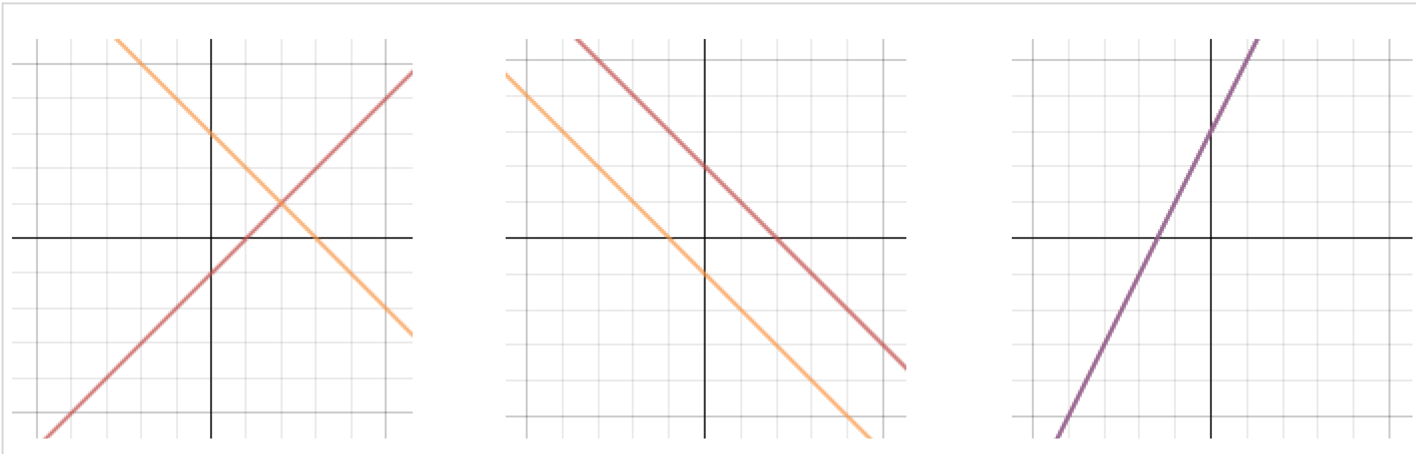
SYSTEMS OF EQUATIONS AND LINEAR EQUALITIES



Unit Overview

A **system of equations** is a collection of equations in the same variables. In this unit you will solve systems of equations using 3 techniques; graphing, substitution, and elimination. At the conclusion of the unit, you will be graphing linear inequalities.

Graphing Systems of Equations

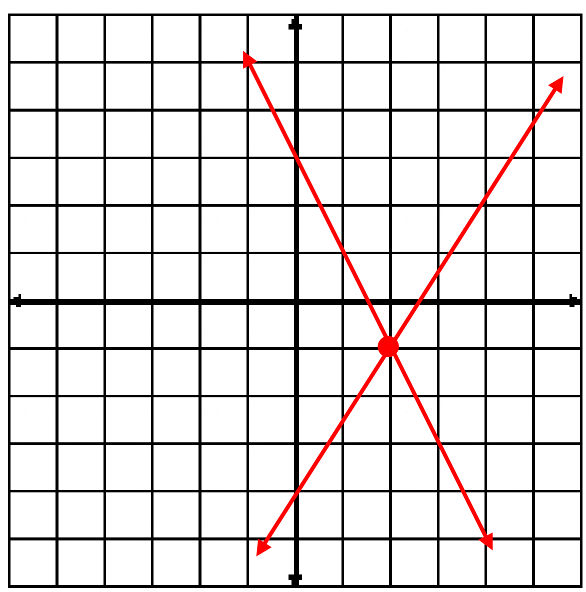
The solution of a system of two linear equations in x and y is any ordered pair, (x, y) that satisfies both equations. The solution (x, y) is also the point of intersection of the graphs.

There are 3 possible solutions.

Number of Solutions	Description of the Solution	Description of the Lines	Classification
One solution	The lines intersect at one point.	The slopes of the lines are different.	Consistent and Independent
Many solutions	The lines intersect at many points.	The lines are exactly the same.	Consistent and Dependent
No solution	The lines do not intersect.	The slopes are the same.	Inconsistent

To graph a line, remember to solve for y and use the slope intercept form of $y = mx + b$. Plot the y-intercept, use the slope ratio of $\frac{rise}{run}$ to plot more points, and then connect the points using a straight edge.

Example #1: $\begin{cases} 2x + y = 3 \\ 3x - 2y = 8 \end{cases}$

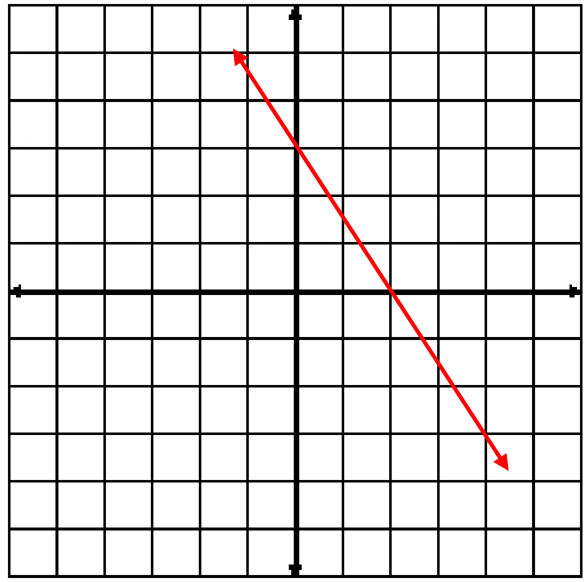


$2x + y = 3$ becomes $y = -2x + 3$

$3x - 2y = 8$ becomes $y = \frac{3}{2}x - 4$

Therefore the solution to this system of equations is **(2, –1)** because that is the point of intersection of the graphs of the two equations. The system would be classified as **consistent and independent**.

Example #2: $\begin{cases} 6x + 4y = 12 \\ 2y = 6 - 3x \end{cases}$

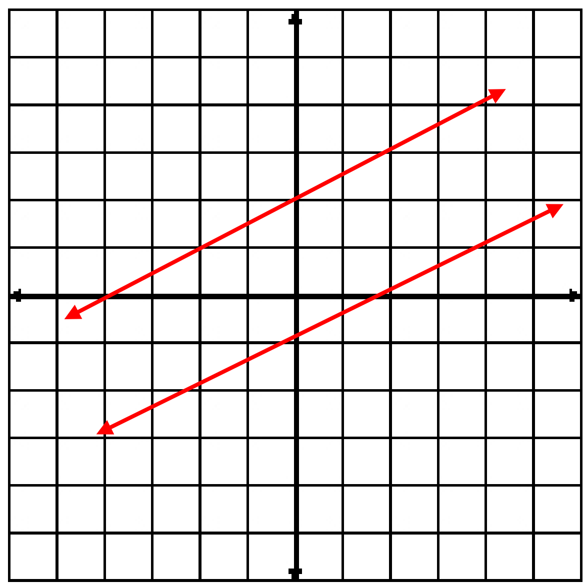


$6x + 4y = 12$ becomes $y = \frac{-3}{2}x + 3$

$2y = 6 - 3x$ becomes $y = \frac{-3}{2}x + 3$

Since these represent the same line they lie on top of each other therefore the solution will be **many solutions** and the system is classified as **consistent and dependent**.

Example #3: $\begin{cases} -2x + 4y = 8 \\ y = \frac{1}{2}x - 1 \end{cases}$



$2x + 4y = 8$ becomes $y = \frac{1}{2}x + 2$

Since these two lines are parallel, there is no intersection; so, the solution is **no solution** and the system is classified as **inconsistent**.

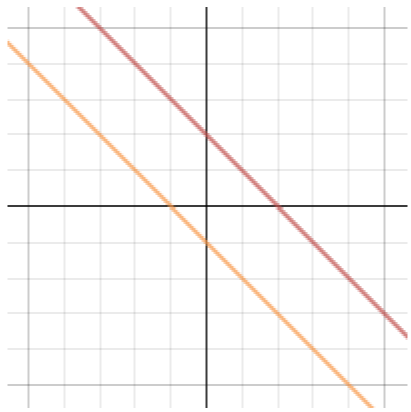
Graphing Systems of Equations -- Free diving (02:59)

Let's review consistent and inconsistent systems.

What is the classification of this system shown below?

“Click here” to check your answer.

The system is inconsistent.



How many solutions does this system have?

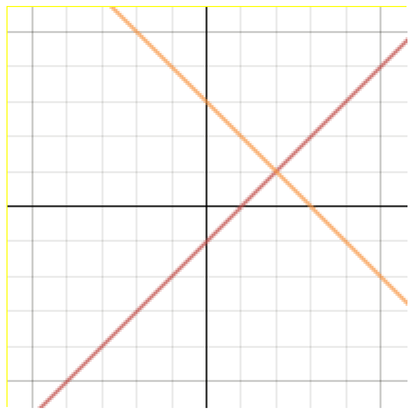
“Click here” to check your answer.

There are no solutions.

What is the classification of the system shown below?

“Click here” to check your answer.

The system is consistent and independent.



How many solutions are there?

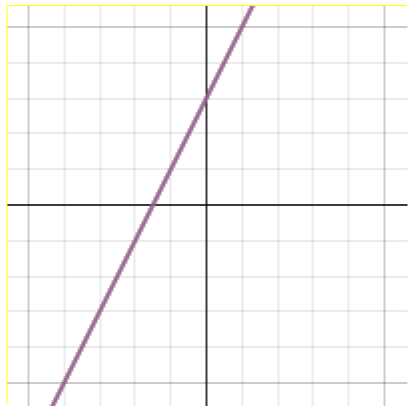
“Click here” to check your answer.

There is one solution.

What is the classification of the system shown below? Note: The graph shows two lines, one coinciding with the other; thus, one lies atop the other.

"Click here" to check your answer.

The system is consistent and dependent.



How many solutions are there?

"Click here" to check your answer.

There are many solutions.

Stop! **Go to Questions #1-5 about this section, then return to continue on to the next section.**

Substitution Method for Solving Systems of Equations

To solve a system by substitution:

1.) solve one of the equations for a variable (hint: solve for a variable that has a coefficient of 1
2.) substitute this value into the other equation to find the value of one of the variables
3.) substitute this value back into either of the equations to find the second variable

Example #1: $\begin{cases} x - y = 3 \\ 2x + 2y = 2 \end{cases}$ Solve the first equation for either x or y. (Solve for x as this is a positive value)

$x = 3 + y$ Substitute $3 + y$ into the second equation for x.

$$\begin{aligned} 2(3 + y) + 2y &= 2 \\ 6 + 2y + 2y &= 2 \\ -6 \qquad\qquad &\; -6 \\ 4y &= -4 \\ y &= -1 \end{aligned}$$

Substitute -1 for y in either equation. (Substitute it into the equation already and solved for x.)

$x = 3 + (-1)$
$x = 2$

Therefore, the solution to this system is $(2, -1)$.

Let's take a look at this problem again, but this time, solve the first equation for y.

Example #2: $\begin{cases} x - y = 3 \\ 2x + 2y = 2 \end{cases}$ Solve the first equation for y.

$-y = 3 - x$ Divide both sides by -1 to have positive y.

$y = -3 + x$ Apply the Commutative Property of Addition.

$y = x - 3$

Substitute $x - 3$ into the second equation for y.

$$\begin{aligned} 2x + 2(x - 3) &= 2 \\ 2x + 2x - 6 &= 2 \\ 4x - 6 &= 2 \\ +6 \quad &\; +6 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

$y = x - 3$ Substitute 2 for x in either equation. (Substitute it into the equation already solved for y.)

$y = 2 - 3$
$y = -1$

Therefore, the solution to this system is (2, –1). This is the same solution as the previous problem, as it should be.

Solving Systems of Equations with Substitution (04:24)

Stop! **Go to Questions #6-8 about this section, then return to continue on to the next section.**

Elimination Method for Solving Systems of Equations

To solve a system by elimination:

1.) the coefficients of the same variable must be the same
2.) if the coefficients **are** the same, either subtract the equations or add the equations to eliminate that variable, depending on the signs (same sign – subtract, different signs – add)
3.) substitute this value back into one of the equations to find the other variable

QuickTime Solving Systems of Equations: Two Approaches (02:59)

Example #1: Solve the system of equations using elimination and express the answer as an ordered pair.

$$\begin{aligned} 2x + y &= 8 \\ x - y &= 10 \end{aligned}$$

Since both the y values have the same coefficient, other than the sign, you can **add the two equations** to eliminate the y.

$$\begin{array}{rr} & 2x + \cancel{y} = 8 \\ + & x - \cancel{y} = 10 \\ \hline & 3x \quad = 18 \end{array}$$

$+\,y - y = 0y$ which equals 0, thus the y is eliminated.

$x = 6$ Solve for x.

$6 - y = 10$ Substitute 6 for x in either equation to find y. (We'll use the second equation here.)

$-y = 4$ $-y = -1y$, so, divide both sides by -1.

$y = -4$

Therefore, the solution to this system is $(6, -4)$.

QuickTime Elimination and the Addition Property (04:33)

Example #2: Solve the system of equations using elimination and express the answer as an ordered pair.

$$\begin{aligned} x + y &= 4 \\ 2x + 3y &= 9 \end{aligned}$$

You have a choice to either eliminate the x or the y.

If you choose to eliminate the x, multiply the top equation by 2 or -2, depending on whether you want to add or subtract.

If you choose to eliminate the y, multiply the top equation by 3 or -3, again depending on whether you want to add or subtract.

In "part a" below, x is eliminated and the system is solved for y. In "part b" below, y is eliminated and the system is solved for x.

a.) eliminating *x* first

$$-2(x + y = 4)$$
$$2x + 3y = 9$$

$$
\begin{array}{rl}
 & -2\cancel{x} - 2y = -8 \\
+ & \underline{\cancel{2x} + 3y = 9} \\
 & y = 1
\end{array}
$$

In situation "a", substitute 1 for *y* to find the value of *x*.

$$x + 1 = 4$$
$$x = 3$$

b.) eliminating *y* first

$$-3(x + y = 4)$$
$$2x + 3y = 9$$

$$
\begin{array}{rl}
 & -3x - \cancel{3y} = -12 \\
+ & \underline{2x + \cancel{3y} = 9} \\
 & -x \quad = \quad -3 \\
 & x = 3
\end{array}
$$

In situation "b", substitute 3 for *x* to find the value of *y*.

$$3 + y = 4$$
$$y = 1$$

Using either method, we arrive at the same answer, (3, 1).

QuickTime Elimination and the Multiplication Property of Equality (04:06)

Example #3: Solve the system of equations using elimination and express the answer as an ordered pair.

$$x - 2y = 2$$
$$3x - 6y = 6$$

$$-3(x - 2y = 2)$$
$$3x - 6y = 6$$

Multiply the first equation by –3 to use addition to eliminate *x*.

$$-3x + 6y = -6$$
$$3x - 6y = 6$$

Add the two equations.

$$
\begin{array}{rl}
 & -3\cancel{x} + 6\cancel{y} = -6 \\
+ & \underline{\cancel{3}x - \cancel{6}y = 6} \\
 & 0 = 0
\end{array}
$$

$-3x + 3x = 0, \quad +6y + -6y = 0$

$-6 + 6 = 0$

Both variables (*x* and *y*) are eliminated and the constant equals zero. The equation 0 = 0 is an identity; thus, any ordered pair that is a solution for equation one is also a solution for equation two.

This system of equations has many solutions.

Example #4: Solve the system of equations using elimination and express the answer as an ordered pair.

$$8x + 2y = 15$$
$$-4x - y = 5$$

$$8x + 2y = 15$$
$$2(-4x - y = 5)$$

Multiply the second equation by 2 to use addition to eliminate *x*.

$$8x + 2y = 15$$
$$-8x - 2y = 10$$

Add the two equations.

$$\begin{array}{r} 8\cancel{x} + 2\cancel{y} = 15 \\ + -\cancel{8}x - \cancel{2}y = 10 \\ \hline 0 = 25 \end{array}$$

$8x + -8x = 0,\quad +2y + -2y = 0$
$15 + 10 = 25$

Since there are no values that will make the equation 0 = 25 true, **there are no solutions for this system of equations.**

Sometimes it is necessary to multiply both equations in a system to eliminate one of the variables. Multiply each equation by values so that one of the variables has the same or opposite coefficient in both equations.

Let's see how this works.

Example #5: Solve the system of equations using elimination and express the answer as an ordered pair.

$$4x + 2y = 14$$
$$7x - 3y = -8$$

To eliminate the *y*'s in this system, multiply the first equation by 3 and the second equation by 2 so that the coefficient of both *y*'s are opposites.

$$3(4x + 2y = 14)$$
$$2(7x - 3y = -8)$$

$$12x + 6y = 42$$
$$14x - 6y = -16$$

Add the two equations.

$$\begin{array}{r} 12x + \cancel{6y} = 42 \\ + \quad 14x - \cancel{6}y = -16 \\ \hline 26x \qquad = 26 \\ x = 1 \end{array}$$

$6y + -6y = 0$

Solve for *x*.

Find the value of *y* by substituting *x* = 1 into either of the original equations.

$$4(1) + 2y = 14$$

Substitute 1 for *x* in the first equation.

$$\begin{aligned} 4(1) + 2y &= 14 \\ 4 + 2y &= 14 \\ 2y &= 10 \\ y &= 5 \end{aligned}$$

Solve for y.

The solution to this system is (1, 5).

Stop! **Go to Questions #9-14 about this section, then return to continue on to the next section.**

Applications of Linear Systems

When solving an application problem using a linear system, write two equations. The system may be solved using either substitution or elimination.

Example #1: Tickets to the annual talent show cost \$5.00 for adults and \$2.00 for students. For the show 285 tickets were sold and \$1065 was collected. Write a system of equations to find how many of each type of ticket were sold.

Solution:

Let x = **number** of adult tickets sold
Let y = **number** of student tickets sold.

Since 285 tickets were sold, the first equation is:

$$\boldsymbol{x + y = 285}$$

Since \$1065 was collected, the second equation represents the amount of money collected.

\$5 per adult ticket ($5x$) \$2 per student ticket ($2y$)

$$\boldsymbol{5x + 2y = 1065}$$

Write and solve the system of equations.

$$\begin{aligned} x + y &= 285 \\ 5x + 2y &= 1065 \end{aligned}$$

First, eliminate the y's by multiplying the first equation by –2.

$$\begin{aligned} -2x + -2y &= 570 \\ 5x + \quad 2y &= 1065 \end{aligned}$$

Now, add the two equations.

$$\begin{array}{r} -2x + -2\cancel{y} = -570 \\ 5x + \quad \cancel{2}y = 1065 \\ \hline 3x \qquad\quad = 495 \end{array}$$

Solve for x.

$$3x = 495 \qquad \text{Divide both sides by 3.}$$
$$x = 165$$

This means 165 adult tickets were sold.

To find the value of y (the number of student tickets sold) substitute $x = 165$ into the first equation.

$$x + y = 285$$
$$\color{red}{165} + y = 285 \qquad \text{Subtract 165 from both sides.}$$
$$y = 120$$

Thus, 120 student tickets were sold.

To ***check*** this solution, substitute the values of x and y into each equation.

$$\begin{aligned} x + y &= 285 \\ 165 + 120 &= 285 \\ 285 &= 285 \checkmark \end{aligned} \qquad \begin{aligned} 5x + 2y &= 1065 \\ 5(165) + 2(120) &= 1065 \\ 825 + 240 &= 1065 \checkmark \end{aligned}$$

There were 165 tickets for adults and 120 tickets for students sold for the talent show.

Example #2: One solution is 20% hydrochloric acid and another solution is 70% hydrochloric acid. How many ounces of each solution should be mixed to obtain 20 ounces of a solution that is 50% hydrochloric acid? Write a system of linear equations and solve.

Solution:

Let x = **ounces** of 20% solution needed

Let y = **ounces** of 70% solution needed

Since 20 ounces are needed, the first equation is:

$$\boldsymbol{x + y = 20}$$

The second equation represents the **amount** of hydrochloric **acid**. To find the amount of hydrochloric acid *multiply* the percentage written as a decimal times the number of ounces.

"one solution is 20% hydrochloric acid"

20% times amount of acid ($0.20x$)

"another solution is 70% hydrochloric acid"

70% times amount of acid ($0.70x$)

"to obtain 20 ounces of a solution that is 50% hydrochloric acid"

50% times 20 ounces of acid (0.50)(20)

Acid in the first solution + acid in the second solution = acid in the mixture

$$0.20x + 0.70y = .50(20) \text{ which simplifies to } 0.2x + 0.7x = 10$$

To clear this equation of decimals (and make the calculations a bit easier), multiply each side of the equation by the equation by 10.

$$10\,(0.2x + 0.7y) = 10(10)$$

$$\mathbf{2x + 7y = 100}$$

Write and solve the system of equations.

$$\begin{aligned} x + \quad y &= 20 \\ 2x + 7y &= 100 \end{aligned}$$

First, eliminate the *x*'s by multiplying the first equation by –2.

$$\begin{aligned} -2x + -2y &= -40 \\ 2x + \ \ 7y &= 100 \end{aligned}$$

Now, add the two equations and solve for *y*.

$$\begin{aligned} -2\cancel{x} + -2y &= -40 \\ \cancel{2}x + \ \ 7y &= 100 \\ \hline 5y &= 60 \\ y &= 12 \end{aligned}$$

Thus, 12 ounces of the 70% solution will be needed.

To find the amount of 20% solution needed, substitute 12 for the *y* in either equation. We'll use the simpler equation:

$$\begin{aligned} x + y &= 20 \\ x + 12 &= 20 \\ x &= 8 \end{aligned}$$

Therefore, 8 ounces of the 20% acid solution will be needed.

To ***check*** this solution, substitute the values of *x* and *y* into each equation.

$$\begin{aligned} x + y &= 20 \\ 8 + 12 &= 20 \\ 20 &= 20 \checkmark \end{aligned} \qquad \begin{aligned} 0.2x + 0.7y &= 10 \\ 0.2(8) + 0.7(12) &= 10 \\ 1.6 + 8.4 &= 10 \\ 10 &= 10 \checkmark \end{aligned}$$

To obtain 20 ounces of a 50% hydrochloric acid, a mixture of 8 ounces of 20% hydrochloric acid and 12 ounces of 70% hydrochloric acid is needed.

In this problem, it may be a little easier to organize all the given information in a table, and then write the system of equations.

Type of Acid	Ounces	Amount of Acid In Solution
20% acid	x	**0.2**x
70% acid	y	**0.7**y
Mixture (50% acid)	**20**	**0.5(20)**

$$x + y = 20 \qquad 0.2x + 0.7y = 10$$

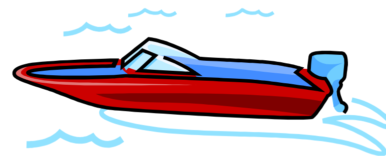
At this point, the solution is the same; that is, solve the system of equations for x and y.

QuickTime Systems of Equations and Algebra -- Scubadiving (02:56)

Example #3: Jacob and Amber rowed their boat 12 miles downstream (with the current) in 2 hours. On the return trip against the current, it took 6 hours. Find the rate of the boat in still water and the rate of the current.

Let: x = rate of the boat in still water

y = rate of the current



Then:

$x + y$ = rate of the boat **with the current** (The speed of the current works with the speed of the boat to go faster than its regular speed downstream)

$x - y$ = rate of the boat **against the current** (the speed of the current pushes against the speed of the boat to make the boat go slower that its regular speed upstream)

Use the distance formula $d = rt$ to model the equations needed to set up the system of equations.

Let's organize our information in a table.

	Downstream	**Upstream**
Distance (*d*)	**12***	**12***
Rate (*r*)	$x + y$	$x - y$
Time (*t*)	**2**	**6**

**Note*: The distance is the same for traveling downstream and upstream, and the rate and time varies.

Since it takes 2 hours to go 12 miles downstream, the first equation is:

$$d = \quad r \quad t$$

$$12 = (x + y)2$$

$$12 = 2x + 2y \qquad \text{Distribute.}$$

$$\frac{12}{2} = \frac{2x + 2y}{2} \qquad \text{Simplfy by dividing both sides by 2.}$$

$$6 = x + y \qquad \text{Apply the Reflexive Property.}$$

$$x + y = 6$$

Since it takes 6 hours to go 12 miles upstream, the second equation is

$$\begin{aligned} d &= \quad r \quad t \\ 12 &= (x-y)6 \\ 12 &= 6x-6y \end{aligned}$$

Distribute

$$\frac{12}{6} = \frac{6x-6y}{6}$$

Simplfy by dividing both sides by 6.

$$2 = x - y$$

Apply the Reflexive Property

$$x - y = 2$$

Write and solve the system of equations.

$$\begin{aligned} x + y &= 6 \\ x - y &= 2 \end{aligned}$$

Add to eliminate the y's and solve for x.

$$\begin{aligned} x + \cancel{y} &= 6 \\ x - \cancel{y} &= 2 \\ \hline 2x &= 8 \\ x &= 4 \end{aligned}$$

The rate of the boat in still water (x) is 4 MPH (miles per hour).

To find rate of the current, substitute 4 for the x in either equation. We'll use the first equation.

$$\begin{aligned} x + y &= 6 \\ 4 + y &= 6 \\ y &= 2 \end{aligned}$$

Therefore, the rate of the current (y) is 2 MPH.

To ***check*** this solution, substitute the values of x and y into each equation.

$$\begin{aligned} x + y &= 6 \\ 4 + 2 &= 6 \\ 6 &= 6 \checkmark \end{aligned} \qquad \begin{aligned} x - y &= 2 \\ 4 - 2 &= 2 \\ 2 &= 2 \checkmark \end{aligned}$$

The rate of the boat in still water is 6 MPH and the rate of the current is 2 MPH.

Word Problem: Rate, Time, and Distance (04:12)

Systems of equations are used in business to determine the break-even point. The break-even point is the point at which income equals the cost.

Example #4: The Jones family plans to rent a car for a one-day trip. The car rental company offers two rental plans. The first plan allows them to rent a car for \$42 per day plus 25 cents per mile. The second plan allows them to rent a car for \$90 a day with unlimited mileage. Write an equation that represents the cost of renting each car. Find the break-even point of the rental cars. If the Jones family plans to drive 150 miles, which plan should they choose?

Let m = the number of miles the car is driven

c = the total cost to rent the car for one day

	Rental Plan 1	**Rental Plan 2**
Daily cost	**\$42**	**\$90**
Mileage Rate	**0.25*m***	**0 (unlimited)**
Total Cost for One Day	***c***	***c***

Since the first plan costs \$42 plus 25 cents per mile, the first equation is:

$c = 42 + 0.25m$

Since the second plan costs \$90 a day with unlimited mileage, the second equation is:

$c = 90$

Write and solve the system of equations.

$c = 42 + 0.25m$
$c = 90$

Since both equations represent c, use substitution to solve the system.

$c = 42 + 0.25m$
$90 = 42 + 0.25m$ Substitute $c = 90$.
$48 = 0.25m$ Subtract 42 from both sides.
$192 = m$ Divide both sides by 0.25

Thus, the cost for the rental plans is the same if they plan to drive 192 miles. This is the break-even point.

If the Jones family plans to drive less than 192 miles the first plan is cheaper. If the Jones family plans to drive more than 192 miles the second plan is cheaper.

Which plan is cheaper for the Jones family outing?

"Click here" to check your answer.

Rental Plan 1 is cheaper since they are only traveling 150 miles.

Stop! **Go to Questions #15-21 about this section, then return to continue on to the next section.**

Linear Inequalities

A solution to a linear inequality in two variables, *x* and *y*, is an ordered pair (*x*, *y*) that satisfies the inequality. The solution to a linear inequality is a region of the coordinate plain and is called a *half-plane* bounded by a *boundary line*.

*If a linear inequality is a < or >, the boundary line will be a dashed line. - - - - - - - - -

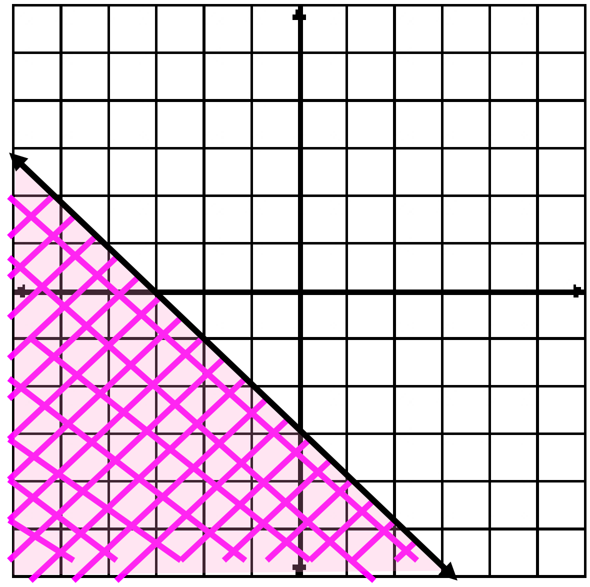
*If a linear inequality is a ≤ or ≥, the boundary line will be a solid line. ________

To graph a linear inequality:

1.) solve the inequality for *y*
2.) plot the *y*-intercept
3.) use the slope $\frac{rise}{run}$ to plot more points
4.) determine whether to connect with a dashed line or a solid line
5.) shade a region of the coordinate plane (this is determined by either <, >, ≤ , or ≥). If < or ≤, shade below the boundary line; if > or ≥, shade above the boundary line.

You can also choose a point to test, such as (0, 0), as it is easy to test if it does not lie on the line. If you plug 0 in for *x* and *y* and get a true statement, shade the portion of the coordinate plane that contains that point. If you plug 0 in for *x* and *y* and get a false statement, shade the portion of the coordinate plane that does not contain that point. If the point (0, 0) is on the line, choose another point as the test point.

Example #1: $y \leq -x - 3$



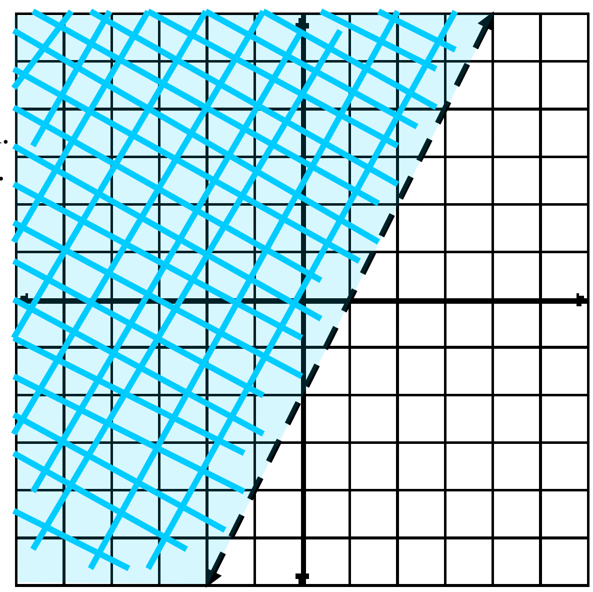
Since this is a ≤ inequality, the line is solid. Shading will occur below the boundary line.

To test the point (0, 0), replace x and y with 0. Since we chose to shade the region that does not include (0, 0), we should get a false statement.

$0 \leq -(0) - 3$
$0 \leq -3$

This is a false statement; therefore, we chose to shade the correct half-plane.

Example #2: $y > 2x - 2$



Since this is >, the boundary line will be dashed. Shading will occur above the line or up and left.

To test the point (0, 0), replace x and y with 0. Since we chose to shade the region that does include (0, 0), we should get a true statement.

$0 > 2(0) - 2$
$0 > -2$

This is a true statement; therefore, we chose to shade the correct region, the one that includes the point (0, 0).

Stop! **Go to Questions #22-30 to complete this unit.**