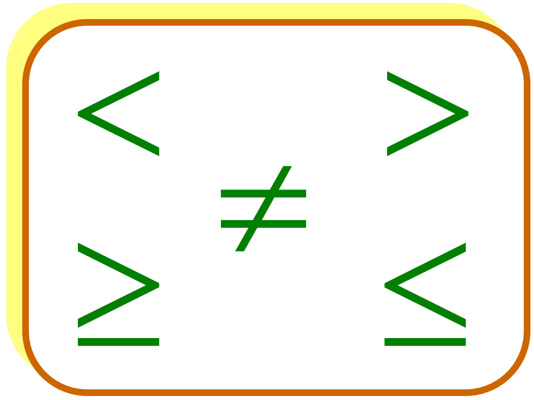
INEQUALITIES AND ABSOLUTE VALUE EQUATIONS



Unit Overview

An inequality is a mathematical statement that compares algebraic expressions using greater than (>), less than (<), and other inequality symbols. A compound inequality is a pair of inequalities joined by *and* or *or*. In this unit, properties of inequalities will be used to solve linear inequalities and compound inequalities in one variable. The unit concludes continues with a study of absolute-value equations and inequalities.

Introduction to Solving Inequalities

Inequality - a mathematical statement that compares algebraic quantities

Inequality Symbols	Meaning	Keyboard Entry
$<$	less than	<
$>$	greater than	>
$\leq$	less than or equal	<=
$\geq$	greater than or equal	>=
$\neq$	not equal	<>

*Solving inequalities is just like solving equations, use opposite operations to isolate the variable.

Example #1: Solve for *x*.

$$3(5x-7)\geq 54$$
$$15x-21\geq 54$$
$$+21\quad +21$$
$$15x\geq 75$$
$$x\geq 5$$

QuickTime Inequalities--Bridge Capacity (02:27)

*When multiplying or dividing by a negative number, the inequality sign must be reversed.

Example #2: Solve for *y*.

$$\begin{array}{l} 2y + 9 < 5y + 15 \\ -5y \quad\quad -5y \\ -3y + 9 < 15 \\ \quad -9 \quad -9 \\ \dfrac{-3y}{-3} < \dfrac{6}{-3} \\ \quad y > -2 \end{array}$$

*Notice that the inequality sign is flipped because of the division by –3.

Example #3: Solve for *x*.

$$\frac{3}{4}(x-7) \le x-3$$

$$(4)\frac{3}{4}(x-7) \le 4(x-3)$$ *Multiply both sides by 4.

$$3(x-7) \le 4(x-3)$$ *Distribute.

$$\begin{array}{l} 3x - 21 \le 4x - 12 \\ -3x \quad\quad -3x \\ -21 \le x - 12 \\ -9 \le x \\ x \ge -9 \end{array}$$

*Rewrite with *x* on left side, inequality sign is reversed.

QuickTime Solving Inequalities: Two Operations (01:25)

Example #4: Jenny has scored 18, 15, 30 and 16 points in her first 4 basketball games. How many points must she score in the next game so that her 5 game average is at least 20 points? Write an inequality and solve.

Let x = points scored in Game 5.

To find the average, add up all five test scores and divide by 5.

$$\frac{\text{sum of the points scored in 5 games}}{5} \geq 20$$ Guide in words

An average of "at least" 20 points means 20 points or higher which can be interpreted mathematically as greater than or equal ($\geq$).

$$\frac{18+15+30+16+x}{5} \geq 20$$ Set up the inequality letting x be points for Game 5.

$18 + 15 + 30 + 16 + x \geq 100$ Multiply each side by 5

$77 + x \geq 100$ Simplify the left side of the equation.

$x \geq 23$ Subtract 77 from each side

Jenny must score at least 23 points.

*Note the answer is not only 23 points, since if Jenny scores more than 23 points she will also have an average of at least 20 points per game.

Check : We will just check to see if 23 points in the fifth game will be enough to give Jenny an average of 20 points.

$$\frac{18+15+30+16+x}{5} = 20$$

$$\frac{18+15+30+16+23}{5} = \frac{77+23}{5} = \frac{100}{5} = 20 \checkmark$$

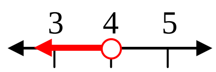
You can represent the solution of an inequality in one variable on a number line.

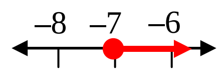
For < and > an open circle is used to denote that the solution number **is not** included in the solution.

For $\leq$ and $\geq$ a closed circle is used to denote that the solution number **is** included in the solution.

Example #5: Graph the solution of each inequality.

$x < 4$ $\qquad\qquad$ $y \geq -7$





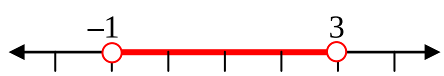
Compound Inequalities

compound inequalities: a pair of inequalities joined by “and” or “or”.

To solve a compound inequality joined with “**and**”, find the values of the variable that satisfy ***both*** inequalities.

*“**and**” means the **intersection** of the solutions

Example #6:
$$\begin{array}{rcr} 2x + 3 > 1 & \text{and} & 5x - 9 < 6 \\ 2x > -2 & \text{and} & 5x < 15 \\ x > -1 & \text{and} & x < 3 \end{array}$$



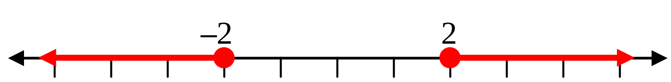
The solution is written $\{x| -1 < x < 3\}$ (set notation) “all numbers *x*, such that –1 is less than *x* is less than 3”.

To solve a compound inequality joined with “**or**”, find the values of the variable that satisfy at **least one** inequality.

“**or**” means the **union** of the solutions

Example #7: Find all solutions for *b*.

$$\begin{array}{rcr} 3b + 7 \leq 1 & \text{or} & 2b - 3 \geq 1 \\ 3b \leq -6 & \text{or} & 2b \geq 4 \\ b \leq -2 & \text{or} & b \geq 2 \end{array}$$



The solution is written $\{b| \ b \leq -2 \text{ or } b \geq 2\}$ (set notation) “all numbers *b* such that *b* is less than or equal to –2 or *b* is greater than or equal to 2”.

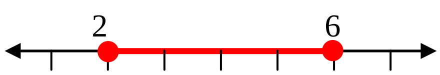
Example #8: Find all solutions for y when $4y - 9 \leq 15$ **and** $4y - 9 \geq -1$.

Since the two inequalities have a common statement (4y – 9) and they are an intersection (**and**) of the two solutions, the inequalities can be written so that the common statement is "sandwiched" between the two inequalities.

$4y - 9 \geq -1$ is the same as $-1 \leq 4y - 9$

$-1 \leq 4y - 9 \qquad 4y - 9 \leq 15$	Make sure the inequality symbols are
$\searrow \quad \swarrow$	both pointing the same direction.
$-1 \leq 4y - 9 \leq 15$	Combine the two inequalities into
	one inequality statement.
$-1 \leq 4y - 9 \leq 15$	
$+9 \qquad +9 \quad +9$	Isolate 4y by adding 9 to all parts.
$8 \leq 4y \leq 24$	
$\frac{8}{4} \leq \frac{4y}{4} \leq \frac{24}{4}$	Isolate y by dividing all parts by 4.
$2 \leq y \leq 6$	

This solution can be interpreted as $y \geq 2$ and $y \leq 6$



The solution is written {y| 2 < y < 6} (set notation) “all numbers y, such that 2 is less than x is less than 6”. In other words, y can be any number in between and including 2 and 6.

QuickTime Solving Compound Inequalities (02:25)

Example #9: Sam expects to make between \$3200 and \$4200 from his summer job. He needs to buy 4 new tires for his car at \$85 each. He estimates his other expenses for the summer at \$550. Write and solve an inequality to determine how much money Sam can expect to save during the summer.

Let x = amount of money Sam can save.

$3200 \le x + 4(85) + 550 \le 4200$ Set up the inequality.

$3200 \le x + 4(85) + 550 \le 4200$ *$3200 \le x + 4(85) + 550$ means the same as $x + 4(85) + 550 \ge 3200$

The inequality symbols show that Sam's expenses and savings will be greater than or equal to 3200
and less than or equal to 4200, all in one nice compact inequality statement.

$3200 \le x + 890 \le 4200$ Simplify

$-890 \quad -890 \quad -890$ Isolate x by subtracting 890 from all of the parts.

$2320 \le x \le 3310$

Sam expects to save between \$2320 and \$3310 this summer.

Stop! **Go to Questions #1-14 about this section, then return to continue on to the next section.**

Solving Absolute Value Equations and Inequalities

absolute value - the distance a number is from zero (always positive).

*Two bars around the number denote absolute value.

$|-5| = 5$ $|6| = 6$

Why do absolute value equations have two solutions?

In a simple absolute value equation such as $|x| = 5$, let's examine what two values of x make the expression true?

Since the $|5| = 5$, then $x = 5$.
Also, since the $|-5| = 5$, then $x = -5$.

Therefore, the solution to $|x| = 5$ is $x = 5$ or $x = -5$.

Let's expand the understanding of an absolute value equation further.

Let's think about the following question: In the absolute value equation, $|3x + 4| = 19$, what two values can $3x + 4$ be and why?

The value of $3x + 4$ in $|3x + 4| = 19$ can be 19 because why?

Click here to check your answer.

The absolute value of $|19| = 19$, therefore $3x + 4$ can equal 19.

The value of $3x + 4$ in $|3x + 4| = 19$ can be -19 because why?

Click here to check your answer.

The absolute value of $|-19| = 19$, therefore $3x + 4$ can equal -19.

What two equations can be written to find the values of x that solve $|3x + 4| = 19$?

Click here to check your answer.

$3x + 4 = 19$ OR $3x + 4 = -19$

Follow these steps to solve absolute value equations, but keep in mind why these steps work by remembering the explanation above.

1.) Rewrite the equation without the absolute value notation.
2.) Rewrite a second time using the opposite of what the original equation was equal to, and connect with the word "**or**".
3.) Solve both equations and check both answers in the original equation.

Example #1: Solve $|2x-1| = 3$

$$\begin{aligned} 2x - 1 &= 3 \quad \text{or} \quad 2x - 1 = -3 \\ 2x &= 4 \quad \text{or} \quad 2x = -2 \\ x &= 2 \quad \text{or} \quad x = -1 \end{aligned}$$

*the -3 is the opposite of what the original was equal to.

Check: $|2(2)-1| = 3$ or $|2(-1)-1| = 3$

$|4-1| = 3$ or $|-2-1| = 3$

$|3| = 3$ or $|-3| = 3$

$3 = 3$ ✓ or $3 = 3$ ✓

Therefore, the solution is $x = 2$ or $x = -1$.

Example #2: Solve $|2x+1| = x + 5$

$$\begin{aligned} 2x + 1 &= x + 5 \quad \text{or} \quad 2x + 1 = -x - 5 \\ x &= 4 \quad \text{or} \quad 3x = -6 \\ & \qquad\qquad\qquad x = -2 \end{aligned}$$

*again use the opposite

Check: $|2(4)+1| = 4 + 5$ or $|2(-2)+1| = -2 + 5$

$|8+1| = 9$ or $|-4+1| = 3$

$|9| = 9$ or $|-3| = 3$

$9 = 9$ ✓ or $3 = 3$ ✓

Therefore, the solution is $x = 4$ or $x = -2$.

Example #3: Solve: $3|x + 4| – 2 = 7$

First, isolate the absolute value expression $|x + 4|$.

$$3|x+4|-2=7$$

$$\begin{aligned} 3|x+4|-2 &= 7 \\ +2 &\ +2 \\ 3|x+4| \quad &= 9 \end{aligned}$$
*Add 2 to each side.

$$\frac{3|x+4|}{3} = \frac{9}{3}$$
*Divide both sides by 3.

$$|x+4|=3$$
*The absolute value expression is isolated on the left side.

$$x+4=3 \text{ OR } x+4=-3$$
*Write the absolute value equation as two separate equations.

$$\begin{aligned} x+4&=3 \qquad & x+4&=-3 \\ x&=-1 & x&=-7 \end{aligned}$$
*Solve each equation for *x*.

Check:

$$\begin{aligned} 3|(-1)+4|-2&=7 \\ 3|3|-2&=7 \\ 9-2&=7 \\ 7&=7\checkmark \end{aligned}$$

$$\begin{aligned} 3|(-7)+4|-2&=7 \\ 3|-3|-2&=7 \\ 9-2&=7 \\ 7&=7\checkmark \end{aligned}$$

Therefore, the solution is x –1 or $x = –7$.

Example #4: Solve: $|x – 1| = 5x + 10$

Write the absolute value equation as two separate equations and solve.

$$x-1=5x+10 \qquad \text{OR} \qquad x-1=-(5x+10)$$

Solve the **first equation** $x-1=5x+10$ for *x*.

$$\begin{aligned}
x - 1 &= 5x + 10 \\
+1 \quad &\quad\; +1
\end{aligned}$$

*Add 1 to each side.

$$\begin{aligned}
x \quad &= 5x + 11 \\
-5x \quad &\; -5x
\end{aligned}$$

*Subtract $5x$ from each side.

$$-4x = 11$$

$$\frac{-4x}{-4} = \frac{11}{-4}$$

*Divide both sides by -4.

$$x = -\frac{11}{4}$$

Solve the **second equation** $x - 1 = -(5x + 10)$ for x.

$$\begin{aligned}
x - 1 &= -(5x + 10) \\
x - 1 &= -5x - 10
\end{aligned}$$

*Rewrite the right side reversing the signof each term since the minus sign is in front of the parenthesis.

$$\begin{aligned}
x - 1 &= -5x - 10 \\
+1 \quad &\quad\; +1
\end{aligned}$$

*Add 1 to each side.

$$\begin{aligned}
x \quad &= -5x - 9 \\
+5x \quad &\; +5x
\end{aligned}$$

*Add $5x$ to each side.

$$6x = -9$$

$$\frac{6x}{6} = \frac{-9}{6}$$

*Divide both sides by 6.

$$x = -\frac{9}{6} = -\frac{3}{2}$$

Check:

$$\left|-\frac{11}{4}-1\right|=5\left(-\frac{11}{4}\right)+10 \qquad \text{OR} \qquad \left|-\frac{3}{2}-1\right|=5\left(-\frac{3}{2}\right)+10$$

$$\left|-\frac{11}{4}-\frac{4}{4}\right|=-\frac{55}{4}+\frac{40}{4} \qquad \text{OR} \qquad \left|-\frac{3}{2}-\frac{2}{2}\right|=-\frac{15}{2}+\frac{20}{2}$$

$$\left|-\frac{15}{4}\right|=-\frac{15}{4} \qquad \text{OR} \qquad \left|-\frac{5}{2}\right|=\frac{5}{2}$$

$$\frac{15}{4}\neq-\frac{15}{4}\ \text{(Doesn't Check)} \qquad \text{OR} \qquad \frac{5}{2}=\frac{5}{2}\checkmark$$

extraneous solution: An extraneous is a solution that does not satisfy the original equation.

Since –11/4 does not satisfy the original equation, –11/4 is an extraneous solution.

*It is *important* to check all solutions in the original equation to determine whether a solution is extraneous.

Therefore, the solution is $x = -3/2$.

absolute value inequalities - an inequality that contains an absolute value.

To solve absolute value inequalities:

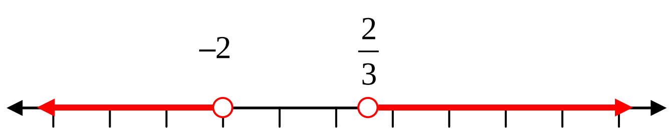
1.) Rewrite the inequality without the absolute value notation.
2.) Rewrite a second time, change the inequality sign, and use opposites.
3.) Solve both inequalities and check both answers in the original inequality.
4.) If the inequality is a $<$ or $\leq$, connect with the word "and".
5.) If the inequality is a $>$ or $\geq$, connect with the word "or".

Example #5: Solve $|3x+2| > 4$

$3x + 2 > 4$ **or** $3x + 2 < -4$ *flip the sign and use the opposite

$3x > 2$ **or** $3x < -6$

$x > \frac{2}{3}$ **or** $x < -2$



Check: $|3x+2| > 4$

To check this problem you will have to choose a number that is less than –2, and then choose a number that is greater than $\frac{2}{3}$.

Check: (–3) or *Check*: (1)

$$|3(-3)+2|>4 \quad \text{or} \quad |3(1)+2|>4$$

$$|-9+2|>4 \quad \text{or} \quad |3+2|>4$$

$$|-7|>4 \quad \text{or} \quad |5|>4$$

$7 > 4$ (true) ✓ $\quad$ $5 > 4$ (true) ✓

Therefore, the solution to this absolute value inequality is $\{x|\ x < -2 \text{ or } x > \frac{2}{3}\}$.

Example #6: Solve $\frac{1}{2}|5x-12|+4 \le 13$ *First, isolate the absolute value expression

$$-4 \quad -4$$

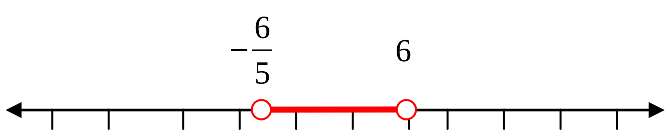
$$(2)\frac{1}{2}|5x-12| \le (2)9$$

$$|5x-12| \le 18$$

$5x-12 \le 18$ **and** $5x-12 \ge -18$ *flip the sign and use opposite

$$5x \le 30 \qquad 5x \ge -6$$

$x \le 6$ **and** $x \ge \frac{-6}{5}$



Check: $\frac{1}{2}|5x-12|+4 \le 13$

To check this problem, you will have to choose a number that is greater than $-\frac{6}{5}$ and also less than 6.

Check: (0)

$$\frac{1}{2}|5x-12|+4\le 13$$

$$\frac{1}{2}|5(0)-12|+4\le 13$$

$$\frac{1}{2}|-12|+4\le 13$$

$$\frac{1}{2}(12)+4\le 13$$

$$6+4\le 13$$

$$10\le 13 \text{ (true)} \checkmark$$

Therefore, the solution to this absolute value inequality is $\{x|\ x > -\frac{6}{5}$ and $x < 6\}$.

Stop! **Go to Questions #15-40 to complete this unit.**