RADICAL EXPRESSIONS AND EQUATIONS

Unit Overview

In this unit you will learn to simplify and manipulate radical expressions so that you can solve radical equations. You will also examine the rational exponents and how they relation to radical expressions.

Domain of a Square Root Function

The domain of a function is the set of all real-number values of *x*; therefore, the domain of a square root function, $f(x) = \sqrt{x}$, does not include negative numbers.

To find the domain:

1.) set the numbers under the radical sign ≥ to 0

2.) solve the inequality

3.) the result will be your domain

Example #1: Find the domain of $h(x) = \sqrt{-4x+7}$.

$-4x + 7 \geq 0$

$-4x \geq -7$

$x \leq \frac{7}{4}$ *this is the domain which means that all of your *x*-values must be less than or equal to $\frac{7}{4}$

Example #2: Find the domain of $g(x) = \sqrt{5x+18}$

$$5x + 18 \geq 0$$
$$5x \geq -18$$
$$x \geq \frac{-18}{5}$$

The domain of $g(x)$ is all *x*-values greater than or equal to $\frac{-18}{5}$.

Stop! **Go to Questions #1-5 about this section, then return to continue on to the next section.**

Simplifying Radical Expressions

In the expression $\sqrt[n]{a^p}$ the $\sqrt{\ }$ is called the **radical**, n is the **index**, a is the **radicand** and p is the **power**.

To simplify:

-divide p by n, this is the exponent of the variable outside the radical sign.

-if there is a remainder, this is the new exponent of the variable under the radical sign.

You may want to remember that when you have an even index, the answer is either positive or negative. Such as $\sqrt{4} = \pm 2$. When this is required, we use an absolute value symbol. In this unit you may want to only consider the positive square root.
For example, $\sqrt{32} = \sqrt{16}\cdot\sqrt{2} = |4|\sqrt{2}$. However, for our purposes, we will not be concerned with the absolute value notation on even roots. This will become more significant in later courses.

Example #1: Evaluate $\frac{2}{3}\sqrt[3]{-27} - 5$.

$$\frac{2}{3}\sqrt[3]{-27} - 5$$

$$\frac{2}{3}(-3) - 5 \qquad \text{*the cubed-root of } -27 = -3$$

$$-2 - 5 = -7$$

Example #2: Express $3\sqrt[4]{80}$ as a simplified radical.

1.) factor 80 into 16 $\times$ 5 $\quad 3\cdot\sqrt[4]{16}\cdot\sqrt[4]{5}$

2.) find the 4th root of 16 $\quad 3\cdot 2\cdot\sqrt[4]{5}$

3.) simplify $\quad 6\sqrt[4]{5}$

Example #3: Simplify $\sqrt{x^3y^4z^6}$.

This is a square root and the index is a 2. When an index is not written in a radical, it is understood to be an index of 2.

Divide each of the exponents by the index 2. This will be the new exponent of the variable outside of the radical.

$$xy^2z^3\sqrt{x}$$

Since there was a remainder of 1 when the exponent 3 was divided by the index 2, there is still an *x* inside the radical sign.

Example #4: Simplify $\sqrt[3]{125x^6yz^5}$.

Since the index is odd, we do not have to worry about absolute value signs.

Take the cubed root of 125 and divide each of the exponents by 3. Any remainders will stay inside the radical sign.

$$5x^2z\sqrt[3]{yz^2}$$

*If the radicand is **not** a perfect root, then we will factor it into perfect roots, if possible.

Example #5: In $\sqrt{50a^3b^4}$, 50 is not a perfect square root; but it can be factored using a perfect square root.

1.) factor 50 into 25 × 2 $\quad \sqrt{25}\cdot\sqrt{2a^3b^4}$

2.) find the square root of 25 $\quad 5\sqrt{2a^3b^4}$

3.) simplify the variables $\quad 5ab^2\sqrt{2a}$

Example #6: In $\sqrt[3]{250r^7s^2t^3}$, 250 is not a perfect cubed root, but it can be factored using a perfect cubed root.

1.) factor 250 into 125 × 2 $\sqrt[3]{125}\cdot\sqrt[3]{2r^7s^2t^3}$

2.) find the cubed root of 125 $5\sqrt[3]{2r^7s^2t^3}$

3.) simplify the variables $5r^2t\sqrt[3]{2rs^2}$

QuickTime Squares and Square Roots (01:52)

QuickTime Higher Roots (05:09)

Stop! **Go to Questions #6-15 about this section, then return to continue on to the next section.**

Product and Quotient Properties of Radicals

If a term has a rational exponent, it can be rewritten in radical form in the following way.

$2^{\frac{4}{5}}$ can be rewritten as $\sqrt[5]{2^4}$.

You can see that the **numerator** became the **exponent** of the radicand and the **denominator** became the **index**.

This process can be reversed.

$\sqrt[3]{13^2}$ can be written using a rational exponent as $13^{\frac{2}{3}}$.

Product Property of Radicals

The product property states that you can multiply two radicals together if they have the same index.

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

Quotient Property of Radicals

The quotient property of radicals states that you can divide radicals if they have the same index.

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Example #1: Find the product of $\sqrt{5ab^5} \cdot \sqrt{12ab^6}$.

$\sqrt{60a^2b^{11}}$ — does 60 contain a perfect square factor? yes

$\sqrt{4}\sqrt{15a^2b^{11}}$ — find the square root of 4 and simplify the variable exponents

$2ab^5\sqrt{15b}$

Example #2: Find the quotient of $\dfrac{9\sqrt[3]{48x^8}}{\sqrt[3]{2x^3}}$.

$$\frac{9\sqrt[3]{48x^8}}{\sqrt[3]{2x^3}} = 9\sqrt[3]{\frac{48x^8}{2x^3}}$$

Divide $\sqrt[3]{48x^8}$ by $\sqrt[3]{2x^3}$.

$9\sqrt[3]{24x^5}$ does 24 contain a perfect cubed factor? yes, simplify

$9\sqrt[3]{8}\cdot\sqrt[3]{3x^5}$ find the cubed root of 8 and simplify the variable exponent.

$9\cdot 2\cdot x\sqrt[3]{3x^2}$ multiply 9 and 2

$18x\sqrt[3]{3x^2}$

QuickTime Fractional Exponents (06:21)

Stop! **Go to Questions #16-20 about this section, then return to continue on to the next section.**

Adding and Subtracting Radicals

To add or subtract radicals:

1.) they must have the same index

2.) they must have the same radicand

3.) add or subtract the numbers in front of the radical term

Example #1: Compute: $3\sqrt{5}+6\sqrt{5}-15\sqrt{5}$.

$$(3+6-15)\sqrt{5}$$

$$-6\sqrt{5}$$

*When adding or subtracting numbers and radicals, you want to combine like terms.

Example #2: Find the difference: $(4+\sqrt{3})-(-6+4\sqrt{3})$.

$4+\sqrt{3}+6-4\sqrt{3}$	change all signs in second quantity
$4+6+\sqrt{3}-4\sqrt{3}$	combine like terms
$10-3\sqrt{3}$	

*If the radicands are not the same in an addition or subtraction problem, check to see if they can be simplified into common radicands.

Example #3: Find the sum.

$(3-\sqrt{24})+(8-\sqrt{96})$	see if 24 and 96 can be factored
	using perfect squares.
$3-(\sqrt{4}\cdot\sqrt{6})+8-(\sqrt{16}\cdot\sqrt{6})$	simplify each of the perfect squares
$3-2\sqrt{6}+8-4\sqrt{6}$	combine like terms
$3+8-2\sqrt{6}-4\sqrt{6}$	simplify
$11-6\sqrt{6}$	

Example #4: Find the difference.

$(4+\sqrt{27})-(-15+\sqrt{48})$	factor 27 and 48 into perfect squares
$4+\sqrt{9}\cdot\sqrt{3}+15-\sqrt{16}\cdot\sqrt{3}$	change all signs of the second quantity
$4+3\sqrt{3}+15-4\sqrt{3}$	simplify each perfect square
$4+15+3\sqrt{3}-4\sqrt{3}$	combine like terms
$19-\sqrt{3}$	

Stop! **Go to Questions #21-23 about this section, then return to continue on to the next section.**

Multiplying Radicals

Example #1: Find the product.

$(-3+5\sqrt{2})(4+2\sqrt{2})$	use the FOIL process to multiply
$-12-6\sqrt{2}+20\sqrt{2}+10\sqrt{4}$	simplify any perfect roots
$-12-6\sqrt{2}+20\sqrt{2}+(10\cdot 2)$	
$-12-6\sqrt{2}+20\sqrt{2}+20$	combine like terms
$-12+20-6\sqrt{2}+20\sqrt{2}$	
$8+14\sqrt{2}$	

Example #2: Multiply.

$3\sqrt{2}(4\sqrt{6}-5\sqrt{3})$	use the distributive property to multiply
$(3\sqrt{2}\cdot 4\sqrt{6})-(3\sqrt{2}\cdot 5\sqrt{3})$	multiply numbers together and radicals together
$12\sqrt{12}-15\sqrt{6}$	simplify the square root of 12 because it contains a perfect square factor of 4
$12\sqrt{4}\cdot\sqrt{3}-15\sqrt{6}$	
$12\cdot 2\sqrt{3}-15\sqrt{6}$	
$24\sqrt{3}-15\sqrt{6}$	

Stop! **Go to Questions #24-25 about this section, then return to continue on to the next section.**

Rationalizing the Denominator

If a rational expression contains a radical in the denominator, it is not completely simplified. A process called **rationalizing the denominator** is used to eliminate the radical from the denominator. We used a similar process in a previous unit when we simplified expressions containing complex numbers. If you remember, we used a process call the **conjugate** of the denominator. The same process is used for radical expressions. Let's first review how to find the conjugate of a complex number.

$\dfrac{4}{2+7i}$ would have a conjugate of $2 - 7i$ because you use the opposite sign.

To simplify this expression you would multiply it by $\dfrac{2-7i}{2-7i}$.

Let's look at a radical expression now.

Example #1: Simplify by rationalizing the denominator.

$$\frac{2}{1+\sqrt{3}}$$

multiply by the conjugate of the denominator: $\dfrac{1-\sqrt{3}}{1-\sqrt{3}}$

$$\frac{2}{1+\sqrt{3}} \cdot \frac{1-\sqrt{3}}{1-\sqrt{3}}$$

$$\frac{2(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})}$$

multiply

$$\frac{2-2\sqrt{3}}{1-\sqrt{3}+\sqrt{3}-\sqrt{9}}$$

simplify

$$\frac{2-2\sqrt{3}}{1-3} = \frac{2-2\sqrt{3}}{-2}$$

there are 3 whole numbers 2, –2, –2

that contain a factor of –2 so these can be simplified

$$\frac{2-2\sqrt{3}}{-2}=\frac{\cancel{2}(1-\sqrt{3})}{-\cancel{2}}$$

$$-(1-\sqrt{3})$$

$-1+1\sqrt{3}$ this is the simplified form of $\frac{2}{1+\sqrt{3}}$

Stop! **Go to Questions #26-31 about this section, then return to continue on to the next section.**

Solving Radical Equations

One Radical Term

-isolate the radical term

-square or cube both sides (depending on what the index is)

-solve and check

Example #1:

$5 = \sqrt{x^2 + 16}$	since this is a square root, square both sides
$(5)^2 = (\sqrt{x^2 + 16})^2$	when squaring a square root, the results are the terms under the radical sign.
$25 = x^2 + 16$	solve for x
$25 - 16 = x^2$	
$9 = x^2$	square root both sides
$\sqrt{9} = \sqrt{x^2}$	
$\pm 3 = x$	

Check: $5 = \sqrt{(\pm 3)^2 + 16}$

$5 = \sqrt{25}$

5 = 5 true; therefore the solution to the equation is ± 3

Two Radical Terms

-make sure that the radicals are on opposite sides of the equal sign, one on each side

-square or cube both sides (depending on what the index is)

-solve and check

Example #1:

$(\sqrt[3]{x-2})-(\sqrt[3]{2x+1})=0$	add $\sqrt[3]{2x+1}$ to both sides
$\sqrt[3]{x-2}=\sqrt[3]{2x+1}$	cube both sides
$(\sqrt[3]{x-2})^3=(\sqrt[3]{2x+1})^3$	
$x-2=2x+1$	solve
$-x=3$	
$x=-3$	

Check: $\sqrt[3]{-3-2}=\sqrt[3]{2(-3)+1}$

$$\sqrt[3]{-5}=\sqrt[3]{-5}$$

Since this is true, the solution to the equation is –3.

Let's try another example.

Example #2:

$\sqrt{3x+4} = \sqrt{x} - 2$	-square both sides
$\left(\sqrt{3x+4}\right)^2=\left(\sqrt{x}-2\right)^2$	-since there are two terms on the right, you need to **FOIL**
$3x+4=\left(\sqrt{x}-2\right)\left(\sqrt{x}-2\right)$	
$3x+4=x-2\sqrt{x}-2\sqrt{x}+4$	-combine like terms
$3x+4=x-4\sqrt{x}+4$	-isolate the radical
$2x=-4\sqrt{x}$	-divide both sides by 2

$x = -2\sqrt{x}$ -square both sides

$x^2 = \left(2\sqrt{x}\right)^2$ -solve

$x^2 = 4x$

$x^2 - 4x = 0$ -factor to solve

$x(x-4) = 0$ -set each factor equal to zero

$x = 0$ and $x - 4 = 0$

$x = 4$

Check both answers.

$\sqrt{3(0)+4} = \sqrt{0} - 2$	$\sqrt{3(4)+4} = \sqrt{4} - 2$
$\sqrt{4} = -2$	$\sqrt{16} = 2 - 2$
$2 = -2$ false	$4 = -2$ false

Since both of these solutions produce false statements, the answer is **no solution**.

Stop! **Go to Questions #32-36 to complete this unit.**