -values stay the same.

Now, add the green equation and see what happens when $k = -3$. Your equation would now be $f(-3x) = (-3x)^3 = -27x^3$.

What happened with the green equation?

[Click here to see the answer.](#)



Did you see that the green equation is a mirror image across the y -axis?

To better see how all of this works, you are encouraged to make up equations and type them in the website we have been using and compare them. If you would prefer, you can also try the graphing calculator at [Desmos](#). It works very well also.

You can see further examples at [Math is Fun](#).

And if you want to go deeper into this topic, try [Parent Functions and Transformations](#).

The graph of $f(x) = x$ is stretched horizontally when x is multiplied by k , a number > 1 .

The graph of $f(x) = x$ is compressed horizontally when x is multiplied by k , a number between 0 and 1, also written as $0 < k < 1$.

When k is negative, the graph is also reflected about the y -axis as well as being stretched or compressed.

In summary, see the table below to remember how to shift the graphs based on their equations and inputs.

In summary:

Transformations of Functions	
$f(x) + k$	shift graph up k units
$f(x) - k$	shift graph down k units
$f(x + k)$	shift graph left k units
$f(x - k)$	shift graph right k units
$k * f(x)$	If $0 < k < 1$, then graph shrinks vertically If $k > 1$, then graph stretches vertically
$f(kx)$	If $0 < k < 1$, then graph shrinks horizontally If $k > 1$, then graph stretches horizontally
$f(-x)$	Graph reflected about the y -axis
$-f(x)$	Graph reflected about the x -axis

Stop! Go to Questions #14-28 to complete this unit.