SOLVING QUADRATIC EQUATIONS

$$y = ax^2 + bx + c$$

$$y = a(x-h)^2 + k$$

Unit Overview

In this unit you will find solutions of quadratic equations by completing the square and using the quadratic formula. You will also graph quadratic functions and rewrite quadratic functions in vertex forms. Many connections between algebra and geometry are noted.

Completing the Square

When a quadratic equation does not contain a perfect square, you can create a perfect square in the equation by *completing the square*. **Completing the square** is a process by which you can force a quadratic expression to factor.

1.) Make sure the quadratic term and the linear term are the only terms on one side of the equation (move the constant term to the other side).

2.) The coefficient of the quadratic term must be one.

3.) Take one-half of the linear term and square it.

4.) Add this number to both sides of the equation.

5.) Factor the perfect square trinomial.

6.) Solve the equation.

Example #1: Complete the given quadratic expression into a perfect square.

$x^2 - 20x$	
$x^2 - 20x + 100$	$\frac{1}{2}(20) = 10, \quad 10^2 = 100$
$(x-10)^2$	Factor (Take square root of 100.)

The completed perfect square is $x^2 - 20x + 100$ *or* $(x-10)^2$.

Check with FOIL

$$\begin{aligned}(x-10)^2 &= (x-10)(x-10)\\ &= x^2-10x-10x+100\\ &= x^2-20x+100\end{aligned}$$

Example #2: Complete the given quadratic expression into a perfect square.

x^2+5x

$x^2+5x+\frac{25}{4}$ $\qquad \frac{1}{2}(5)=\frac{5}{2}, \quad \left(\frac{5}{2}\right)^2=\frac{25}{4}$

$\left(x+\frac{5}{2}\right)^2$ $\qquad$ Factor (Take square root of $\frac{25}{4}$.)

The completed perfect square is $x^2+5x+\frac{25}{4}$ *or* $(x+\frac{5}{2})^2$.

Let's practice finding perfect square trinomials.

What value of c will make the expression a perfect square trinomial?

$x^2+22x+c$

Click here" to check your answer.

c = 121 (1/2 of 22 = 11, and then 11^2 equals 121)

What value of c will make the expression a perfect square trinomial?

$x^2-18x+c$

Click here" to check your answer.

c = 81 (1/2 of –18 = –9, and then $(-9)^2$ equals 81)

What value of c will make the expression a perfect square trinomial?

$$x^2 + 11x + c$$

Click here" to check your answer.

c = 121/4 (1/2 of 11 = 11/2, and then $(11/2)^2$ equals 121/4)

Example #3: Solve $x^2 + 6x = 16$ for x by completing the square.

$x^2 + 6x = 16$ — Take $\frac{1}{2}$ of 6 = 3, then $3^2 = 9$

$x^2 + 6x \ \underline{+9} = 16 \ \underline{+9}$ — Add 9 to both sides of the equation.

$$\frac{1}{2}(6) = 3 \rightarrow 3^2 = 9$$

$(x+3)^2 = 25$ — $x^2 + 6x + 9 = (x+3)^2 \qquad 16 + 9 = 25$

$\sqrt{(x+3)^2} = \sqrt{25}$ — Take the square root of both sides.

$x + 3 = \pm 5$ — $\sqrt{(x+3)^2} = (x+3) \qquad \sqrt{25} = \pm 5$

$$x + 3 = 5 \qquad x + 3 = -5$$
$$x = 2 \quad \text{or} \quad x = -8$$

QuickTime Completing the Square Algebraically (03:56)

Example # 4: Solve $x^2 - 10x + 18 = 0$ for x by completing the square.

$x^2 - 10x + 18 = 0$

$x^2 - 10x = -18$ — Move the constant term to the other side .(Subtract 18 from both sides.)

$x^2 - 10x \ \underline{+25} = -18 \ \underline{+25}$ — Take $\frac{1}{2}$ of –10 = –5, then $(-5)^2$ = 25. Add 25 to both sides of the equation.

$$(x-5)^2 = 7 \qquad x^2 - 10x + 25 = (x-5)^2 \quad -18+25=7$$

$$\sqrt{(x-5)^2} = \sqrt{7} \qquad \text{Take the square root of both sides.}$$

$$x-5 = \pm\sqrt{7} \qquad \text{Add 5 to both sides}$$

$$x = 5 \pm \sqrt{7}$$

$$x = 5+\sqrt{7} \quad \text{or} \quad x = 5-\sqrt{7}$$

*If the coefficient of the quadratic term is not 1, **divide all terms** by the coefficient to make it one. Let's see how this works!

Example #5: Solve $3x^2 - 6x = 5$ for x by completing the square.

$$3x^2 - 6x = 5$$

$$\frac{3x^2}{3} - \frac{6x}{3} = \frac{5}{3} \qquad \text{Divide all terms by 3}$$

$$x^2 - 2x = \frac{5}{3} \qquad \text{Simplify}$$

$$x^2 - 2x \underline{+1} = \frac{5}{3} \underline{+1} \qquad \text{Take } \frac{1}{2} \text{ of } -2 = -1\text{, then } (-1)^2 = 1.$$

Add 1 to both sides of the equation.

$$(x-1)^2 = \frac{8}{3} \qquad x^2 - 2x + 1 = (x-1)^2 \qquad \frac{5}{3} + \frac{3}{3} = \frac{8}{3}$$

$$\sqrt{(x-1)^2} = \sqrt{\frac{8}{3}} \qquad \text{Take the square root of both sides.}$$

$$x - 1 = \pm\sqrt{\frac{8}{3}} \qquad \text{Add 1 to both sides.}$$

$$x = 1 \pm \sqrt{\frac{8}{3}} \qquad \text{Add 1 to both sides.}$$

$$x = 1 + \sqrt{\frac{8}{3}} \text{ and } x = 1 - \sqrt{\frac{8}{3}}$$

$$x \approx 1 + 1.63 \text{ and } x \approx 1 - 1.63$$

$$x \approx 2.63 \quad \text{and} \quad x \approx -.63$$

Stop! **Go to Questions #1-8 about this section, then return to continue on to the next section.**

The Quadratic Formula

The quadratic formula is used to solve any quadratic equation in standard form, $ax^2 + bx + c = 0$. The quadratic formula is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the quadratic formula

1.) make sure the equation is in standard form
2.) label the values of *a*, *b*, and *c*
3.) replace the values into the equation and solve

Example #1: Use the quadratic formula to solve the given quadratic for "*x*".

$x^2 - 16x - 36 = 0$ $\quad a = 1 \quad b = –16 \quad c = –36$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a = 1,\ b = -16,\ c = -36$

$$x = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(1)(-36)}}{2(1)}$$

$$x = \frac{16 \pm \sqrt{256 + 144}}{2}$$

$$x = \frac{16 \pm \sqrt{400}}{2}$$

$$x = \frac{16 \pm 20}{2}$$

$$x = \frac{16 + 20}{2} \quad \text{or} \quad x = \frac{16 - 20}{2}$$

$$x = \frac{36}{2} \quad \text{or} \quad x = \frac{-4}{2}$$

$$x = 18 \quad \text{or} \quad x = -2$$

QuickTime The Quadratic Formula (06:38)

Example #2: Use the quadratic formula to solve the given quadratic for "*x*".

$x^2 + 4x - 18 = 0$ $\quad a = 1 \quad b = 4 \quad c = -18$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a = 1$, $b = 4$, $c = -18$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-18)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 + 72}}{2}$$

$$x = \frac{-4 \pm \sqrt{88}}{2}$$

$$x = \frac{-4 + \sqrt{88}}{2} \quad \text{or} \quad x = \frac{-4 - \sqrt{88}}{2}$$

*These expressions can be expressed in a simplified radical form, and this will be addressed in a later unit.

Example #3: Use the quadratic formula to solve the given quadratic for "*x*".

$2x^2 + 4x - 5 = 0$ $\quad a = 2 \quad b = 4 \quad c = -5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute $a = 2$, $b = 4$, $c = -5$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(2)(-5)}}{2(2)}$$

$$x = \frac{-4 \pm \sqrt{16 + 40}}{4}$$

$$x = \frac{-4 \pm \sqrt{56}}{4}$$

$$x = \frac{-4 + \sqrt{56}}{4} \quad \text{or} \quad x = \frac{-4 - \sqrt{56}}{4}$$

*These expressions can be expressed in a simplified radical form, and this will be addressed in a later unit.

Stop! **Go to Questions #9-11 about this section, then return to continue on to the next section.**

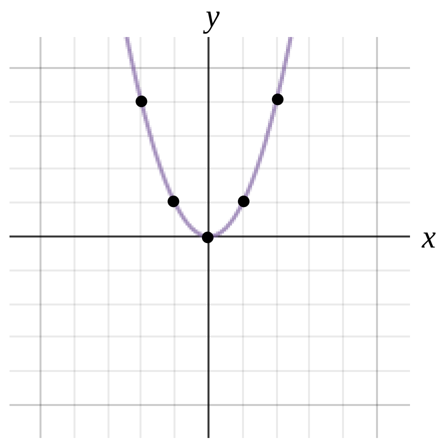
Graphing Quadratic Functions

The graph of a quadratic function is a parabola. All parabolas are related to the graph of $f(x) = x^2$. This makes $f(x) = x^2$ the parent graph of the family of parabolas. Any graph of a quadratic function is a **transformation** of the graph of $f(x) = x^2$. These transformations are similar to the transformations studied in a previous, using the parent graph $f(x) = |x|$.

Graphing a Quadratic Function of the Form $f(x) = ax^2$

Graph of the parent graph $f(x) = x^2$.

x	$f(x)$
–2	4
–1	1
0	0
1	1
2	4

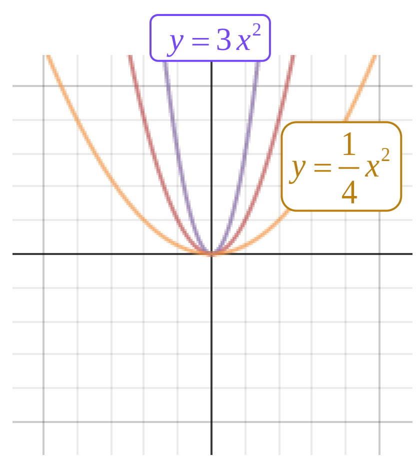


In this graph, the vertex is (0, 0) and the axis of symmetry is $x = 0$.

Now, let's take a look at how the graph changes when a changes for $f(x) = ax^2$.

In $y = \frac{1}{4}x^2$, the graph is *wider* than the parent function, $y = x^2$. The vertex is (0, 0) and the axis of symmetry is $x = 0$.

In $y = 3x^2$, the graph is *more narrow* than the parent function, $y = x^2$. The vertex is (0, 0) and the axis of symmetry is $x = 0$.

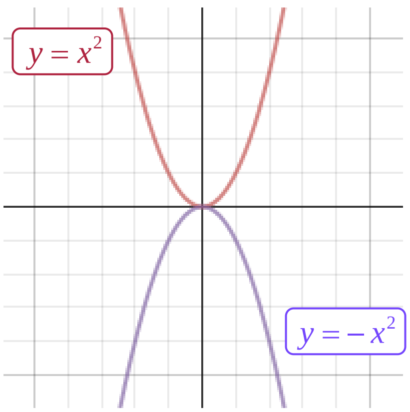


For all parabolas in the form $f(x) = ax^2$ the vertex is (0,0) and the axis of symmetry is $x = 0$.

If $|a| > 1$ then the graph of $f(x) = ax^2$ is **stretched** vertically and the graph is narrower than the parent graph. The larger the value of $|a|$ is the narrower the graph.

If $0 < a < 1$ then the graph of $f(x) = ax^2$ is **compressed** vertically and graph is wider than the parent graph. The smaller the absolute value of a is the wider the graph.

If a is negative, then the graph is opens downward instead of upward.



In $f(x) = x^2$, the graph opens upward, and therefore has a minimum value. The vertex is (0, 0) and the axis of symmetry is $x = 0$.

In $f(x) = -x^2$, the graph opens downward, and therefore has a maximum value. The vertex is (0, 0) and the axis of symmetry is $x = 0$.

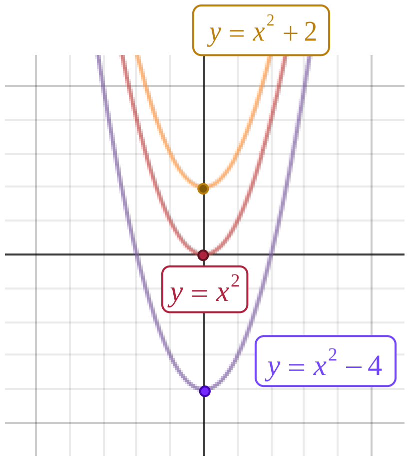
Graphs of $f(x) = ax^2$ and $f(x) = -ax^2$ are reflections of each other across the x-axis.

If $a > 0$ (a is positive), the parabola opens up, and thus has a minimum value.

If $a < 0$ (a is negative), the parabola opens down and thus has a maximum value.

Graphing Translations of $f(x) = x^2$

Vertical Translations

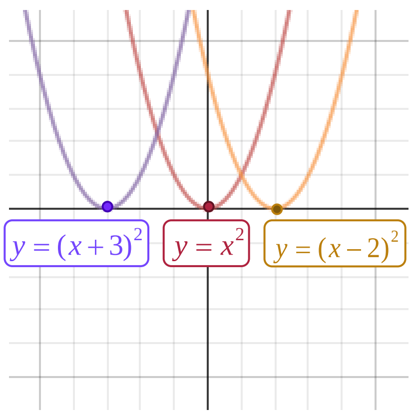


In $f(x) = x^2$, the vertex is **(0, 0)** and the axis of symmetry is $x = 0$.

In $f(x) = x^2 + 2$, the vertex is **(0, 2)** and the axis of symmetry is $x = 0$. The graph of $f(x) = x^2$ is translated *up* 2 units.

In $f(x) = x^2 - 4$, the vertex is **(0, –4)** and the axis of symmetry is $x = 0$. The graph of $f(x) = x^2$ is translated *down* 4 units.

Horizontal Translations



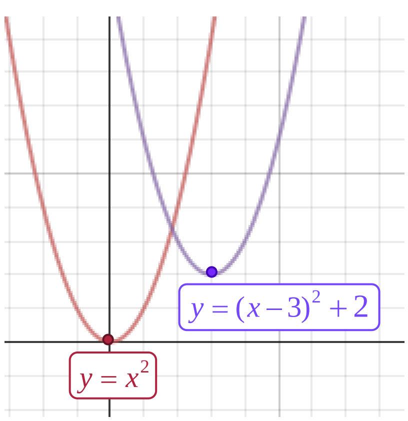
In $f(x) = x^2$, the vertex is **(0, 0)** and the axis of symmetry is $x = 0$.

In $f(x) = (x-2)^2$, the vertex is **(2, 0)** and the axis of symmetry is $x = 2$. The graph of $f(x) = x^2$ is translated *right* 2 units.

In $f(x) = (x+3)^2$, the vertex is **(–3, 0)** and the axis of symmetry is $x = -3$. The graph of $f(x) = x^2$ is translated *left* 3 units.

*Notice that it seems like the graph is sliding in the opposite direction as it should be translated; however, this will be explained in the discussion of the next two sections.

Vertical and Horizontal Translation



In $f(x) = x^2$, the vertex is **(0, 0)** and the axis of symmetry is $x = 0$.

In $f(x) = (x-3)^2 + 2$, the vertex is **(3, 2)** and the axis of symmetry is $x = 3$. The graph of $f(x) = x^2$ is translated 3 units to the *right* and 2 units *up*.

QuickTime Quadratic Polynomials (04:03)

Stop! **Go to Questions #12-16 about this section, then return to continue on to the next section.**

Using Vertex Form

Expressing quadratic equations in vertex form can be very useful in determining how the equation relates to its graph. It can be used to identify the vertex and *y*-intercept quickly along with other characteristics of its graph such as its maximum or minimum point and how wide or narrow it is.

The **vertex form** of a quadratic function is:

$$f(x) = a(x-h)^2 + k$$

The **vertex** is located at (h, k).

The **domain** of quadratics is the real numbers.

The **range** of quadratics is $y \geq k$ when the vertex is a minimum and $y \leq k$ when the vertex is a maximum.

The **axis of symmetry** is $x = h$.

The vertical stretch is a and can be used to determine if the parabola opens **upward** (when a is positive) or **downward** (when a is negative).

Example #1: State the vertex, the axis of symmetry, the maximum or minimum value, and the domain and the range for $f(x) = 4(x-2)^2 - 3$.

Compare the quadratic function with the general equation for vertex form and identify *a*, *h*, and *k*.

$$f(x) = a(x-h)^2 + k$$

$$f(x) = 4(x-2)^2 - 3$$

$$f(x) = 4(x-2)^2 + -3 \qquad \text{(–3 is the same as + –3)}$$

$a = 4$, $h = 2$, $k = -3$

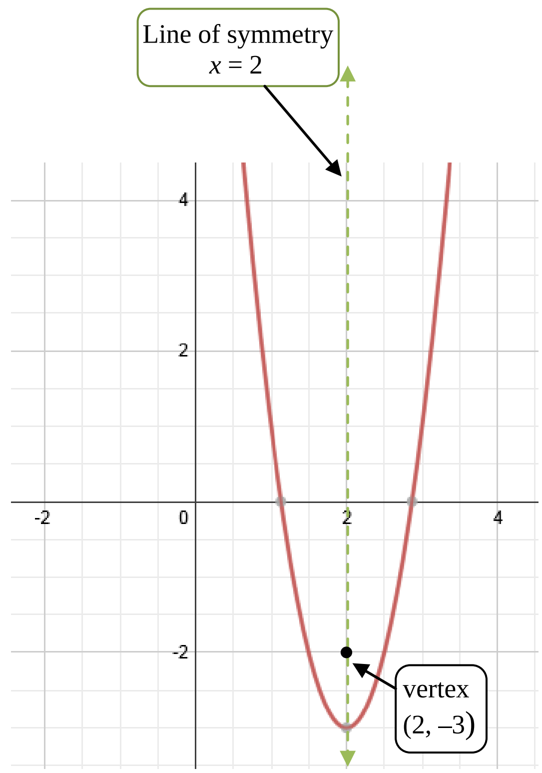
The **vertex** is $(h, k) = (2, -3)$.

The **axis of symmetry** is $x = h$ or $x = 2$.

Since $a > 0$, the parabola opens upward, and therefore the function has a minimum value. The **minimum value** of $f(x)$ is *k*, so the minimum value is –3.

Domain: All real numbers. (The domain for all quadratic functions is the real numbers.)

Range: $y \geq -3$ since the minimum value of the function is –3. There is *no* value in the set of points for the parabola where the *y*-value is less than –3.



Example #2: State the vertex, the axis of symmetry, the maximum or minimum value, and the domain and the range for $f(x) = -2(x+4)^2$.

Compare the quadratic function with the general equation for vertex form and identify *a*, *h*, and *k*.

$$f(x) = a(x-h)^2 + k$$

$$f(x) = -2(x+4)^2$$

$$f(x) = -2(x - -4)^2 + 0 \qquad \text{(+4 is the same as } - -4,\ k = 0)$$

$a = -2$, $h = -4$, $k = 0$

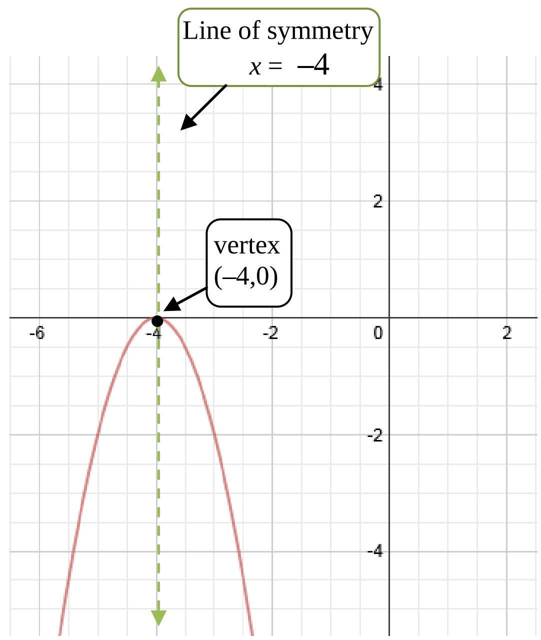
The **vertex** is $(h, k) = (-4, 0)$.

The **axis of symmetry** is $x = h$ or $x = -4$.

Since $a < 0$, the parabola opens downward, and therefore the function has a maximum value. The **maximum value** of $f(x)$ is k, so the maximum value is 0.

Domain: All real numbers. (The domain for all quadratic functions is the real numbers.)

Range: $y \le 0$ since the maximum value of the function is 0. There is *no* value in the set of points for the parabola where the *y*-value is greater than 0.



QuickTime Finding the Maximum Value of a Quadratic Function (02:12)

Let's practice identifying parts of a graph of a quadratic function when the function is expressed in vertex form.

What is the vertex of the given quadratic function?

$$y = 2(x-3)^2 + 4$$

Click here" to check your answer.

The vertex is (3, 4).

Is the vertex of the function a maximum or minimum point? Explain why.

$$y = 2(x-3)^2 + 4$$

Click here" to check your answer.

The vertex is a minimum point because *a* (2) is positive.

What is the vertex of the given quadratic function?

$$y = -4(x+5)^2 + 2$$

Click here" to check your answer.

The vertex is (–5, 2).

Is the vertex of the function a maximum or minimum point? Explain why.

$$y = -4(x+5)^2 + 2$$

Click here" to check your answer.

The vertex is a maximum point because *a* (–4) is negative.

What is the axis of symmetry of the graph for the given quadratic function?

$$y = -(x+1)^2 + 2$$

Click here" to check your answer.

The axis of symmetry is $x = -1$.

What is the axis of symmetry of the graph for the given quadratic function?

$$y = -5(x-4)^2 - 2$$

Click here" to check your answer.

The axis of symmetry is $x = 4$.

Example #3: Graph $f(x) = -(x+2)^2 + 3$.

Identify the constants for this graph:

$$f(x) = a(x-h)^2 + k$$

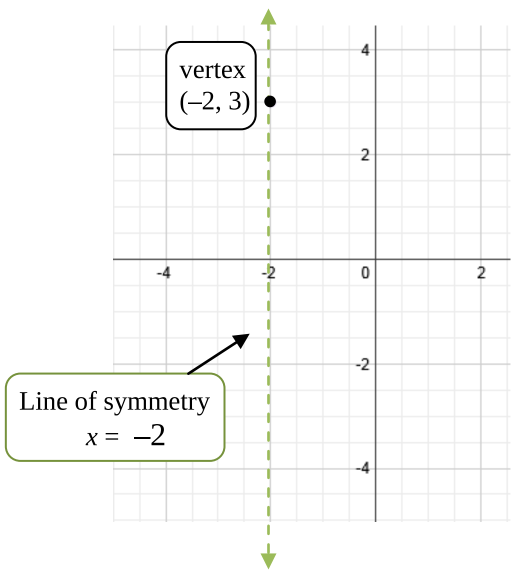
$$f(x) = -(x+2)^2 + 3$$

$$f(x) = -1(x - -2)^2 + 3$$ (a is understood to be –1, +2 = – –2)

a = –1, h = –2, k = 3

Since a < –1, the parabola opens downward.

Plot the vertex (h, k) = (–2, 3) and draw the axis of symmetry x = –2.



Plot two points: Let x = –1 since it is near the line of symmetry.

$$\begin{aligned} f(x) &= -(x+2)^2 + 3 \\ f(-1) &= -(-1+2)^2 + 3 \\ &= -(1)^2 + 3 \\ &= -1 + 3 \\ &= 2 \end{aligned}$$

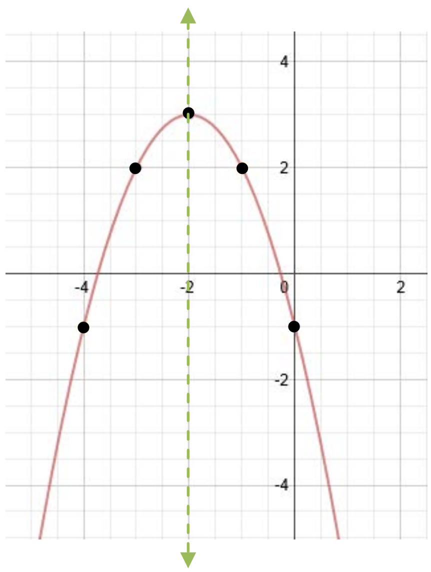
Plot (–1, 2) and the symmetric point (–3, 2).

Plot two additional points:

$$f(x) = -(x+2)^2 + 3$$
$$f(0) = -(0+2)^2 + 3$$
$$= -(2)^2 + 3$$
$$= -4 + 3$$
$$= -1$$

Plot (0, –1,) and the symmetric point (–4, –1).

Sketch the curve.



Locating the vertex first proved very useful in deciding which additional points to graph. The line of symmetry helped in determining the symmetrical points which occur in parabolas.

Stop! **Go to Questions #17-22 about this section, then return to continue on to the next section.**

Write Quadratic Functions in Vertex Form

To write a quadratic function in vertex form, complete the square first, using the quadratic and linear terms only, if the coefficient of the quadratic term is 1.

The **vertex form** of a quadratic function is:

$$f(x) = a(x-h)^2 + k$$

Example #1: Write the given quadratic function in vertex form, and then state the coordinates of the function's vertex and the axis of symmetry.

$g(x) = x^2 + 6x + 5$ — Notice that $g(x) = x^2 + 6x + 5$ is not a perfect square.

$g(x) = (x^2 + 6x \underline{+9}) + 5 \underline{-9}$ — Complete the square by taking half of the linear term (6) which is 3, and square it to get 9. **Balance this addition by subtracting 9.**

$g(x) = (x+3)^2 - 4$ — Factor x^2 +6x + 9 into (x + 3)^2 and combine +5 – 9 = –4.

The vertex form of the given quadratic functions is $g(x) = (x+3)^2 - 4$.

$$f(x) = a(x-h)^2 + k$$
$$g(x) = (x - -3)^2 + -4 \qquad (+3 = --3, \quad -4 = +-4)$$

The vertex of this quadratic function is located at (–3, –4) and the axis of symmetry is x = –3.

*If the leading coefficient (a) is not one, factor the coefficient (a) out of the quadratic and linear terms only and adjust to keep the equation in balance.

Example #2: Write the given quadratic function in vertex form, and then state the coordinates of the function's vertex and the axis of symmetry.

$f(x) = 2x^2 + 12x + 13$

$f(x) = 2(x^2 + 6x) + 13$ — Group the quadratic and linear terms only, $2x^2 + 12x$, and factor, dividing by 2. Factor a 2 out of the quadratic and linear terms.

$f(x) = 2(x^2 + 6x \underline{+9}) + 13 \underline{-18}$ — Complete the square inside the parentheses and keep the equation in balance.

This is a little different because when you take one-half of 6 and then square it you get 9, but the parentheses are multiplied by 2, so you are really **adding 18** so to balance this addition you must **subtract 18**.

$f(x) = 2(x+3)^2 - 5$ Factor $x^2 + 6x + 9$ into $(x + 3)^2$ and combine $+13 - 18 = -5$.

The vertex form of the given quadratic functions is $f(x) = 2(x+3)^2 - 5$.

$$f(x) = a(x-h)^2 + k$$
$$g(x) = 2(x - -3)^2 + -5 \qquad (+3 = --3, \quad -5 = +-5)$$

The vertex of this quadratic function is located at (–3, –5) and the axis of symmetry is $x = -3$.

*If a quadratic function is in standard form, $ax^2 + bx + c = y$, then it is possible to locate the axis of symmetry by using the following formula:

Axis of Symmetry: $x = \dfrac{-b}{2a}$

Example #3: Find the axis of symmetry for the given quadratic function.

$f(x) = 2x^2 + 8x + 19$ $a = 2$, $b = 8$, $c = 19$

The axis of symmetry is $x = \dfrac{-b}{2a} = \dfrac{-8}{2(2)} = \dfrac{-8}{4} = -2.$

The axis of symmetry is $x = -2$.

The axis of symmetry also refers to the x-value of the vertex.

To find the y-value of the vertex:

1.) Replace the value of x into the equation
2.) Solve for y

Example #4: Find the vertex of the parabola for the quadratic function, $y = 2x - 2 + x^2$.

$y = 2x - 2 + x^2$

$y = x^2 + 2x - 2$ Put in standard form.

$a = 1, \quad b = 2, \quad c = -2$ Identify a, b, and c.

$x = \frac{-b}{2a} = \frac{-2}{2(1)} = \frac{-2}{2} = -1$ Find the axis of symmetry.

The axis of symmetry is $x = -1$

$$y = (-1)^2 + 2(-1) - 2$$
$$y = 1 - 2 - 2$$
$$y = -3$$

Replace all x values with –1 and solve for y.

Therefore, the vertex of this parabola is located at (–1, –3).

Stop! **Go to Questions #23-30 to complete this unit.**