RATIONAL NUMBERS AND EXPONENTS

Unit Overview

In this unit, you will learn about rational numbers. You will learn how to simplify rational expressions and solve equations that contain rational numbers. This unit will also introduce you to the properties of exponents so you will be able to simplify polynomials in a later unit.

Rational Numbers

A rational number is any number that can be expressed as a ratio of two integers, $\dfrac{a}{b}$, where $b \neq 0$.

> *Example #1*:
>
> 4 is a rational number because it can be written as $\dfrac{4}{1}$.

> *Example #2*:
>
> –8.3 is a rational number because it can be written as $\dfrac{-83}{10}$.

Therefore, in general, a rational number may be written in the form of a fraction or a decimal.

Review: In this course, we will not explain how to add, subtract, multiply, or divide rational numbers extensively as you should have learned these processes in earlier courses. Therefore, we will only go through a couple of examples to refresh your memory.

Adding/Subtracting:

*Must have a common denominator, and then add or subtract the numerators.
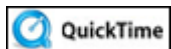
Example #1: $\dfrac{-5}{6} + \dfrac{2}{3}$

$\dfrac{-5}{6} + \dfrac{4}{6}$

$\dfrac{-5+4}{6} = \dfrac{-1}{6}$

Example #2: $\dfrac{-5}{8} - \left(\dfrac{-1}{3}\right)$

$\dfrac{-15}{24} + \left(\dfrac{8}{24}\right)$

$\dfrac{-15+8}{24} = \dfrac{-7}{24}$

QuickTime Adding and Subtracting Fractions--Running and Driving (03:36)

Multiplying:
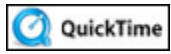
*Cancel any common factors and multiply straight across.

Example #1: $\dfrac{1}{5} \cdot \left(\dfrac{-2}{9}\right)$

$= \dfrac{-2}{45}$

Example #2: $\dfrac{4}{5} \cdot \left(\dfrac{-10}{12}\right)$

$\dfrac{\overset{1}{\cancel{4}}}{\underset{1}{\cancel{5}}} \cdot \left(\dfrac{\overset{}{\cancel{-10}}}{\underset{3}{\cancel{12}}}\right)^{\!2}$

$$\frac{1 \cdot -2}{1 \cdot 3} = \frac{-2}{3}$$
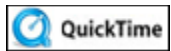
Dividing:

*Multiply by the reciprocal.

Example #1: $\dfrac{3}{5} \div \dfrac{-9}{7}$

$$\frac{\overset{1}{\cancel{3}}}{5} \cdot \frac{-7}{\underset{3}{\cancel{9}}}$$

$$\frac{1 \cdot -7}{5 \cdot 3} = \frac{-7}{15}$$

Stop! **Go to Questions #1-8 about this section, then return to continue on to the next section.**

Solving Equations with Rational Numbers

When solving equations involving rational numbers, you will use the same principle of opposite operations as you did when solving equations with integers. One exception is that when dividing by a rational fraction, remember to multiply by the reciprocal.

Let's try a few examples:

Example #1: Solve $x - \dfrac{5}{6} = \dfrac{7}{8}$ for *x*.

Step #1: Add $\dfrac{5}{6}$ to both sides.

$$x - \dfrac{5}{6} + \dfrac{5}{6} = \dfrac{7}{8} + \dfrac{5}{6}$$

Step #2: Determine the common denominator (24).

$$x = \dfrac{21}{24} + \dfrac{20}{24}$$

Step #3: Add the rational numbers.

$$x = \dfrac{41}{24} \quad \text{or} \quad 1\dfrac{17}{24}$$

Example #2: Solve $y + 6.8 = 2.5$ for *y*.

Step #1: Subtract 6.8 from both sides.

$$y + 6.8 - 6.8 = 2.5 - 6.8$$

Step #2: Simplify and rewrite the subtraction problem on the right side as an addition problem.

$$y = 2.5 + (-6.8)$$

Step #3: Combine the rational numbers.

$$y = -4.3$$

Example #3: $\dfrac{5}{6}=\dfrac{-2}{3}x$

Step #1: Multiply both sides by the reciprocal of $\dfrac{-2}{3}$ which is $\dfrac{-3}{2}$.

$$\dfrac{-3}{2}\left(\dfrac{5}{6}\right) \;=\; \left(\dfrac{-2}{3}x\right)\dfrac{-3}{2}$$

Step #2: Simplify.

$$\dfrac{-\cancel{3}^{\,1}}{2}\left(\dfrac{5}{\cancel{6}^{\,2}}\right) \;=\; \left(\dfrac{-\cancel{2}^{\,1}}{\cancel{3}^{\,1}}x\right)\dfrac{-\cancel{3}^{\,1}}{\cancel{2}^{\,1}}$$

Step #3: Multiply.

$$\dfrac{-1\cdot 5}{2\cdot 2}=x$$

$$x=\dfrac{-5}{4}\quad\text{or}\quad x=-1\dfrac{1}{4}$$

Example #4: $\dfrac{4}{9}q-7=\dfrac{8}{9}q$

Step #1: Subtract $\dfrac{4}{9}q$ from both sides.

$$\cancel{\dfrac{4}{9}q}-\cancel{\dfrac{4}{9}q}-7=\dfrac{8}{9}q-\dfrac{4}{9}q$$

Step #2: Simplify and subtract.

$$-7=\dfrac{4}{9}q$$

Step #3: Multiply both sides by the reciprocal of $\dfrac{4}{9}$ which is $\dfrac{9}{4}$.

$$\dfrac{9}{4}\left(\dfrac{-7}{1}\right) \;=\; \left(\dfrac{4}{9}x\right)\dfrac{9}{4}$$

Step #4: Simplify.

$$\frac{9}{4}\left(\frac{-7}{1}\right) \;=\; \left(\frac{\cancel{4}}{\cancel{9}}x\right)\frac{\cancel{9}}{\cancel{4}}$$

Step #5: Multiply.

$$\frac{9(-7)}{4\cdot 1} = x$$

$$x = \frac{-63}{4} \quad\text{or}\quad x = -15\frac{3}{4}$$

Example #5: $\dfrac{2}{3}t + 8 = t - \dfrac{3}{4}$

Step #1: Subtract t from both sides.

$$\frac{2}{3}t - t + 8 = t - t - \frac{3}{4}$$

Step #2: Subtract and Simplify.

$$-\frac{1}{3}t + 8 = -\frac{3}{4} \qquad\qquad \left(\frac{2}{3}t - t = \frac{2}{3}t - 1t = \frac{2}{3}t - \frac{3}{3}t = -\frac{1}{3}t\right)$$

Step #3: Subtract 8 from both sides.

$$-\frac{1}{3}t + 8 - 8 = -\frac{3}{4} - 8 \qquad\qquad \left(-\frac{3}{4} - 8 = -\frac{3}{4} - \frac{8}{1} = -\frac{3}{4} - \frac{32}{4} = -\frac{35}{4}\right)$$

Step #4: Subtract and Simplify.

$$-\frac{1}{3}t = -\frac{35}{4}$$

Step #5: Multiply both sides by the reciprocal of $-\dfrac{1}{3}$ which is $\dfrac{-3}{1}$.

$$\left(-\frac{3}{1}\right)\left(-\frac{1}{3}\right)t = \left(-\frac{35}{4}\right)\left(-\frac{3}{1}\right)$$

Step #6: Simplify.

$$\left(-\frac{\cancel{3}}{1}\right)\left(-\frac{1}{\cancel{3}}\right)t = \left(-\frac{35}{4}\right)\left(-\frac{3}{1}\right)$$

Step #7: Multiply.

$$t = \frac{(-35)(-3)}{4 \cdot 1}$$

$$t = \frac{105}{4} \quad \text{or} \quad t = 26\frac{1}{4}$$

Stop! **Go to Questions #9-15 about this section, then return to continue on to the next section.**

Exponent Properties

Recall that exponents tell us how many times we use the base as a factor. For example, $3^4 = 3 \cdot 3 \cdot 3 \cdot 3$. An expression written with exponents is said to be in **exponential form**.

When simplifying algebraic expressions involving exponents, there are a few rules (properties) that apply when completely simplifying the expression. Those properties are listed below with examples.

#1) Product Property

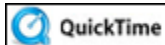When multiplying like bases, **add** the exponents.

$$a^m \cdot a^n = a^{m+n}$$

Example #1: Simplify $x^7 \cdot x^3$.

$$x^7 \cdot x^3 \;=\; x^{7+3} \;=\; x^{10} \qquad\qquad \text{*Add the exponents.}$$

Example #2: Simplify $z^5 \cdot z^2 \cdot z$.

$$z^5 \cdot z^2 \cdot z \;=\; z^5 \cdot z^2 \cdot z^1 \;=\; z^{(5+2+1)} \;=\; z^8$$

QuickTime Multiplying Exponents (09:40)

#2) Power of a Power

When a base to a power is raised to another power, **multiply** the exponents.

$$\left(a^m\right)^n \;=\; a^{mn}$$

Example #3: Simplify $\left(x^4\right)^3$.

$$\left(x^4\right)^3 \;=\; x^{4 \times 3} \;=\; x^{12} \qquad\qquad \text{*Multiply the exponents.}$$

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#3) Power of a Product

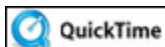When a product is raised to a power, **both** or all bases are raised to the power.

$$(ab)^n = a^n b^n$$

</div>

Example #4: Simplify $(2x)^3$.

$$(2x)^3 = 2^3 \cdot x^3 = 8x^3$$ *Raise all bases to the power.

Example #5: Simplify $(-5m^3n^2)^4$.

$$(-5m^3n^2)^4 = (-5)^4 \cdot m^{3 \times 4} \cdot n^{2 \times 4} = 625m^{12}n^8$$

QuickTime Finding the Power of a Power (09:40)

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#4) Power of a Quotient

When a quotient is raised to a power, **both** the numerator and denominator are raised to the power.

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

</div>

Example #6: Simplify $\left(\frac{2}{3}\right)^2$.

$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$$ *Raise numerator and denominator to the power.

Example #7: Simplify $\left(\dfrac{p^5}{6}\right)^3$.

$$\left(\dfrac{p^5}{6}\right)^3 = \dfrac{p^{5\,\times\,3}}{6^3} = \dfrac{p^{15}}{216}$$

#5) Quotient Property

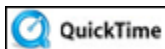When dividing like bases, **subtract** the exponents.

$$\dfrac{a^m}{a^n} = a^{m-n}$$

Example #8: Simplify $\dfrac{x^8}{x^3}$.

$$\dfrac{x^8}{x^3} = x^{8-3} = x^5 \qquad\qquad \text{*Subtract the exponents.}$$

Example #9: Simplify $\dfrac{k}{k}$.

$$\dfrac{k}{k} = \dfrac{k^1}{k^1} = k^{1-1} = k^0 = 1$$

QuickTime Dividing Exponents (08:18)

*Exponential expressions are not considered completely simplified unless the exponents are positive; for this, the next property is especially important.

#6) Negative Exponent Property

When a base has a negative exponent, use the **reciprocal** of the base to produce a positive exponent.

$$a^{-m} = \frac{1}{a^m} \qquad \text{or} \qquad \left(\frac{1}{a}\right)^{-m} = a^m$$

Example #10: Simplify x^{-3}.

$$x^{-3} = \frac{1}{x^3}$$ *The reciprocal of x is $\frac{1}{x}$.

Example #11: Simplify $\left(\dfrac{1}{x}\right)^{-3}$.

$$\left(\frac{1}{x}\right)^{-3} = x^3$$ *The reciprocal of $\frac{1}{x}$ is x.

Example #12: Simplify $\dfrac{q^2}{q^5}$.

$$\frac{q^2}{q^5} = q^{2-5} = q^{-3} = \frac{1}{q^3}$$

Below is a link to Practice with Exponents, a great website that gives you an opportunity to practice some problems. You can check your answer immediately.

- Start with problems labeled *Easier*. You will get a chance to try the harder problems a little later in this unit.
- Use scrap paper as you work out the expressions.
- Try numerous problems, but only spend about 15 minutes doing so.
- Make sure you understand the properties of exponents and how they are used so that you will be able to complete the problems in the questions area.

Click <u>here</u> to begin the practice.

Stop! **Go to Questions #16-22 about this section, then return to continue on to the next section.**

Multiplying and Dividing Monomials

In the previous section, each of the exponent properties used a term called a **monomial.** A monomial can be a number, a variable, or products of numbers and variables. For example, 3, x, and $3xy$ are all considered monomials.

When multiplying or dividing monomials, the properties of exponents must be followed as shown in the examples below:

Example #1: Simplify $(3x^2)(4x^6)$

Step #1: Arrange the terms so the numbers are together and the bases of x are together.

$(3 \cdot 4)(x^2 \cdot x^6)$

Step #2: Multiply the numbers and use the exponent properties to simplify the exponents.

$12x^{2+6} = 12x^8$

Example #2: Simplify $\dfrac{15k^3m^2}{12km}$

Step #1: Simplify the rational number $\dfrac{15}{12}$.

$\dfrac{5k^3m^2}{4km}$

Step #2: Use the properties of exponents to simplify.

$\dfrac{5}{4}k^{3-1}m^{2-1} = \dfrac{5}{4}k^2m$

We will use this concept in future units when we multiply monomials with polynomials.

Below is a link to Practice with Exponents, a great website that gives you an opportunity to practice some problems. You can check your answer immediately.

- Start with problems labeled *Medium.*
- Move to more difficult problems as you answer questions correctly.
- Use scrap paper as you work out the expressions.
- Try numerous problems, but only spend about 15 minutes doing so.

- Make sure you understand the properties of exponents and how they are used so that you will be able to complete the problems in the questions area.

Click here to begin the practice.

Stop! **Go to Questions #23-24 about this section, then return to continue on to the next section.**

Rational Exponents

Did you know that there is more than one way to write a root?

An alternate way to express roots is to use rational (fractional) exponents and can be very useful when dealing with more complicated expressions.

First, let's review the terms associated with radicals.

Base	Index	Exponent
$\sqrt[n]{m^b}$	$\sqrt[n]{m^b}$	$\sqrt[n]{m^b}$
The base is denoted as m.	The index is denoted as n.	The exponent of the base is denoted as b.

Now, let's examine the rational exponent rule.

$$\sqrt[n]{m^b} = \left(\sqrt[n]{m}\right)^b = m^{\frac{b}{n}}$$

This rule can be interpreted as the nth root of m raised to the power of b equals the m (the base) raised to the $\dfrac{b}{n}$ power.

As you can see, when the exponent is a rational (fractional) number, this is actually a root and vice versa.

Let's look at some examples below.

$\sqrt[n]{m} = m^{\frac{1}{n}}$ *The denominator of the rational exponent is the root (index of the radical).

$\sqrt[3]{8} = 8^{\frac{1}{3}}$ *$\sqrt[3]{8}$ can be thought of as $\sqrt[3]{8^1}$

$\sqrt{8} = 8^{\frac{1}{2}}$ *$\sqrt{8}$ can be though of as $\sqrt[2]{8^1}$
(Notice that when the index is square root, the 2 is assumed as the index and not written.)

$$\sqrt[4]{8} = 8^{\frac{1}{4}}$$

$$\sqrt[5]{x} = x^{\frac{1}{5}}$$

$$\sqrt[3]{4x} = (4x)^{\frac{1}{3}}$$

What is the rational exponent expression for $\sqrt{x}$?

Click here to check the property.

x raised to the 1/2 power

In the examples above, the b is a one as described in the rule $\sqrt[n]{m^b} = m^{\frac{b}{n}}$.

Let's examine some examples when b is not a one.

$$\sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 8^{\frac{2}{3}}$$

$$\sqrt{8^5} = (\sqrt{8})^5 = 8^{\frac{5}{2}}$$ *The index is understood to be 2.

$$\sqrt[4]{8^3} = (\sqrt[4]{8})^3 = 8^{\frac{3}{4}}$$

$$\sqrt[4]{x^3} = (\sqrt[4]{x})^3 = x^{\frac{3}{4}}$$

What is the rational exponent expression for $\sqrt[5]{x^3}$?

Click here to check the property.

x raised to the 3/5 power

Now, let's evaluate using our new knowledge of rational exponents.

Example 1: $16^{\frac{1}{2}} = ?$

Solution: $16^{\frac{1}{2}} = \sqrt{16}$ *The square root of 16 = 4.

               $16^{\frac{1}{2}} = 4$

Example 2: $25^{\frac{3}{2}} = ?$

Solution: $25^{\frac{3}{2}} = \sqrt{25^3} = (\sqrt{25})^3$ *The rational exponent rule.

                  $= (5)^3$ *The square root of 25 is 5.

               $25^{\frac{3}{2}} = 125$

Example 3: Simplify the expression $27^{\frac{2}{3}}$.

Solution: $27^{\frac{2}{3}} = \sqrt[3]{27^2}$ * This means what number times itself 3 times is 27, then square the answer.

                                         *The third root can be called the "cube" root.

Now, apply the rational exponent rule.

          $\sqrt[3]{27^2} = (\sqrt[3]{27})^2$

                  $= (3)^2$ *The cube root of 27 is 3 because $3 \times 3 \times 3 = 27$.

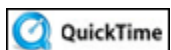 $= 9$

          $27^{\frac{2}{3}} = 9$

Evaluate $\sqrt[3]{8^5}$?

Click here to check the property.

The cube root of 8 to the 5th power = 2 × 2 × 2 × 2 × 2 = 32

QuickTime Strategies for Simplifying Other Roots and Rational Exponents (04:49)

Now, let's apply some of the rational exponents using the property of exponents.

Example 4: Simplify the expression $4^{\frac{1}{2}} \times 4^{\frac{1}{2}}$.

Solution:

$$4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^{\frac{1}{2}+\frac{1}{2}}$$ *When multiplying expressions with exponents, ADD the exponents.

$$= 4^1$$ To add fractions, use a common denominator.

$$= 4$$ $\left(\dfrac{1}{2} + \dfrac{1}{2} = \dfrac{1+1}{2} = \dfrac{2}{2} = 1 \right)$

Example 5: Simplify the expression $3\, x^{\frac{2}{5}} \times 2\, x^{\frac{3}{4}}$.

Solution:

This expression means $3 \times x^{\frac{2}{5}} \times 2 \times x^{\frac{3}{4}}$, so multiply the coefficients, and then apply the exponent property for multiplying expressions with exponents.

$$3 \times 2 \times x^{\frac{2}{5}} \times x^{\frac{3}{4}}$$ *Apply the commutative property.

$$6 \times x^{\frac{2}{5}} \times x^{\frac{3}{4}}$$ *Multiply the coefficents.

$$6\, x^{\frac{2}{5}+\frac{3}{4}}$$ * When multiplying expressions with exponents, ADD the exponents.

$$6\, x^{\frac{23}{20}}$$ *Add the fractions using a common denominator. LCD = 20

$$\left(\frac{2}{5} + \frac{3}{4} \ = \ \frac{8}{20} + \frac{15}{20} \ = \ \frac{23}{20} \right)$$

***Note: In the study of algebra, improper fractions do NOT have to be reduced to mixed numbers.

Simplify $-5\,x^{\frac{2}{3}} \times 3\,x^{\frac{7}{10}}$.

Click here to check the property.

(–5)(3) *x* raised to the (20/30 + 21/30) = –15*x* raised to the 41/30th power

Example 6:   Simplify the expression $\left(n^{\frac{1}{2}} \right)^{10}$.

Solution:

$$\left(n^{\frac{1}{2}} \right)^{10} = n^{\frac{1}{2} \times 10}$$ *When a power is raised to a power,

MULTIPLY the exponents.

$$= n^5 \qquad \left(\frac{1}{2} \times 10 = \frac{1}{2} \times \frac{10}{1} = \frac{10}{2} = 5 \right)$$

Simplify $\left(x^{\frac{2}{3}} \right)^{9}$.

Click here to check the property.

x raised to the (2/3 × 9) = *x* raised to the 6th power

Example 7: Simplify the expression $\dfrac{x^{\frac{3}{4}}}{x^{\frac{1}{2}}}$.

(This expression means $x^{\frac{3}{4}} \div x^{\frac{1}{2}}$.)

Solution:

$$x^{\frac{3}{4}} \div x^{\frac{1}{2}} = x^{\frac{3}{4} - \frac{2}{4}}$$ *When dividing expressions with exponents, SUBTRACT the exponents.

$$= x^{\frac{1}{4}}$$ Use a common denominator.
$$\left(\frac{3}{4} - \frac{1}{2} \ = \frac{3}{4} - \frac{2}{4} \ = \frac{3-2}{4} \ = \frac{1}{4} \right)$$

Example 8: Simplify the expression $\dfrac{20\,x^{\frac{5}{6}}}{5\,x^{\frac{1}{3}}}$.

Solution:

This expression means $\dfrac{20}{5} \times \dfrac{x^{\frac{5}{6}}}{x^{\frac{1}{3}}}$, so divide the coefficients, and then apply the exponent property for dividing expressions with exponents.

$$\frac{20}{5} \times (x^{\frac{5}{6}} \div x^{\frac{1}{3}})$$

$$4 \times (x^{\frac{5}{6}} \div x^{\frac{1}{3}})$$ *Divide the coefficents.

$$4\,x^{\frac{5}{6} - \frac{1}{3}}$$ * When dividing expressions with exponents, SUBTRACT the exponents.

$$4\,x^{\frac{1}{2}}$$ *Use the common denominator of 6.

$$\left(\frac{5}{6} - \frac{1}{3} \ = \ \frac{5}{6} - \frac{2}{6} \ = \ \frac{3}{6} = \ \frac{1}{2} \right)$$

✔ Simplify $\dfrac{15\, x^{\frac{7}{8}}}{5\, x^{\frac{1}{2}}}$.

Click here to check the property.

(15/5) *x* **raised to (7/8 – 4/8) = 3***x* **raised to the 3/8 power**

Stop! **Go to Questions #25-40 to complete this unit.**