

## FACTORING

$$\begin{array}{l} x^2 - x - 6 = 0 \\ (x - 3)(x + 2) = 0 \\ x - 3 = 0 \quad \text{or} \quad x + 2 = 0 \\ \begin{array}{r} +3 \quad +3 \\ \hline x = 3 \end{array} \quad \uparrow \quad \begin{array}{r} -2 \quad -2 \\ \hline x = -2 \end{array} \end{array}$$

### Unit Overview

In this unit, you will review prime factorization of positive integers, and then extend this concept to factoring polynomials and other algebraic expressions such as quadratics. Factoring helps in simplifying fractions and solving equations.

## Prime Factorization

**Factors:** numbers or expressions whose product is a given number or expression. If  $xy = z$ , then  $x$  and  $y$  are factors of  $z$ .

*Example:*  $3 \times 6 = 18$  so 3 and 6 are factors of 18.

**Multiple:** a number that may be divided by one of its factors with no remainder. If  $xy = z$ , then  $z$  is a multiple of  $x$  and also a multiple of  $y$ .

*Example:* 18 is a multiple of 3 because  $3 \times 6 = 18$ .

**Common Factor:** a number that is a factor of two or more given numbers.

*Example:* 3 is a common factor of 12 and 15

**Prime Number:** a number whose factors are one and itself.

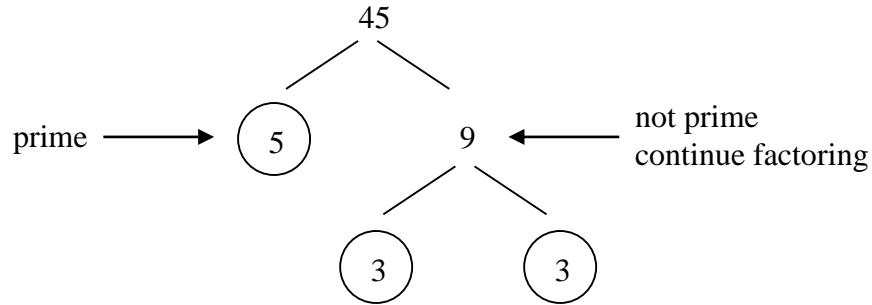
*Example:* 2, 3, 5, 7, 11, 13 ... are all prime because the only factors of each number are one and itself.

**Composite Number:** any number that has more than one and itself as factors; the non-prime numbers.

When a number is composite, it can be factored into primes. This is done by using a factor tree. Study the examples below.

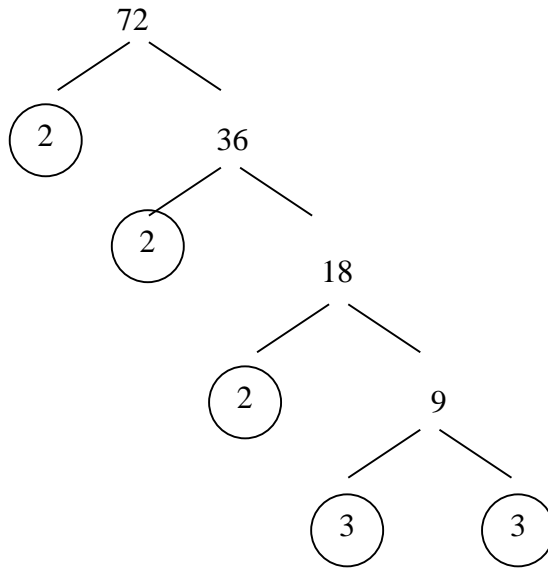
- a.) Decide if the given number is divisible by one of the smaller prime numbers (start with 2, 3, 5, and so on until you find a prime number that is a factor of the given number).
- b.) List that prime number with its factor underneath the given number.
- c.) Repeat this process until all factors are prime numbers.

Example #1: Factor 45 into primes.



$5 \cdot 3 \cdot 3 = 5 \cdot 3^2$  is the prime factorization of 45.

Example #2: Factor 72 into primes.



$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2$  is the prime factorization of 72.

Prime factorization is written as a multiplication problem of prime numbers.

**Stop!** Go to Questions #1-8 about this section, then return to continue on to the next section.

## Monomial Factors

When asked to factor a polynomial, the first thing you want to do is look at the terms for a greatest common factor (GCF).

You will then factor out this term (actually divide all terms by the GCF) and write the expression. This process is actually undoing the distributive property.

Study the examples below.

*Example #1:* Factor  $4y^2 + 12y$ .

- a.) each term contains a  $4y$ , so this is the GCF.
- b.) divide each term by  $4y$  to produce the following:

$$4y(y + 3)$$

*Example #2:* Factor  $3x^4 - 12x^3 + 6x^2$ .

- a.) the GCF is  $3x^2$ .
- b.) divide each term by  $3x^2$ .

$$3x^2(x^2 - 4x + 2)$$

*Example #3:* Factor  $24p^3q - 30pq^3$ .

- a.) the GCF is  $6pq$ .
- b.) divide each term by  $6pq$ .

$$6pq(4p^2 - 5q^2)$$



*Example #4:* Factor  $x(y + 2) + z(y + 2)$ .

\*Notice that both terms  $x(y + 2)$  and  $z(y + 2)$  contain a common factor of  $(y + 2)$ . This is called a common binomial factor. Use  $(y + 2)$  to write the following.

$$x(y + 2) + z(y + 2)$$

$$(y + 2)(x + z)$$

*Example #5:* Factor  $b(a - 5) - 4(a - 5)$ .

The common binomial factor is  $(a - 5)$  so:

$$b(a - 5) - 4(a - 5)$$

$$(a - 5)(b - 4)$$

Sometimes, you can not factor a polynomial using the method from the examples above, but instead by **grouping** terms together. This method is generally used when there are more than 3 terms in the polynomial. Study the examples below.

*Example #6:* Factor  $xy - 3y + 2x - 6$  by grouping.

a.) Group the terms that contain a common factor.

$$(xy - 3y) + (2x - 6)$$

b.) Factor out the GCF from each group.

$$y(x - 3) + 2(x - 3)$$

c.)  $(x - 3)$  is the common binomial factor so:

$$xy - 3y + 2x - 6 \text{ factors into}$$

$$(x - 3)(y + 2)$$

*Example #7:* Factor  $m^2 + 6m + 4m + 24$

$$(m^2 + 6m) + (4m + 24) \quad \text{group terms together}$$

$$m(m + 6) + 4(m + 6) \quad \text{factor out the GCF}$$

$$(m + 6)(m + 4) \quad \text{common binomial factor } (m + 6)$$

***Stop!*** Go to Questions #9-15 about this section, then return to continue on to the next section.

## Factoring Trinomials

Polynomials in the form of  $x^2 + bx + c$  are factored in a different way because there are no common factors between the terms. In this section, we are going to focus on quadratics whose leading coefficient is 1 (the coefficient of  $x^2$ ). Remember that a polynomial whose degree is two (2) is called a quadratic.

When factoring a polynomial of the form  $x^2 + bx + c$ , there is a pattern to discover.

\* Look for factors of the constant term “ $c$ ” whose sum is the coefficient of the linear term “ $b$ ”.

*Example #1:* Factor  $x^2 + 5x + 6$ .

a.) List the factors of the constant term (6).

$$1 \cdot 6$$

$$2 \cdot 3$$

b.) Find the sum of each set of factors.

$$1 + 6 = 7$$

$$2 + 3 = 5$$

c.) The sum of 2 and 3 is the coefficient of the linear term ( $5x$ ); so, we are going to use the factors of 2 and 3 to factor the trinomial.

d.) The trinomial  $x^2 + 5x + 6$  is factored as follows:

$$(x + 2)(x + 3)$$

factors of  $x^2$       factors of 6 whose sum is 5

$$\begin{aligned} \text{Check: } (x+2)(x+3) &= x^2 + 3x + 2x + 6 \\ &= x^2 + 5x + 6 \end{aligned}$$

*Example #2:* Factor  $x^2 + 11x + 18$ .

| factors of 18 | sum |
|---------------|-----|
| 1, 18         | 19  |
| 2, 9          | 11  |
| 3, 6          | 9   |

$$(x + 2)(x + 9) \text{ or } (x + 9)(x + 2)$$

*Example #3:* Factor  $x^2 + 13x + 30$

| factors of 30 | sum |
|---------------|-----|
| 1, 30         | 31  |
| 2, 15         | 17  |
| 3, 10         | 13  |
| 5, 6          | 11  |

$$(x + 3)(x + 10) \text{ or } (x + 10)(x + 3)$$



\*Not all trinomials will contain (+) signs, so you must be very careful when finding factors of the constant term and choosing your sum.

*Example #4:* Factor  $x^2 - 10x + 16$ .

**\*Notice in this trinomial that you are trying to find factors of  $c$  (+16) whose sum is (-10). This means that both factors will have to be negative because to get +16, you must multiply two positive or two negative numbers.**

| factors of 16 | sum |
|---------------|-----|
| -1, -16       | -17 |
| -2, -8        | -10 |
| -4, -4        | -8  |

$(x - 2)(x - 8)$  or  $(x - 8)(x - 2)$

\*Sometimes, it will be necessary to find factors of a negative constant. In this case, you will have to list all possibilities in order to determine which sum is correct.

*Example #5:* Factor  $x^2 - 2x - 8$ .

| factors of -8 | sum |
|---------------|-----|
| -1, 8         | 7   |
| 1, -8         | -7  |
| -2, 4         | 2   |
| 2, -4         | -2  |

**\*In this case, you have to be very careful as to which factor is (+) and which factor is (-) because you are looking for the sum of two factors to equal the linear coefficient.**

$$(x + 2)(x - 4) \text{ or } (x - 4)(x + 2)$$

*\*In this case, make sure the negative sign stays with the 4.*

The examples above lead us into a discussion of signs within the trinomial and how to determine what signs will be used with the factors.

- a.) If both signs are (+), then both factors will be (+).

$$x^2 + bx + c$$

$$(x + \underline{\quad})(x + \underline{\quad})$$

- b.) If the last sign is (+) and the middle sign is (-), then both factors will be (-).

$$x^2 - bx + c$$

$$(x - \underline{\quad})(x - \underline{\quad})$$

- c.) If the last sign is (-), then one factor is (+) and one factor is (-). You have to determine which is which depending on the sum you are trying to find.

$$x^2 + bx - c$$

or

$$x^2 - bx - c$$

$$(x + \underline{\quad})(x - \underline{\quad})$$

$$(x + \underline{\quad})(x - \underline{\quad})$$

Example #6: Factor  $x^2 - 13xy - 48y^2$ .

If the last sign is ( - ), then one factor is (+) and one factor is ( - ). You have to determine which is which depending on the sum -13.

| factors of -48 | sum |
|----------------|-----|
| -1, 48         | 47  |
| 1, -48         | -47 |
| -2, 24         | 22  |
| 2, -24         | -22 |
| -3, 16         | 13  |
| 3, -16         | -13 |
| -4, 12         | 8   |
| 4, -12         | -8  |
| -6, 8          | 2   |
| 6, -8          | -2  |

**\*Be very careful as to which factor is (+) and which factor is (-) because you are looking for the sum of two factors to equal the linear coefficient. Notice that the last term is not a constant but rather  $-48y^2$ . Therefore, the factor will contain a y as well as a coefficient.**

$$(x + 3y)(x - 16y) \text{ or } (x - 16y)(x + 3y)$$

*\*In this case, make sure the negative sign stays with the 16.  
You can check your answer by using FOIL to multiply it out and see that it matches the original problem.*



Factoring Trinomials

**Activity**

**Stop!** Go to Questions #16-20 about this section, then return to continue on to the next section.

## Factoring Trinomials in the form of $ax^2 + bx + c$ ( $a > 1$ )

The process of factoring a trinomial, whose leading coefficient is something other than one (1), involves the factors of the first and last terms. It is called trial and error because you are trying different combinations of factors until you find the correct combination that will correctly factor the trinomial. Study the examples below.

*Example #1:* Factor  $2x^2 + 11x + 14$ .

*(Since all signs are positive, this means that all factors will be positive.)*

a.) List factors of  $2x^2$ .     $2x, x$

b.) List factors of  $+14$ .     $1, 14$   
    $2, 7$

c.) Use trial and error to determine the combination needed so that the product of the outside factors **plus** the product of the inside factors result is the linear term (middle term).

$$\begin{array}{c} \text{outside product} \\ \quad \quad \quad \color{blue}{28x} \\ \quad \quad \quad \frown \\ (2x + 1)(x + 14) \\ \quad \quad \quad \smile \\ \text{inside product} \\ \quad \quad \quad \color{blue}{1x} \end{array}$$

sum of products  $28x + 1x = 29x$   
**this is not the linear term so try again  
by switching the factors around**

$$(2x + 14)(x + 1)$$

sum of products  $2x + 14x = 16x$

$$(2x + 2)(x + 7)$$

sum of products  $14x + 2x = 16x$

$$(2x + 7)(x + 2)$$

sum of products  $4x + 7x = \mathbf{11x}$

Since the last combination of factors resulted in the correct linear term, the factored form of  $2x^2 + 11x + 14$  is  $(2x + 7)(x + 2)$ .

Example #2: Factor  $6m^2 - 7m - 5$ .

(Since the last term is negative, this means that one factor is positive and the other is negative.)

a.) List factors of  $6m^2$ .  $1m, 6m$   
 $2m, 3m$

b.) List factors of  $-5$ .  $-1, 5$   
 $1, -5$

c.) Use trial and error to find the correct combination.

$(6m - 1)(m + 5)$  products  $30m - 1m = 29m$

$(6m + 5)(m - 1)$  products  $-6m + 5m = -1m$

$(6m + 1)(m - 5)$  products  $-30m + 1m = -29m$

$(6m - 5)(m + 1)$  products  $6m - 5m = 1m$

Since there were no correct combinations with  $6m$  and  $m$  used as the factors of  $6m^2$ , try combinations with  $2m$  and  $3m$  as factors.

$(2m - 1)(3m + 5)$  products  $10m - 3m = 7m$  (need a  $-7m$ )

$(2m + 1)(3m - 5)$  products  $-10m + 3m = -7m$

The correct factored form of  $6m^2 - 7m - 5$  is  $(2m + 1)(3m - 5)$

Example #3: Factor  $4c^2 + 3c - 10$ .

(Since the last term is negative, this means that one factor is positive and the other is negative.)

a.) List factors of  $4c^2$ .     $1c, 4c$   
    $2c, 2c$

b.) List factors of  $-10$ .     $-1, 10$   
    $1, -10$   
    $-2, 5$   
    $2, -5$

c.) Use trial and error to find the correct combination.

$(4c - 1)(c + 10)$         products  $40c - 1c = 39c$

$(4c + 1)(c - 10)$         products  $-40c + 1c = -39c$

$(4c - 10)(c + 1)$         products  $4c - 10c = -6c$

$(4c + 10)(c - 1)$         products  $-4c + 10c = 6c$

$(4c - 2)(c + 5)$         products  $20c - 2c = 18c$

$(4c + 2)(c - 5)$         products  $-20c + 2c = -18c$

$(4c - 5)(c + 2)$         products  $8c - 5c = 3c$

Since the last combination of factors resulted in the correct linear term, the factored form of  $4c^2 + 3c - 10$  is  $(4c - 5)(c + 2)$ .

*Example #4: Factor  $9x^2 - 6x + 1$ .*

*(Since the last term is positive and the middle term is negative, this means that both factors will be negative.)*

a.) List factors of  $9x^2$ .     $9x, 1x$   
    $3x, 3x$

b.) List factors of +1         $-1, -1$  (Only the negative factors are needed.)

c.) Use trial and error to find the correct combination.

$(9x - 1)(x - 1)$         products  $-9x - 1x = -10x$

$(3x - 1)(3x - 1)$         products  $-3x - 3x = -6x$

Since the last combination of factors resulted in the correct linear term, the factored form of  $9x^2 - 6x + 1$  is  $(3x - 1)(3x - 1)$  or  $(3x - 1)^2$ .

As you practice this type of factoring, you will be able to notice if certain factors will combine without having to write each combination out.

*Example #5:* Factor  $25x^2 - 81$ .

This is a special case of factoring. Notice that both terms are perfect squares and can be written as follows.

$$\begin{aligned} 25x^2 - 81 \\ (5x)^2 - 9^2 \end{aligned}$$

This case is called the “**difference of two squares.**”

To quickly factor,

- (1) take the square root of both terms

$$\sqrt{25x^2} = 5x \quad \sqrt{81} = 9$$

- (2) Write the square roots as a product of two binomial factors, one connected with a plus and the other connected with a minus.

$$(5x + 9)(5x - 9)$$

The factored form of  $25x^2 - 81$  is  $(5x + 9)(5x - 9)$ .

Multiply to check.

$$\begin{aligned} (5x + 9)(5x - 9) &= \\ &= 25x^2 - 45x + 45x - 81 \\ &= 25x^2 \cancel{-45x} \cancel{+45x} - 81 \\ &= 25x^2 - 81 \end{aligned}$$

***Stop!*** Go to Questions #21-26 about this section, then return to continue on to the next section.



## Solving Quadratic Equations

The process of factoring trinomials is used in solving quadratic equations. Quadratic equations are solved in order to determine what the zeros of a quadratic graph are. The zeros are the places at which the graph crosses the  $x$ -axis and will be discussed in more detail in later units. **The Zero Product Property** is used when solving quadratic equations.

### Zero Product Property

If  $a$  and  $b$  are two real numbers and  $ab = 0$ , then either  $a = 0$  or  $b = 0$ .

**\*Make sure that before you factor, the equation is set equal to zero.**

**\*Also, remember that when you solve a quadratic equation, the solutions are called the zeros because the solutions represent where the graph crosses the  $x$ -axis ( $x, 0$ ).**

*Example #1:* Solve  $x^2 - 11x - 12 = 0$ .

a.) Factor the trinomial.

$$(x - 12)(x + 1) = 0$$

Remember, this means that one factor must equal zero. So, either  $(x - 12)$  must equal zero or  $(x + 1)$  must equal zero. Therefore, we can write the following equations.

b.) Set each factor equal to zero and solve the equations.

$$x - 12 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = 12 \quad \text{or} \quad x = -1$$

c.) The solution to this quadratic equation is  $x = 12$  or  $x = -1$ .

*Example #2:* Solve  $3x^2 + 11x = 4$ .

- a.) Set the equation equal to zero by subtracting 4 from each side.

$$3x^2 + 11x - 4 = 0$$

- b.) Factor the trinomial using trial and error.

$$(3x - 1)(x + 4) = 0$$

- c.) Set each factor equal to zero and solve the equations.

$$3x - 1 = 0 \quad \text{or} \quad x + 4 = 0$$

$$3x = 1 \quad \text{or} \quad x = -4$$

$$x = \frac{1}{3}$$

***Stop!*** Go to Questions #27-34 to complete this unit.