LOGARITHMIC FUNCTIONS

Unit Overview

This unit is about the properties of exponential and logarithmic functions and solving equations using these properties. Common logarithmic functions are used to solve exponential and logarithmic equations. The change-of-base formula is used to evaluate exponential and logarithmic equations. Applications of logarithmic functions include the pH scale in chemistry, sound intensity, the Richter scale for earthquakes, and Newton's law of cooling.
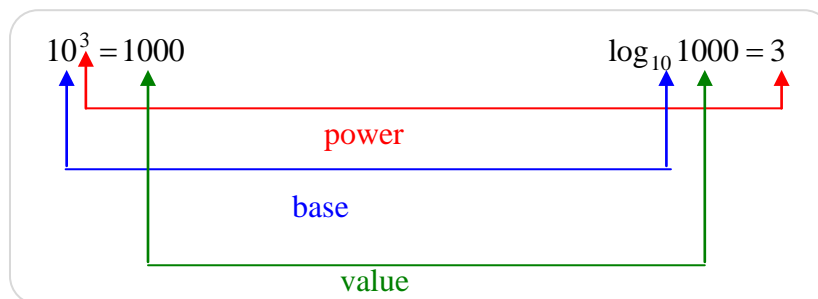
Note: The picture shows San Francisco in 1906 after a devastating earthquake.

Logarithmic Functions

The logarithm of a number is the exponent to which a fixed value, called the base must be raised to produce that number. For example, the log of 1000 to base 10 is 3 , because 1000 is 10 to the third power.
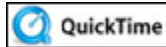
exponential form   **logarithmic form**

$$10^3 = 1000 \qquad\qquad \log_{10} 1000 = 3$$

power

base

value

$$base^{power} = value \qquad\qquad \log_{base} value = power$$

In general, the logarithmic function is the inverse of the exponential function.
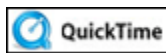
QuickTime Introduction to Logarithms (02:53)

A logarithm base b of a positive number x satisfies the following definition.
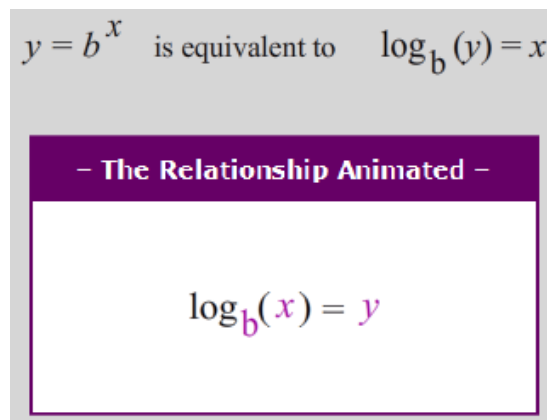
> A logarithm base b of a positive number x satisfies the following definition. When $b > 0$ and $b \neq 1$,
>
> $$\log_b x = y \text{ if and only if } b^y = x$$

QuickTime Defining Logarithms (05:45)

Here is an animation that will help with understanding what a log is.

$$y = b^x \quad \text{is equivalent to} \quad \log_b (y) = x$$

– The Relationship Animated –

$$\log_b(x) = y$$

The expression, $\log_b x$, is read as "log base b of x." In other words, the logarithm y is the exponent to which b must be raised to get x.

*Because $y = \log_b x$ is the inverse of $y = b^x$ and $y = \log_b x$ is a function, the exponential function $y = b^x$ is a one-to-one function**.

Meaning: For each element in the domain (x-values), there is exactly one corresponding element in the range (y-values).
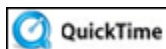
QuickTime Logs and Exponents (05:11)

Example #1: $5^3 = 125$ becomes $\log_5 125 = 3$

More examples: The black (dark) print represents what is given; the red (light) print represents the results.

Exponential	$2^5 = 32$	$10^3 = 1000$	$3^{-2} = \dfrac{1}{9}$	$16^{\frac{1}{2}} = 4$
Logarithmic	$\log_2 32 = 5$	$\log_{10} 1000 = 3$	$\log_3 \dfrac{1}{9} = -2$	$\log_{16} 4 = \dfrac{1}{2}$

Stop! **Go to Questions #1-4 about this section, then return to continue on to the next section.**

One-to-One Property of Exponents

Let's examine how the One-to-One Property of Exponents can help in solving logarithmic equations expressed in exponential form.

> If the base (b) is a positive real number, and $b \neq 1$, then
>
> if $b^x = b^y$, then $x = y$.

Here is an animation that will help with understanding the One-to-one property.

If the base (b) is a positive real number, and $b \neq 1$, then

if $b^x = b^y$, then $x = y$.

Example #1: Solve $\log_2 1 = r$ for r.

$\log_2 1 = r$	-put in exponential form
$2^r = 1$	-ask yourself "2 to what power is 1?"
$2^r = 2^0$	-since $2^0 = 1$, $r = 0$
$r = 0$	

Example #2: Solve $\log_v 32 = 5$ for v.

$\log_v 32 = 5$	-put in exponential form
$v^5 = 32$	-ask "Can 32 can be written using a base to the 5th power?"
$v^5 = 2^5$	-the base 2 raised to the 5th power equals 32
$v = 2$	-since $2^5 = 32$, $v = 2$

Example #3: Solve $\log_b 9 = \dfrac{1}{2}$ for b.

$\log_b 9 = \dfrac{1}{2}$ -put in exponential form

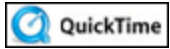$b^{\frac{1}{2}} = 9$ -multiply both exponents by 2

$b^{(\frac{1}{2})(2)} = 9^{(1)(2)}$ -Simplify the variable to the first power (b):

$$b^{(\frac{1}{2})(2)} = b^1 = b$$

$b^1 = 9^2$

$b = 81$

Let's practice evaluating a few logarithms by applying the One-to-One Property of Exponents.

Evaluate $\log_2 8 = x$

Click here" to check your answer.

$x = 3$ because $2^3 = 8$

Evaluate $\log_9 9 = v$

Click here" to check your answer.

$v = 1$ because $9^1 = 9$

Evaluate $\log_7 49 = q$

Click here" to check your answer.

$q = 2$ because $7^2 = 49$

Evaluate $\log_2 \dfrac{1}{25} = m$

Click here" to check your answer.

$m = -2$ because $5^{-2} = 1/25$

Stop! **Go to Questions #5-8 about this section, then return to continue on to the next section.**
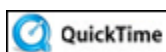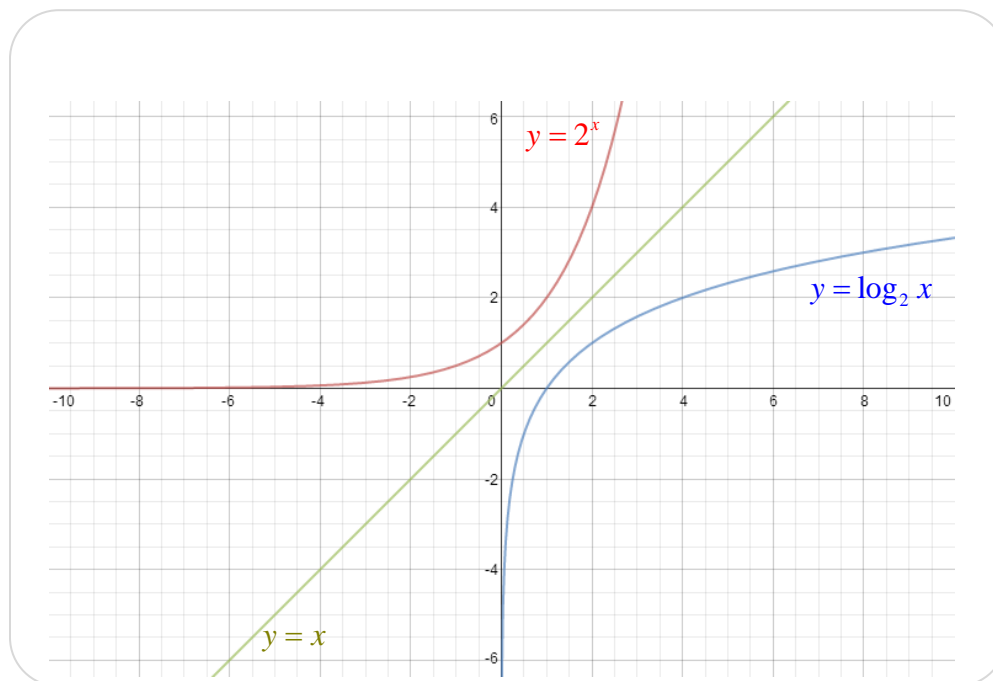
Graphs of Logarithmic Functions

To sketch the graph of a logarithmic function, use the fact that the logarithmic function is the inverse of the exponential function.

Remember that the inverse of a function is obtained by *switching* the x and y coordinates. This reflects the graph about the line $y = x$.

The table below shows how the x and y coordinates of the points on the exponential curve $y = 2^x$ can be switched to find the coordinates of the points on the logarithmic curve $y = \log_2 x$.

Points on the *Exponential* Curve $y = 2^x$	Points on the *Logarithmic* Curve $y = \log_2 x$
(–3, 1/8)	(1/8, –3)
(–2, 1/4)	(1/4, –2)
(–1, 1/2)	(1/2, –1)
(0, 1)	(1, 0)
(1, 2)	(2, 1)
(2, 4)	(4, 2)
(3, 8)	(8, 3)

Obtaining the Image of an Inverse of a Function (04:59)

Let's examine the logarithmic function closely.

Domain (*x*-coordinates) of a Logarithmic Function

Look at the left side of the logarithmic graph (blue). Notice that the *x*-coordinates of the points are very close to zero, but never equal zero or cross over the *y*-axis. On the right side of the graph, the *x*-coordinates of the points grow infinitely.

Thus, the domain for the logarithmic function is $x > 0$.

Range (*y*-coordinates) of a Logarithmic Function

Since the *y*-coordinates of the points are unrestricted and can be infinitely small and infinitely larger, the range of a logarithmic function is **all real numbers**.

Features of a Logarithmic Function

-There is no *y*-intercept.
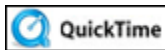
-The vertical asymptote is $x = 0$.

Note: An asymptote is a line that a graph approaches as *x* or *y* increases in absolute value.

-A logarithmic function has no horizontal asymptote.

-The *x*-intercept is 1.

-The function is increasing over the domain.

Note: If the base is between 0 and 1, the function decreases over the domain.

QuickTime Log Graphs (01:54)

Translating a Logarithmic Function

Knowing the shape of a logarithmic graph, it can then be shifted vertically and/or horizontally, stretched or compressed, and reflected.

The function $y = \log_b x$ is the parent graph for the logarithmic function.

When working with the logarithmic function, $y = \log_b(x - h) + k$ the graph of the parent function, $y = \log_b x$, can be translated horizontally by *h* units and vertically by *k* units.

The *a* in the standard form of a logarithmic function $y = a\log_b(x-h)+k$ results in a stretch (if $|a| > 0$), a compression (if $0 < |a| < 1$) or a reflection (if $a < 0$).

Example #1: How does the graph of $y = \log_3(x+2)+4$ compare to the graph of the parent function $y = \log_3 x$? State the domain, range and identify the asymptote (the vertical line that the graph approaches).

$y = a\log_b(x-h)+k$ -standard form of a logarithmic function

$y = \log_3(x+2)+4$ -given function

$y = \log_3(x-(-2))+4$ -rewrite the given function in standard form

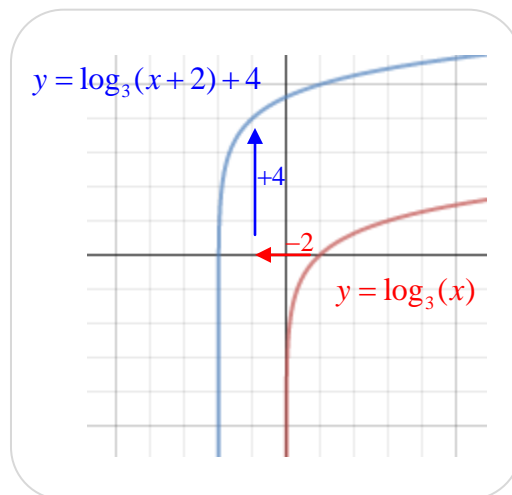$h = -2$ so the graph translates 2 units to the left

$k = 4$ so the graph translates 4 units up

The domain changes from $x > 0$ to $x > -2$.

The vertical asymptote is $x = -2$.

The range remains all real numbers.

The graph below shows the horizontal (–2) and vertical (+4) shifts.

Example #2: How does the graph of $y = 3\log_4(x)-2$ compare to the graph of the parent function $y = \log_4 x$? State the domain, range and identify the asymptote (the vertical line that the graph approaches).

$$y = a\log_b(x-h)+k$$ -standard form of a logarithmic function

$$y = 3\log_4(x)-2$$ -given function

$$y = 3\log_4(x-0)+(-2)$$ -rewrite the given function in standard form

$a = 3$ so the graph is stretched by a factor of 3

$h = 0$ so there is no horizontal shift

$k = -2$ so the graph shifts down 2 units

Because there is no horizontal shift, the domain remains $x > 0$, and thus, the vertical asymptote is $x = 0$.

The range remains all real numbers.

The graph below compares the parent function (red) to a function that shows just the compression $a = 3$ (blue), and then the entire given function (green) which is compressed by 3 and shifted down two units.
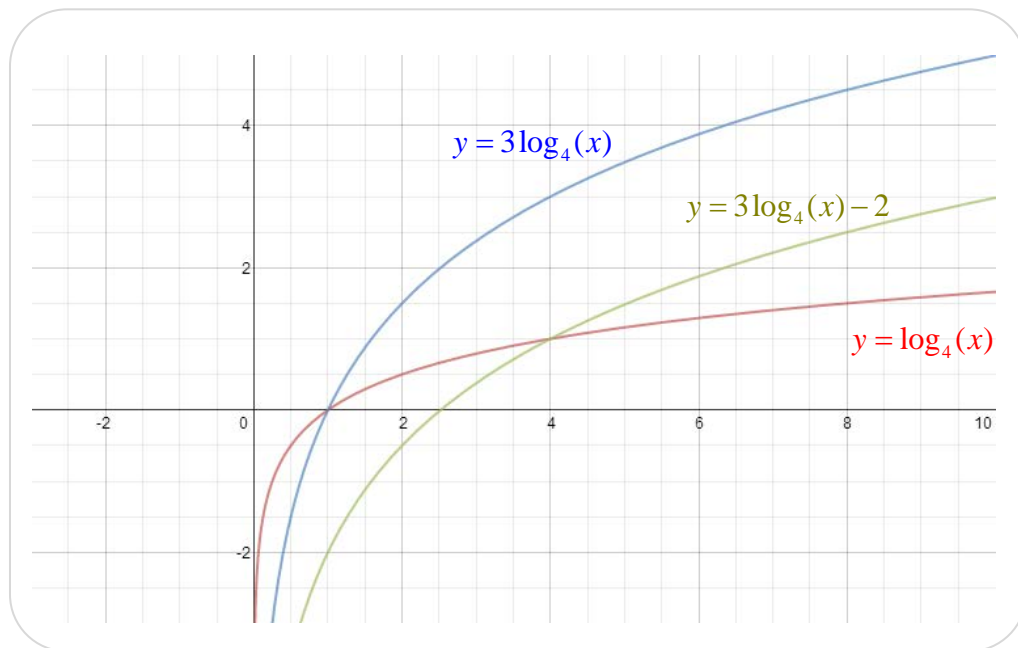
Stop! **Go to Questions #9-12 about this section, then return to continue on to the next section.**

Properties of Logarithmic Functions

Logarithms, by definition, are *exponents*; thus, the properties of logarithms are similar to the properties of exponents.

Product & Quotient Properties of Logarithms

For $m > 0$, $n > 0$ and $b \neq 1$

Product property $\qquad\qquad \log_b(mn) = \log_b m + \log_b n$

Quotient property $\qquad\qquad \log_b\left(\dfrac{m}{n}\right) = \log_b m - \log_b n$

Product Property: The log of a product is equal to the sum of the log of the first base and the log of the second base.

Quotient Property: The log of a quotient is equal to the difference of the logs of the numerator and denominator of the fraction that represents the division.

Example #1: Rewrite the log expression $\log_5 2 + \log_5 6x$ as one log.

$\qquad \log_5 2 + \log_5 6x$
$\qquad\qquad = \log_5 2(6x) \qquad$ Apply the Product Property
$\qquad\qquad = \log_5 12x$

$\qquad$ *Notice that the addition turned into multiplication.

Example #2: Rewrite the log expression $\log_8 12 - \log_8 4$ as one log.

$\qquad \log_8 12 - \log_8 4$
$\qquad\qquad = \log_8 \dfrac{12}{4} \qquad$ Apply the Quotient Property
$\qquad\qquad = \log_8 3$

$\qquad$ *Notice that the subtraction turned into division.

Example #3: Rewrite the log expression $\log_b 4x - \log_b 3y + \log_b y$ as one log.

$$\log_b 4x - \log_b 3y + \log_b y$$

$$= \log_b \frac{4x}{3y} + \log_b y \qquad\qquad \text{Apply the Quotient Property}$$

$$= \log_b \frac{4x}{3y} \cdot \frac{y}{1} \qquad\qquad \text{Apply the Multiplication Property}$$

$$= \log_b \frac{4x}{3\cancel{y}_1} \cdot \frac{\cancel{y}^1}{1} \qquad\qquad \text{Cancel the } y\text{'s}$$

$$= \log_b \frac{4x}{3} \qquad\qquad \text{Simplify}$$

*Notice that the subtraction turned into division and the addition turned into multiplication.

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Power Property of Logarithms

For $m > 0$, $b > 0$, $b \neq 1$, and any real number p

$$\log_b m^p = p \log_b m$$

</div>

Power Property: The logarithm of a power is equal to the power times the log of the base.

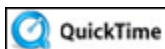*Example #4*: Evaluate: $\log_3 27^8$

$$\log_3 27^8$$

$$\quad = 8\log_3 27 \qquad\qquad \text{Move the power in front of the log}$$
$$\qquad\qquad\qquad\qquad\qquad \text{Multiply 8 times } \log_3 27$$
$$\quad = 8 \cdot 3 \qquad\qquad \text{Since } 3^3 = 27, \ \log_3 27 \text{ is equal to 3.}$$
$$\quad = 24$$

QuickTime Simplifying Logarithmic Expressions Using Basic Log Laws (05:00)

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Exponential-Logarithmic Inverse Properties

For $b > 0$ and $b \neq 1$

$$\log_b b^x = x \quad \text{and} \quad b^{\log_b x} = x \ \text{ for } x > 0$$

</div>

An exponential function and logarithmic function with the same base are inverses of each other.

$$\log_b b^x = x \qquad\qquad b^{\log_b x} = x$$

Example #5: Simplify: $3^{\log_3 4} + \log_5 25$

    Step #1: since the bases are the same for $3^{\log_3 4}$ the result is 4

        $4 + \log_5 25$

    Step #2: 25 needs to be written using a base of 5

        $4 + \log_5 5^2$

    Step #3: Now that the bases are the same for $\log_5 5^2$ the result is 2

        $4 + 2 = 6$

Therefore the value of $3^{\log_3 4} + \log_5 25$ is 6.

Example #6: Simplify: $\log_2 32 - 5^{\log_5 3}$

    Step #1: rewrite 32 using a base of 2

        $\log_2 2^5 - 5^{\log_5 3}$
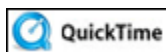
    Step #2: since the bases are the same in $5^{\log_5 3}$ the result is 3

        $\log_2 2^5 - 3$

    Step #3: since the bases are the same in $\log_2 2^5$ the result is 5

        $5 - 3 = 2$

Therefore the value of $\log_2 32 - 5^{\log_5 3}$ is 2.

QuickTime Log Rules and Properties (04:34)

One-to-One Property of Logarithms

If $\log_b x = \log_b y$, then $x = y$

Here is an animation that will help with understanding the One-to-one Property of Logarithms.

One-to-One Property of Logarithms

$$\text{if } \log_b x = \log_b y, \text{ then}$$

If the bases of equivalent logarithms are the same, then the values are equal.

Example #7: Solve for x: $\log_2(2x^2 + 8x - 11) = \log_2(2x + 9)$

Step #1: Since the bases are the same, we can set the expressions equal to each other and solve.

$$2x^2 + 8x - 11 = 2x + 9$$

$$2x^2 + 6x - 20 = 0$$

$$2(x^2 + 3x - 10) = 0$$

$$2(x + 5)(x - 2) = 0$$

$$x + 5 = 0 \text{ and } x - 2 = 0$$

$$x = -5 \text{ or } x = 2$$

Step #2: Both of these numbers are put back into the original logarithmic equation to check the solution. If either number produces a negative logarithm, this answer must be excluded because the *domain* of logarithmic functions excludes negative numbers.

Check: (–5)

$$\log_2(2x^2 + 8x - 11) \quad = \quad \log_2(2x + 9)$$
$$\log_2(2 \cdot (-5)^2 + 8(-5) - 11) \quad = \quad \log_2(2 \cdot (-5) + 9)$$
$$\log_2(2 \cdot 25 + -40 - 11) \quad = \quad \log_2(-10 + 9)$$
$$\log_2(50 + -40 - 11) \quad = \quad \log_2(-10 + 9)$$
$$\log_2(-1) \quad = \quad \log_2(-1)$$

Since –5 results in a negative logarithm, it cannot be a solution.

Check: (2)

$$\log_2(2x^2 + 8x - 11) = \log_2(2x + 9)$$
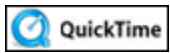$$\log_2(2\cdot 2^2 + 8\cdot 2 - 11) = \log_2(2\cdot 2 + 9)$$
$$\log_2(2\cdot 4 + 8\cdot 2 - 11) = \log_2(2\cdot 2 + 9)$$
$$\log_2(8 + 16 - 11) = \log_2(4 + 9)$$
$$\log_2(13) = \log_2(13)$$

$$\log_2 13 = \log_2 13 \checkmark$$

Since 2 results in a positive logarithm, then the solution to this problem is 2.

QuickTime Using Special Procedures to Solve Equations (07:15)

Stop! **Go to Questions #13-24 about this section, then return to continue on to the next section.**

Applications of Common Logarithms

A **common logarithm** is a logarithm with base 10. Common logarithms, $\log_{10} x$ can be written simply as $\log x$, without showing the 10.

To solve equations where x is an exponent:

$$7^x = 20$$
$$\log 7^x = \log 20 \qquad \text{1.) take the common logarithm of both sides}$$
$$x \log 7 = \log 20 \qquad \text{2.) apply the power property}$$
$$x = \frac{\log 20}{\log 7} \qquad \text{3.) solve for } x \text{ by dividing by } \log 7 \text{ (calculator)}$$
$$x = 1.5395$$

Example #1: Solve for x: $\;3^{x-4} = 5^{x-1}$

$$\log 3^{x-4} = \log 5^{x-1} \qquad \text{1.) tale the log of both sides}$$

$$(x-4)\log 3 = (x-1)\log 5 \qquad \text{2.) use the power property to rewrite}$$

$$x\log 3 - 4\log 3 = x\log 5 - \log 5 \qquad \text{3.) distribute the } x-4 \text{ and the } x-1$$

$$x\log 3 - x\log 5 = 4\log 3 - \log 5 \qquad \text{4.) combine like terms on one side (Put the } x\text{'s on the left side and the constants on the right side.}$$
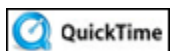
$$x(\log 3 - \log 5) = 4\log 3 - \log 5 \qquad \text{5.) on left, factor out the } x$$

$$x = \frac{(4\log 3 - \log 5)}{(\log 3 - \log 5)} \qquad \text{6.) divide both sides by } \log 3 - \log 5$$

$$x = -5.45 \qquad \text{7.) use graphing calculator to solve*}$$

QuickTime Solving Exponential Equations (08:04)

*Use a graphing calculator (like TI-83 or TI-84) to solve logarithmic equations. Make sure to put parentheses around the entire numerator and the entire denominator when entering numerators or denominators.

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Change-of-base Formula

$$\log_b x = \frac{\log x}{\log b}$$

</div>

The **change-of-base formula** works well with graphing calculators because the default base for logs on a calculator is 10.   The value of a log in a different base (b) can be found by taking the log of the value (x) and divide it by the log of the base (b).

Example #2: Evaluate: $\log_8 97$

$$\log_8 97 = \frac{\log 97}{\log 8} \approx 2.2$$

Example #3: Evaluate: $\log_7 \dfrac{1}{50}$

$$\log_7 \frac{1}{50} = \frac{\log \dfrac{1}{50}}{\log 7} \approx -2.01$$

*Use the graphing calculator to find the value of an expression using the change-of-base formula.

Logarithms and Earthquakes

The Richter scale is a logarithmic scale used to express the total amount of energy released by an earthquake. Each number increase on the Richter scale indicates an intensity ten times stronger. For example, an earthquake of magnitude 5 is ten times stronger than an earthquake of magnitude 4. An earthquake of magnitude 6 is 10 x 10, or 100 times stronger. In general, a measure of under 5 on the Richer scales is considered a minor earthquake, while a measure over 7 indicates major destruction.

The formula, $\log \dfrac{I_1}{I_2} = M_1 - M_2$ compares the intensity levels of earthquakes where I is the intensity level determined by a seismograph, and M is the magnitude on the Richter scale.
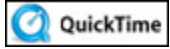
Using Logarithms with the Richter Scale (11:06)

Example #4: In 1906, an earthquake in San Francisco was estimated to have a magnitude of 7.8. In 1989, an earthquake of magnitude 6.9 shook San Francisco. How many times more intense was the 1906 earthquake than the 1989 earthquake?

Note: The overview picture shows San Francisco in 1906 after the earthquake.

$$\log \frac{I_1}{I_2} = M_1 - M_2 \qquad \text{Use the formula}$$

$$\log \frac{I_1}{I_2} = 7.8 - 6.9 \qquad \text{Substitute } M_1 = 7.8 \text{ and } M_2 = 6.9$$

$$\log \frac{I_1}{I_2} = 0.9 \qquad \text{Simplify}$$

$$\frac{I_1}{I_2} = 10^{0.9} \qquad \text{Apply the definition of common logarithm}$$

(*Note*: The base 10 is understood when just log is written.)

$$10^{0.9} \approx 8 \qquad \text{Use a calculator}$$

The 1906 San Francisco earthquake was 8 times more intense than the 1989 earthquake.

Stop! **Go to Questions #25-37 about this section, then return to continue on to the next section.**
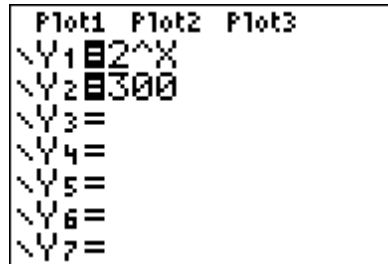
Solving an Exponential Equation with a Graph

A graphing calculator can also be used to solve exponential equations.

    Example #1:   Solve $2^x = 300$ using a graphing calculator.

        Let $Y_1 = 2^x$

        Let $Y_2 = 300$

```
Plot1  Plot2  Plot3
\Y1■2^X
\Y2■300
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

    Adjust the window to find the point of intersection.

    The solution is $x \approx 8.23$.

Use the questions below as a guide, if necessary, to set up the window on the graphing calculator.

To determine the point of intersection, trace to the intersecting point or do the following:

    -On the graphing calculator, enter 2ND CALC 5: intersect.

    -Press ENTER to accept the default for the First curve. Press ENTER again to accept the default of the Second curve. Press ENTER a third time and accept the calculator's Guess. Press ENTER one final time to view the coordinates of the point of intersection.

    The solution is $x \approx 8.23$. This is the *x*-coordinate of the point of intersection.

          ✔ What are reasonable domain values?

          Click here" to check your answer.

          From Xmin = 0 to Xmax = 20

          ✔ What are reasonable range values?

          Click here" to check your answer.

From Ymin = 0 to Ymax = 400
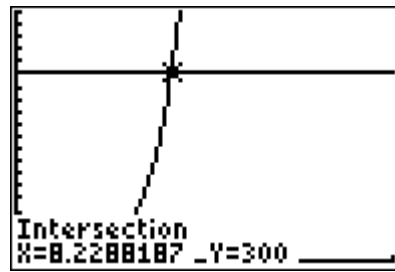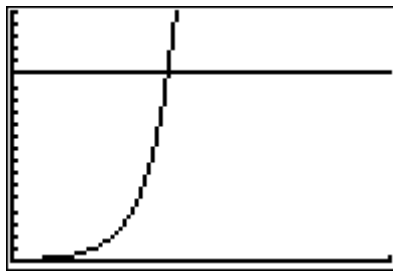
✓ What are reasonable scale values?

Click here" to check your answer.

From Xscl = 20 to Yscl = 20

✓ Check the graph.

Click here" to check your answer.


```
Intersection
X=8.2288187 _Y=300
```

Stop! **Go to Questions #38-40 to complete this unit.**