MORE ON QUADRATICS

Unit Overview

In this unit, you will learn how to write a quadratic equation in vertex form by using a process called completing the square. Completing the square is a process that turns a quadratic into a perfect square trinomial. You will also solve equations using this process when factoring is not possible. The unit will conclude with the Quadratic Formula that is also used to solve quadratic equations and determine what type of solutions there will be.

Completing the Square

In the previous unit, you were introduced to the vertex form of a quadratic ($y = a(x-h)^2 + k$) and used this form to determine the direction of opening, vertex, and axis of symmetry of a parabola. In this unit, you will learn a process called "completing the square," which is used to make a perfect square trinomial so that it can be put into the vertex form. Follow the example below.

Example #1: Write $y = x^2 - 8x + 7$ in vertex form, and then determine the vertex of the parabola.

a.) Take half the linear term (–8) and square it.

$$\frac{1}{2}(-8) = -4$$

$$(-4)^2 = 16$$

b.) Add this to make the trinomial a perfect square.

$$y = (x^2 - 8x + \underline{16}) + 7$$

c.) Subtract this value (16) from the constant (7) to keep the equation equal. We do this to keep the equation balanced. Remember that adding 16 changes the original problem, so subtracting 16 after adding 16 keeps it the same (+16 – 16 = 0).

$$y = (x^2 - 8x + \underline{16}) + 7 - \underline{16}$$

d.) Factor the trinomial in parentheses and combine the 7 – 16.

$$y = (x^2 - 8x + \underline{16}) + 7 - \underline{16}$$

$$y = \underbrace{(x-4)(x-4)}\ -9$$

$$y = (x-4)^2 - 9$$

e.) This is now in vertex form and the **vertex** is (4, –9)
Keep in mind that because you are creating a perfect square trinomial, you will easily factor by using the term that you squared above in part a (–4). You should not have to figure out the factors by trial and error, in this case.

Example #2: Write $y = x^2 + 6x + 4$ in vertex form by completing the square, and then determine the vertex of the parabola.

a.) Take $\frac{1}{2}$ the linear term (6) and square it.

$\frac{1}{2}(6) = 3$

$(3)^2 = 9$

b.) Add this to the equation to produce a perfect square trinomial.

$y = (x^2 + 6x + \underline{9}) + 4$

c.) Subtract this value from the constant to keep the equation equal.

$y = (x^2 + 6x + \underline{9}) + 4 - \underline{9}$

d.) Factor the trinomial in parentheses.

$y = (x^2 + 6x + \underline{9}) + 4 - \underline{9}$

$y = \underbrace{(x + 3)(x + 3)} - 5$

$y = (x + 3)^2 - 5$

e.) The **vertex** of this quadratic is (–3, –5).

Example #3: Write $y = x^2 - 9x - 10$ in vertex form by completing the square, and then determine the vertex of the parabola.

a.) Take $\frac{1}{2}$ the linear term (–9) and square it.

$$\frac{1}{2}(9) = \frac{9}{2}$$
$$(\frac{9}{2})^2 = \frac{81}{4}$$

b.) Add this to the equation to produce a perfect square trinomial.

$$y = (x^2 - 9x + \frac{81}{4}) - 10$$

c.) Subtract this value from the constant to keep the equation equal.

$$y = (x^2 - 9x + \frac{81}{4}) - 10 - \frac{81}{4}$$

d.) Factor the trinomial in parentheses.

$$y = (x^2 - 9x + \frac{81}{4}) - \frac{40}{4} - \frac{81}{4} \qquad {}^*10 = \frac{10}{1} = \frac{40}{4}$$

$$y = \underbrace{(x - \frac{9}{2})(x - \frac{9}{2})} - \frac{121}{4}$$

$$y = (x - \frac{9}{2})^2 - \frac{121}{4}$$

e.) The **vertex** of this quadratic is $(\frac{9}{2}, -\frac{121}{4})$ or $(4\frac{1}{2}, -30\frac{1}{4})$.

Wrap up for completing the square:

- Find half the linear term and square it.
- Add this number to form a perfect square trinomial.
- Subtract this number from the constant to keep the equation equal.
- Factor the trinomial.
- Determine the vertex.

Stop! **Go to Questions #1-7 about this section, then return to continue on to the next section.**

Solving Equations

Completing the square can be used to solve equations in the form of $ax^2 + bx + c = 0$ when factoring is not possible. You need to make sure that each equation is set equal to 0 before you begin to complete the square.

Example #1: Solve $x^2 - 6x - 27 = 0$ by completing the square.

a.) Find half the linear term and square it.

$$\frac{1}{2}(-6) = -3$$

$$(-3)^2 = 9$$

b.) Add this to the equation to form a perfect square trinomial.

$$(x^2 - 6x + \underline{9}) - 27 = 0$$

c.) Subtract the value (9) from the constant (–27) to keep the equation equal (this can actually be done with step *b* if you want).

$$(x^2 - 6x + \underline{9}) - 27 - \underline{9} = 0$$

d.) Factor the trinomial and combine the constant terms.

$$(x-3)^2 - 36 = 0$$

e.) Isolate the squared quantity (because that is where the variable is).

$$(x-3)^2 = 36$$

f.) Take the square root of both sides and solve.

$$\sqrt{(x-3)^2} = \sqrt{36}$$

$$x - 3 = \pm 6$$

$$x = \pm 6 + 3$$

$x = 6 + 3$ and $x = -6 + 3$

$x = 9$ and $x = -3$

The exact solutions are $x = 9$ and $x = -3$.

Example #2: Solve $x^2 - 10x + 14 = 0$ by completing the square.

a.) Find half the linear term and square it.

$$\frac{1}{2}(-10) = -5$$
$$(-5)^2 = 25$$

b.) Add this to the equation to form a perfect square trinomial.

$$(x^2 - 10x + \underline{25}) + 14 = 0$$

c.) Subtract the value (25) from the constant 14 to keep the equation equal (this can actually be done with step *b* if you want).

$$(x^2 - 10x + \underline{25}) + 14 - \underline{25} = 0$$

d.) Factor the trinomial and combine the constant terms.

$$(x-5)^2 - 11 = 0$$

e.) Isolate the squared quantity (because that is where the variable is).

$$(x-5)^2 = 11$$
$$\sqrt{(x-5)^2} = \sqrt{11}$$
$$x - 5 = \pm\sqrt{11}$$
$$x = 5 \pm \sqrt{11}$$

The exact solutions are $x = 5 + \sqrt{11}$ and $x = 5 - \sqrt{11}$.
Unless asked to do so, you can leave your answer in this form. Using your calculator to find $\sqrt{11}$ will give a decimal that needs to be rounded. Therefore, leaving in this form, gives an exact, rather than a rounded solution.

Example #3: Solve $x^2 + 12x = -20$ by completing the square.

$x^2 + 12x + 20 = 0$ *add 20 to both sides first

$(x^2 + 12x + \underline{36}) + 20 - \underline{36} = 0$

$(x+6)^2 - 16 = 0$

$(x+6)^2 = 16$

$\sqrt{(x+6)^2} = \pm\sqrt{16}$

$(x+6) = \pm 4$

$x = \pm 4 - 6$

$x = 4 - 6$ and $x = -4 - 6$

$x = -2$ and $x = -10$

The exact solutions are $x = -2$ and $x = -10$.

Example #4: Solve $x^2 - 9x - 10 = 0$ by completing the square.

$(x^2 - 9x + \frac{81}{4}) - 10 - \frac{81}{4} = 0$ Take half the linear term (-9), square it, and then add it and subtract it to "complete the square."

$(x^2 - 9x + \frac{81}{4}) - \frac{40}{4} - \frac{81}{4} = 0$ Write 10 in 4ths $(10 = \frac{10}{1} = \frac{40}{4})$.

$(x - \frac{9}{2})^2 - \frac{121}{4} = 0$ Simplify.

$(x - \frac{9}{2})^2 = \frac{121}{4}$ Isolate the squared quantity.

$\sqrt{(x - \frac{9}{2})^2} = \sqrt{\frac{121}{4}}$ Take the square root of both sides.

$x - \frac{9}{2} = \pm\frac{11}{2}$ Solve for *x*.

$x = \pm\frac{11}{2} + \frac{9}{2}$

$x = +\frac{11}{2} + \frac{9}{2}$ and $x = -\frac{11}{2} + \frac{9}{2}$

$x = \frac{20}{2} = 10$ and $x = \frac{-2}{2} = -1$

The exact solutions are $x = 10$ and $x = -1$.

Stop! **Go to Questions #8-11 about this section, then return to continue on to the next section.**

The Quadratic Formula

Another process used to solve quadratic equations in the form of $ax^2 + bx + c = y$ is the Quadratic Formula.

Quadratic Formula

If a quadratic is in the form of $ax^2 + bx + c = y$, then the quadratic formula can be used to solve the equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic formula can be used to solve any quadratic equation but is commonly used when the quadratic is **not factorable**.

To use the quadratic formula, first identify *a*, *b*, and *c*, and then substitute these values into the formula and solve.

Example #1: Solve $x^2 - 4x - 10 = 0$ using the quadratic formula.

a.) Identify *a*, *b*, and *c*. $a = 1$ $b = -4$ $c = -10$

b.) Substitute the above values into the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-10)}}{(2)(1)}$$

c.) Solve. $x = \frac{4 \pm \sqrt{16 - (-40)}}{2}$

$$x = \frac{4 \pm \sqrt{56}}{2}$$

The solutions to this quadratic are $x = \frac{4+\sqrt{56}}{2}$ and $x = \frac{4-\sqrt{56}}{2}$.

Example #2: Solve $y = 3x^2 - 2x - 4$ using the quadratic formula.

a = 3, b = –2, c = –4

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-4)}}{(2)(3)}$$

$$x = \frac{2 \pm \sqrt{4-(-48)}}{6}$$

$$x = \frac{2 \pm \sqrt{52}}{6}$$

$$x = \frac{2+\sqrt{52}}{6} \text{ and } x = \frac{2-\sqrt{52}}{6}$$

Part of the quadratic formula is useful in determining how many real solutions the equation will have. In other words, it is helpful in determining how many values x has to make this equation true. This part is called the **discriminant** and is the value under the radical sign, $b^2 - 4ac$.

- If the value of $b^2 - 4ac$ is **less than** zero, there are **no real solutions**.
- If the value of $b^2 - 4ac$ is **equal to** zero, there is **one real solution.**
- If the value of $b^2 - 4ac$ is **greater than** zero, there are **two real solutions**.

Sometimes, one may think the radical goes with the value of the discriminant, but it **DOES NOT.**

Example #3: Use the value of the discriminant to determine how many real solutions the following equation will have.

$y = x^2 - 8x + 7$

a.) Identify a, b, and c. a = 1, b = –8, c = 7

b.) Substitute the values into the discriminant $b^2 - 4ac$.

$(-8)^2 - 4(1)(7)$

$64 - 28$

$= 36$

The value of the discriminant of this quadratic equation is **greater than zero**, and therefore, the equation has **two real solutions**.

Example #4: Use the value of the discriminant to determine how many real solutions the following equation will have.

$4x^2 - 5x + 11 = 0$

a.) Identify *a*, *b*, and *c*.

$a = 4$, $b = -5$, $c = 11$

b.) Substitute the values into the discriminant $b^2 - 4ac$.

$(-5)^2 - 4(4)(11)$

$25 - 176$

$= -151$

The value of the discriminant of this quadratic equation is **less than zero**, and therefore, the equation has **no real solutions**.

Example #5: Use the value of the discriminant to determine how many real solutions the following equation will have.

$9x^2 - 30x + 25 = 0$

a.) Identify *a*, *b*, and *c*.

$a = 9$, $b = -30$, $c = 25$

b.) Substitute the values into the discriminant $b^2 - 4ac$.

$$(-30)^2 - 4(9)(25)$$

$$900 - 900$$

$$= 0$$

The value of the discriminant of this quadratic equation **equals zero**, and therefore, the equation has **one real solution**.

Stop! **Go to Questions #12-31 to complete this unit**