

# RATIONAL EXPRESSIONS

## Unit Overview

In this unit, you will learn how to add, subtract, multiply and divide rational expressions. The unit will conclude with solving rational equations which can be used to model such quantities, such as time, work, and average cost.

# Operations with Rational Expressions

## Multiplying

For both multiplying and dividing rational expressions, we are going to assume that no denominator is equal to zero, so we will not have to find any restrictions at this time.

- Factor the numerators and denominators if possible.
- Cancel any common factors.
- Multiply straight across.

*Example #1:* Find the product.  $\frac{1}{20a} \cdot \frac{5a}{6}$

- In this case, you do not have to factor anything.
- Cancel common factors and use exponent properties to simplify.

$$\frac{1}{\cancel{20}^4 \cancel{a}^1} \cdot \frac{\cancel{5}^1 \cancel{a}^1}{6}$$

A factor of 5 is cancelled diagonally in one direction and a factor of  $a$  is cancelled diagonally in the other direction.

- Multiply straight across.

$$= \frac{1}{24}$$

*Example #2:* Find the product.  $\frac{24xy^2}{8x^4y^3} \cdot \frac{12x^3y^2}{36x^2y}$

- In this case, you do not have to factor anything.
- Cancel common factors.

$$\frac{\cancel{2}^2 \cancel{24}^3 xy^2}{\cancel{8}^2 x^4 y^3} \cdot \frac{\cancel{12}^3 x^3 y^2}{\cancel{36}^3 x^2 y}$$

A factor of 12 is cancelled between the 24 and 36.  
A factor of 4 is cancelled between the 12 and 8.

$$\frac{\cancel{1}^1 \cancel{24}^2 xy^2}{\cancel{1}^2 \cancel{8}^1 x^4 y^3} \cdot \frac{\cancel{1}^1 \cancel{12}^2 x^3 y^2}{\cancel{1}^1 \cancel{36}^3 x^2 y}$$

The left fraction reduces the 2/2 to 1 and the right fraction reduces the 3/3 to 1.

\*Use exponent properties to simplify.

$$\frac{x^4 y^4}{x^6 y^4}$$

The  $x$  and  $y$  terms have been combined using the product property.

$$\frac{\cancel{x^4}^1 \cancel{y^4}^1}{\cancel{x^6}^1 x^2 \cancel{y^4}^1}$$

$$= \frac{1}{x^2}$$

*Example #3:* Find the product.  $\frac{z^2 - 25}{4} \cdot \frac{z}{z^2 + 5z}$

a.) Factor the numerators and denominators.

$$\frac{(z+5)(z-5)}{4} \cdot \frac{z}{z(z+5)}$$

b.) Cancel common factors.

$$\frac{\cancel{(z+5)} (z-5)}{4} \cdot \frac{\cancel{z}}{\cancel{z} \cancel{(z+5)}}$$

c.) Multiply straight across.

$$= \frac{z-5}{4}$$

*Example #4:* Find the product.  $\frac{3v-12}{2v+8} \cdot \frac{3v+12}{v-4}$

a.) Factor the numerators and denominators.

$$\frac{3(v-4)}{2(v+4)} \cdot \frac{3(v+4)}{v-4}$$

b.) Cancel common factors.

$$\frac{\cancel{3(v-4)}}{2(v+4)} \cdot \frac{3(v+4)}{\cancel{v-4}}$$

c.) Multiply straight across.

$$= \frac{9}{2}$$

*Example #5:* Find the product.  $\frac{x^2+6x+9}{x^2-10x+25} \cdot \frac{x^2-x-20}{x^2+7x+12}$

a.) Factor the numerators and denominators.

$$\frac{(x+3)(x+3)}{(x-5)(x-5)} \cdot \frac{(x-5)(x+4)}{(x+3)(x+4)}$$

b.) Cancel common factors.

$$\frac{\cancel{(x+3)}(x+3)}{\cancel{(x-5)}(x-5)} \cdot \frac{\cancel{(x-5)}\cancel{(x+4)}}{\cancel{(x+3)}\cancel{(x+4)}}$$

c.) Multiply straight across.

$$= \frac{x+3}{x-5}$$

***Stop!*** Go to Questions #1-8 about this section, then return to continue on to the next section.

## Dividing

- Factor the numerators and denominators.
- Change to multiplication and multiply by the reciprocal.
- Cancel common factors.
- Multiply straight across.

*Example #6:* Find the quotient.  $\frac{6k^2}{7} \div \frac{9k}{14}$

- There is no need to factor.
- Multiply by the reciprocal.

$$\frac{6k^2}{7} \times \frac{14}{9k}$$

- Cancel common factors.

$$\frac{\overset{2}{\cancel{6}}k^2}{\cancel{7}^1} \times \frac{\overset{2}{\cancel{14}}}{\overset{3}{\cancel{9}}k}$$

- Multiply straight across.

$$\frac{4k^2}{3k}$$

\*Use exponent properties to simplify.

$$\frac{4\cancel{k^2}^k}{3\cancel{k}}$$

$$= \frac{4k}{3}$$

*Example #7:* Find the quotient.  $\frac{5c^3}{10d^3} \div \frac{15d^3}{20c^3}$

a.) There is no need to factor.

b.) Multiply by the reciprocal.

$$\frac{5c^3}{10d^3} \times \frac{20c^3}{15d^3}$$

c.) Cancel common factors.

$$\frac{\overset{1}{\cancel{5}}c^3}{\underset{1}{\cancel{10}}d^3} \times \frac{\overset{2}{\cancel{20}}c^3}{\underset{3}{\cancel{15}}d^3}$$

d.) Multiply straight across.

$$= \frac{2c^6}{3d^6}$$

*Example #8:* Find the quotient.  $\frac{1}{p+3} \div \frac{1}{p-3}$

a.) Multiply by the reciprocal.

$$\frac{1}{p+3} \times \frac{p-3}{1}$$

b.) Multiply across.

$$= \frac{p-3}{p+3}$$

*Example #9:* Find the quotient.  $\frac{18}{y^2-64} \div \frac{9}{y+8}$

a.) Factor the denominators.

$$\frac{18}{(y+8)(y-8)} \div \frac{9}{y+8}$$

b.) Multiply by the reciprocal.

$$\frac{18}{(y+8)(y-8)} \times \frac{y+8}{9}$$

c.) Cancel common factors.

$$\frac{\cancel{18}^2}{\cancel{(y+8)}(y-8)} \times \frac{\cancel{(y+8)}}{\cancel{9}^1}$$

d.) Multiply straight across.

$$= \frac{2}{y-8}$$

*Example #10:* Find the quotient.  $\frac{5m}{m+1} \div \frac{25m^2}{m^2+2m+1}$

a.) Factor the numerators and denominators.

$$\frac{5m}{m+1} \div \frac{25m^2}{(m+1)(m+1)}$$

b.) Multiply by the reciprocal.

$$\frac{5m}{m+1} \times \frac{(m+1)(m+1)}{25m^2}$$

c.) Cancel common factors.

$$\frac{\cancel{5m}}{\cancel{m+1}} \times \frac{\cancel{m+1}(m+1)}{\cancel{25m^2} \cancel{5m}}$$

d.) Multiply straight across.

$$= \frac{m+1}{5m}$$

*Example #11:* Find the quotient.  $\frac{x^2 - x - 6}{x^2 + 8x + 15} \div \frac{x^2 - 4x - 5}{x^2 - 25}$

a.) Factor the numerators and denominators.

$$\frac{(x-3)(x+2)}{(x+5)(x+3)} \div \frac{(x-5)(x+1)}{(x+5)(x-5)}$$

b.) Multiply by the reciprocal.

$$\frac{(x-3)(x+2)}{(x+5)(x+3)} \times \frac{(x+5)(x-5)}{(x-5)(x+1)}$$

c.) Cancel common factors.

$$\frac{(x-3)(x+2)}{\cancel{(x+5)}(x+3)} \times \frac{\cancel{(x+5)}\cancel{(x-5)}}{\cancel{(x-5)}(x+1)}$$

d.) Multiply straight across.

$$= \frac{(x-3)(x+2)}{(x+3)(x+1)}$$

*Example #12:* Find the quotient.  $\frac{x^2 - 144}{5x} \div (x-12)$

a.) Factor the numerator.

$$\frac{(x+12)(x-12)}{5x} \div (x-12)$$

b.) Multiply by the reciprocal.

$$\frac{(x+12)(x-12)}{5x} \times \frac{1}{(x-12)}$$

c.) Cancel common factors.

$$\frac{(x+12)\cancel{(x-12)}}{5x} \times \frac{1}{\cancel{(x-12)}}$$



d.) Multiply straight across.

$$= \frac{x+12}{5x}$$

***Stop!*** Go to Questions #9-16 about this section, then return to continue on to the next section.

## Adding and Subtracting

Adding and subtracting rational expression can be a little more challenging than multiplication and division. When adding or subtracting rational expressions, it is necessary to find a common denominator. Follow along with the steps below and the examples that follow.

- Factor the denominators if possible.
- Determine the least common denominator (LCD).
- Change each expression to have a common denominator.
- Add or subtract the numerators.
- Simplify.

*Example #1:* Find the difference.  $\frac{19}{20x} - \frac{4}{4x}$

- There is no need to factor the denominators.
- Determine the least common denominator (what number is the smallest number that  $20x$  and  $4x$  will go into?)

$$\text{LCD} = 20x$$

- Change  $\frac{4}{4x}$  into a fraction with  $20x$  as the denominator (multiply the entire fraction by  $\frac{5}{5}$ ).

$$\frac{19}{20x} - \frac{4}{4x} \left( \frac{5}{5} \right)$$

$$\frac{19}{20x} - \frac{20}{20x}$$

- Combine the numerators and put over one denominator.

$$\frac{19 - 20}{20x}$$

$$= \frac{-1}{20x}$$

Example #2: Compute.  $\frac{7k}{k} + \frac{k}{k^2} - \frac{4k}{5k}$

- a.) There is no need to factor the denominators.
- b.) Determine the least common denominator (what number is the smallest number that  $k$ ,  $k^2$ , and  $5k$  will go into?)

$$\text{LCD} = 5k^2$$

- c.) Change each fraction to a fraction with  $5k^2$  as the denominator.

$$\frac{7k}{k} \cdot \frac{5k}{5k} + \frac{k}{k^2} \cdot \frac{5}{5} - \frac{4k}{5k} \cdot \frac{k}{k}$$

$$\frac{35k^2}{5k^2} + \frac{5k}{5k^2} - \frac{4k^2}{5k^2}$$

- d.) Combine the numerators and put over one denominator.

$$\frac{35k^2 + 5k - 4k^2}{5k^2}$$

- e.) Combine like terms.

$$\frac{31k^2 + 5k}{5k^2}$$

- f.) Simplify.

$$\frac{k(31k + 5)}{5k^2}$$

\*Factor the numerator.

$$\frac{\cancel{k}(31k + 5)}{5\cancel{k}^2 k}$$

\*Use the exponent properties to simplify.

$$= \frac{31k + 5}{5k}$$

*Example #3:* Find the difference.  $\frac{2}{3y} - \frac{2}{5xy}$

- a.) There is no need to factor the denominators.
- b.) Determine the least common denominator (what number is the smallest number that 3y and 5xy will go into?)

$$\text{LCD} = 15xy$$

- c.) Change each fraction to a fraction with 15xy as the denominator.

$$\frac{2}{3y} \cdot \frac{5x}{5x} - \frac{2}{5xy} \cdot \frac{3}{3}$$

$$\frac{10x}{15xy} - \frac{6}{15xy}$$

- d.) Combine the numerators and put over one denominator.

$$= \frac{10x - 6}{15xy}$$

\*Even though the numerator can be factored to  $2(5x - 3)$ , no further simplification can be done, so leave the numerator as is.

*Example #4:* Find the sum.  $\frac{2x+6}{x^2-9} + \frac{1}{x+3}$

- a.) Factor the denominators.

$$\frac{2x+6}{(x+3)(x-3)} + \frac{1}{x+3}$$

- b.) Determine the LCD (what number will both denominators go into?)

-Since the denominator of the first rational expression is  $(x+3)(x-3)$ , you will need to use at least this.

-Ask yourself if the second denominator is within the first one.

-Since the second denominator is within the first, the LCD of these two expressions is  $(x + 3)(x - 3)$ .

- c.) Change the second expression into one that has a denominator of  $(x + 3)(x - 3)$  by multiplying  $\frac{1}{x+3}$  by  $\left(\frac{x-3}{x-3}\right)$ .

$$\frac{2x+6}{(x+3)(x-3)} + \frac{1}{x+3} \left(\frac{x-3}{x-3}\right)$$

$$\frac{2x+6}{(x+3)(x-3)} + \frac{1(x-3)}{(x+3)(x-3)}$$

- d.) Combine the numerators and put them over one denominator.

$$\frac{2x+6+x-3}{(x+3)(x-3)}$$

- e.) Combine like terms.

$$\frac{3x+3}{(x+3)(x-3)}$$

\*Even though the numerator can be factored to  $3(x + 1)$ , no further simplification can be done, so leave the numerator as is.

*Example #5:* Find the difference.  $\frac{4x-1}{8x} - \frac{2}{x}$

- a.) There is no need to factor the denominators.  
b.) Determine the least common denominator (what number is the smallest number that  $8x$  and  $x$  will go into?)

$$\text{LCD} = 8x$$

- c.) Change the second fraction to a fraction with  $8x$  as the denominator, so multiply the second rational expression by  $\frac{8}{8}$ .

$$\frac{4x-1}{8x} - \frac{2}{x} \left(\frac{8}{8}\right)$$

$$\frac{4x-1}{8x} - \frac{16}{8x}$$

d.) Combine numerators.

$$\begin{aligned} & \frac{4x-1-16}{8x} \\ &= \frac{4x-17}{8x} \end{aligned}$$

*Example #6:* Find the sum.  $\frac{7}{y} + \frac{2}{y+3}$

a.) The LCD is  $y(y+3)$ .

b.) Multiply each term by the appropriate factor to produce the new denominator.

$$\frac{7}{y} \left( \frac{y+3}{y+3} \right) + \frac{2}{y+3} \left( \frac{y}{y} \right)$$

c.) Distribute the numerators.

$$\frac{7y+21}{y(y+3)} + \frac{2y}{y(y+3)}$$

d.) Combine numerators.

$$\begin{aligned} & \frac{7y+21+2y}{y(y+3)} \\ &= \frac{9y+21}{y(y+3)} \end{aligned}$$

\*Even though the numerator can be factored to  $3(3y+7)$ , no further simplification can be done, so leave the numerator as is.

*Example #7:* Find the difference.  $\frac{7}{x-y} - \frac{3x}{x^2-y^2}$

a.) Factor the denominator of the second fraction.

$$\frac{7}{x-y} - \frac{3x}{(x+y)(x-y)}$$

b.) The LCD is  $(x+y)(x-y)$ .

c.) Multiply the first term by the appropriate binomial to produce the new denominator.

$$\frac{7}{x-y} \left( \frac{x+y}{x+y} \right) - \frac{3x}{(x+y)(x-y)}$$

d.) Distribute the numerator of the first fraction.

$$\frac{7x+7y}{(x+y)(x-y)} - \frac{3x}{(x+y)(x-y)}$$

e.) Combine like terms.

$$\begin{aligned} & \frac{7x+7y-3x}{(x+y)(x-y)} \\ &= \frac{4x+7y}{(x+y)(x-y)} \end{aligned}$$

\*You need to be very careful when simplifying a subtraction problem involving more than one term in the numerator. If the rational expression you are subtracting contains multiple terms, you need to make sure that you subtract **all** terms. Follow the example below.

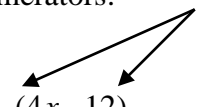
*Example #8:* Find the difference.  $\frac{5}{x-3} - \frac{4}{x+1}$

d.) The LCD is  $(x-3)(x+1)$ .

e.) Multiply each term by the appropriate binomial to produce the new denominator.

$$\frac{5}{x-3} \left( \frac{x+1}{x+1} \right) - \frac{4}{x+1} \left( \frac{x-3}{x-3} \right)$$

f.) Distribute the numerators.

$$\frac{5x+5}{(x-3)(x+1)} - \frac{(4x-12)}{(x+1)(x-3)}$$


\*Notice the parentheses that are needed because you are subtracting the entire quantity.

d.) Distribute the minus sign.

$$\frac{5x+5-4x+12}{(x-3)(x+1)}$$

e.) Combine like terms.

$$= \frac{x+17}{(x-3)(x+1)}$$

**Stop!** Go to Questions #17-29 about this section, then return to continue on to the next section.



## Solving Rational Equations

Quantities such as time, work, and average cost can be modeled with rational equations. To solve a rational equation, follow the steps below.

- a.) Determine the least common denominator.
- b.) Multiply **each** term by the LCD.
- c.) Cancel any common factors.
- d.) Distribute or multiply.
- e.) Combine like terms and solve.

**\*After solving, make sure that the value found for the variable does not produce a zero in the denominator.**

*Example #1:* Solve.  $\frac{x}{6} + \frac{4x}{9} = 5$

- a.) LCD = 18
- b.) Multiply each term by 18.

$$\left(\frac{x}{6}\right)18 + \left(\frac{4x}{9}\right)18 = (5)18$$

- c.) Cancel common factors.

$$\left(\frac{x}{\cancel{6}}\right)\cancel{18}^3 + \left(\frac{4x}{\cancel{9}}\right)\cancel{18}^2 = 90$$

- d.) Multiply.

$$(3 \cdot x) + (2 \cdot 4x) = 90$$

$$3x + 8x = 90$$

e.) Solve the equation.

$$11x = 90$$

$$x = \frac{90}{11}$$

\*In this problem, since there is no variable in the denominator of the original equation, the value for  $x$  will not produce a zero in the denominator.

*Example #2:* Solve.  $\frac{q-2}{5} - \frac{3q-4}{5} = 8$

a.) LCD = 5

b.) Multiply each term by 5.

$$\left(\frac{q-2}{5}\right)5 - \left(\frac{3q-4}{5}\right)5 = (8)5$$

c.) Cancel common factors.

$$\left(\frac{q-2}{\cancel{5}}\right)\cancel{5} - \left(\frac{3q-4}{\cancel{5}}\right)\cancel{5} = (8)5$$

d.) Multiply.

$$q - 2 - (3q - 4) = 40$$

$$q - 2 - 3q + 4 = 40$$

\*Distribute the minus sign over  $3q - 4$ .

e.) Solve the equation.

$$-2q + 2 = 40$$

$$-2q = 38$$

$$q = -19$$

\*In this problem, since there is no variable in the denominator of the original equation, the value for  $q$  will not produce a zero in the denominator.

*Example #3:* Solve.  $4 - \frac{3a-4}{10} = a + \frac{11a+5}{15}$

a.) LCD = 30

b.) Multiply each term by 30.

$$(4)30 - \left(\frac{3a-4}{10}\right)30 = (a)30 + \left(\frac{11a+5}{15}\right)30$$

c.) Cancel common factors.

$$(4)30 - \left(\frac{3a-4}{\cancel{10}}\right)\cancel{30}^3 = (a)30 + \left(\frac{11a+5}{\cancel{15}}\right)\cancel{30}^2$$

d.) Multiply.

$$120 - 3(3a - 4) = 30a + 2(11a + 5)$$

$$120 - 9a + 12 = 30a + 22a + 10 \quad \text{*Distribute.}$$

e.) Solve the equation.

$$132 - 9a = 52a + 10 \quad \text{*Collect like terms.}$$

$$-9a = 52a - 122$$

$$-61a = -122$$

$$a = 2$$

\*In this problem, since there is no variable in the denominator of the original equation, the value for  $a$  will not produce a zero in the denominator.

Example #4: Solve.  $\frac{3}{4n} + \frac{1}{n} = \frac{7}{8}$

a.) LCD =  $8n$

b.) Multiply each term by  $8n$ .

$$\left(\frac{3}{4n}\right)8n + \left(\frac{1}{n}\right)8n = \left(\frac{7}{8}\right)8n$$

c.) Cancel common factors.

$$\left(\frac{3}{\cancel{4n}}\right)\cancel{8n}^2 + \left(\frac{1}{\cancel{n}}\right)\cancel{8n} = \left(\frac{7}{\cancel{8}}\right)\cancel{8n}$$

d.) Multiply.

$$(3 \cdot 2) + (1 \cdot 8) = (7 \cdot n)$$

$$6 + 8 = 7n$$

e.) Solve the equation.

$$14 = 7n$$

$$2 = n$$

\*Replace any “ $n$ ” in the equation with 2 to check and make sure the denominator does not equal zero.

$$\frac{3}{4(2)} + \frac{1}{2} = \frac{7}{8}$$

In this case, it does not, so the solution to this rational equation is  $n = 2$ .

Example #5:  $\frac{1}{m} + \frac{4m}{m+1} = 4$

a.) LCD is  $m(m + 1)$ .

b.) Multiply each term by the LCD.

$$\left(\frac{1}{m}\right)m(m+1) + \left(\frac{4m}{m+1}\right)m(m+1) = (4)m(m+1)$$

c.) Cancel any common factors.

$$\left(\frac{1}{\cancel{m}}\right)\cancel{m}(m+1) + \left(\frac{4\cancel{m}}{\cancel{m+1}}\right)m\cancel{m+1} = (4)m(m+1)$$

d.) Multiply the terms together.

$$1(m+1) + 4m(m) = 4m(m+1)$$

$$m+1 + 4m^2 = 4m^2 + 4m$$

e.) Solve.

$$m+1 + \cancel{4m^2} = \cancel{4m^2} + 4m \quad \text{*Subtract } 4m^2 \text{ from both sides.}$$

$$m+1 = 4m$$

$$1 = 3m$$

$$\frac{1}{3} = m$$

Again, check to make sure that the value found does not produce a zero in the denominator.

$$\frac{1}{\frac{1}{3}} + \frac{4m}{\frac{1}{3} + 1} = 4$$

In this case, it does not; therefore, the solution to this rational expression is  $m = 1/3$ .

Sometimes, it will be necessary to factor the denominators to determine the LCD. It may also be necessary to use the zero product property when solving. You will use this if there is a squared term in the equation. Remember to use the zero product property you will factor the expression, and then set each factor equal to zero and solve. Again, remember to check your answers in the denominator.

Example #6:  $\frac{3}{x-2} - \frac{6}{x^2-2x} = 1$

a.) Factor the denominators.

$$\frac{3}{x-2} - \frac{6}{x(x-2)} = 1$$

The LCD will be  $x(x-2)$

b.) Multiply each term by the LCD  $x(x-2)$ .

$$\left(\frac{3}{x-2}\right)x(x-2) - \left(\frac{6}{x(x-2)}\right)x(x-2) = 1x(x-2)$$

c.) Cancel common terms.

$$\left(\frac{3}{\cancel{x-2}}\right)x\cancel{(x-2)} - \left(\frac{6}{\cancel{x}\cancel{x-2}}\right)\cancel{x}\cancel{(x-2)} = 1x(x-2)$$

d.) Multiply.

$$3(x) - 6 = x(x-2)$$

$$3x - 6 = x^2 - 2x$$

e.) Set the equation equal to zero to factor and use the zero product property. Keep the squared term positive.

$$0 = x^2 - 5x + 6$$

$$0 = (x-3)(x-2)$$

$$x-3=0 \quad x-2=0$$

$$x=3 \quad x \neq 2^*$$

\*If a number produces zero in any denominator of the equation, the number is not a solution.

$$\frac{3}{\cancel{2-2}_0} - \frac{6}{x^2-2x} = 1$$

**Stop!** Go to Questions #30-35 to complete this unit.