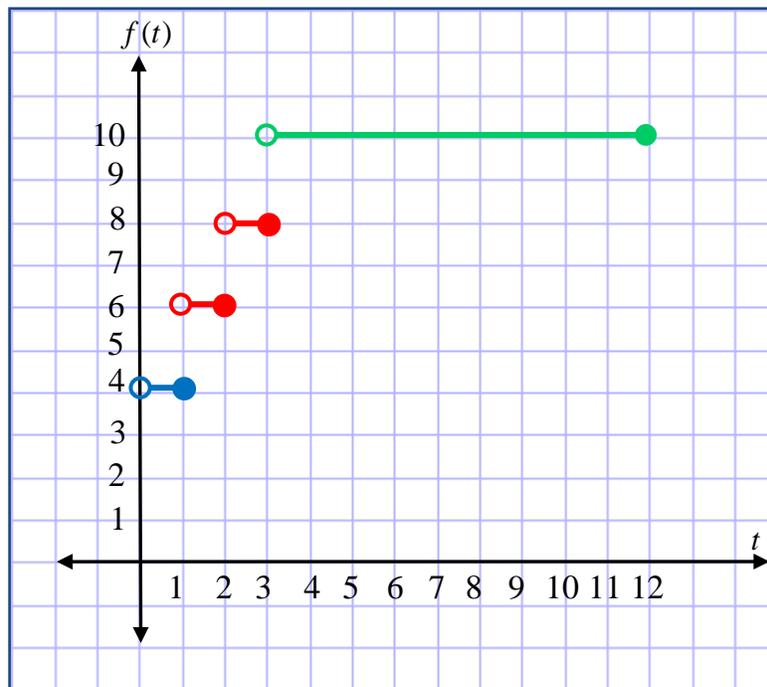# SPECIAL FUNCTIONS AND TRANSFORMATIONS

Unit Overview

Some real-world relationships can be modeled with special functions. For example, the dosage of medicine given to a child can be modeled with a step-function. This unit will look at "special" functions – piecewise, step and absolute-value functions. The unit concludes with an overview of basic transformations of the graphs of functions.

Special Functions

Piecewise Functions

piecewise function: consists of different function rules for different parts of the domain.

To graph:
1.) make a table of values for each domain
2.) substitute values into each given equation
3.) plot points
4.) decide whether to use open points ($<$ or $>$) or closed points ($\leq$, $\geq$); and whether or not to use a line segment or a continuous ray, and then connect the points.

Example #1: $f(x) = 3x$ if $0 < x \leq 2$

$\qquad\qquad\quad f(x) = 6$ if $2 < x \leq 4$

$\qquad\qquad\quad f(x) = -x + 10$ if $4 < x \leq 6$

$f(x) = 3x$	$f(x) = 6$	$f(x) = -x + 10$
x values are greater than 0 (open circle) and less than or equal to 2 (closed)	x values are greater than 2 (open circle) and less than or equal to 4 (closed)	x values are greater than 4 (open circle) and less than or equal to 6 (closed)

x	y		x	y		x	y	
0	0	(open)	2	6		4	6	
1	3		3	6		5	5	
2	6	(closed)	4	6	(closed)	6	4	(closed)

After establishing the domain in each of the functions, find the *y* values by replacing *x* with each of the domain and solve for *y*.

Graph each function separately.
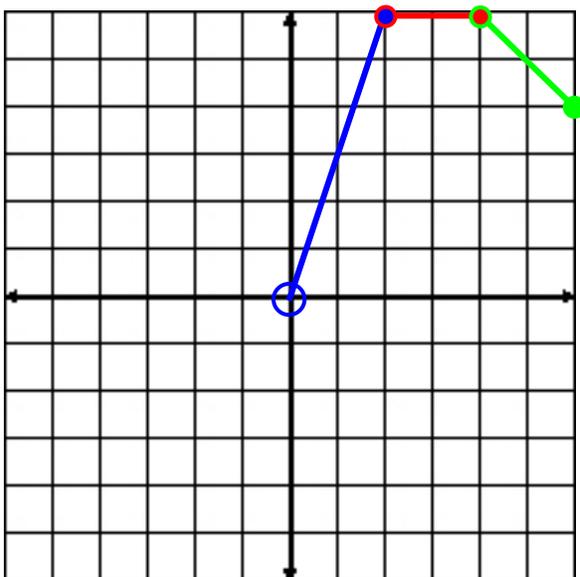
Step Functions

A piecewise function where the graph looks like a set of steps is called a step function.

A **step function** pairs every number in an interval with a single value.

*Recall that that the **set of integers** are: {... –3, –2, –1, 0, 1, 2, 3 ...}.

The **greatest-integer function** $f(x) = [x]$ is a step function where $[x]$ represents the *greatest* integer less than or equal to x. This does not mean to round on truncate the number. For example, $[7.8] = 7$ because 7 is the greatest integer less than 7.8. Similarly, $[-2.9] = -3$ because –3 is the greatest integer less than –2.9.

*Note the answer to $[-2.9] \neq -2$ since –2 is not less than –2.9. Also, $[4] = 4$, since 4 is the integer equal to 4.
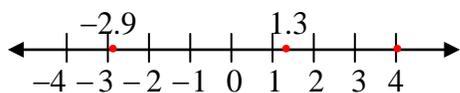
$$[-2.9] = -3 \qquad [1.3] = 1 \quad [4] = 4$$

$$-3 < -2 \qquad\qquad 1 < 2$$

The greatest integer value will be the first integer to the **left of the given decimal value on the number line.

What is the value of [12.6]?

"Click here" to check your answer.

12 because 12 is the greatest integer less than 12.7

What is the value of [0.5]?

"Click here" to check your answer.

0 because 0 is the greatest integer less than 0.5

What is the value of [– 0.3]?

"Click here" to check your answer.

–1 because –1 is the greatest integer less than –0.3

*Note: The first integer to the left of –0.3 on the number line is –1.

What is the value of [– 4.2]?

"Click here" to check your answer.

–5 because –5 is the greatest integer less than –4.2

*Note: The first integer to the left of –4.2 on the number line is –5.

*Example #2*: Graph $f(x) = [x]$ ***notice the notation of the brackets [ ]**

To graph
1.) choose values for *x*, make sure to chose rational numbers
2.) all rational numbers will map back to the smaller integer
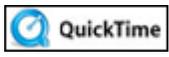
$f(x) = [x]$

All values from 0.1 to 0.9
will map back to 0.

When you put in the next
integer, the rational values
will map back to that integer
until you choose the next
integer.

x	y
0	0
0.1	0
0.2	0
0.9	0
1	1
1.1	1
1.2	1
1.9	1
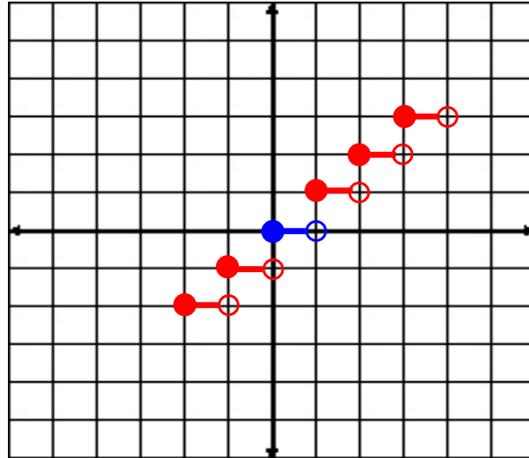
Each piece of the graph is a horizontal line segment that is missing the right endpoint. The open circle shows that the right endpoint is not part of the graph.

QuickTime Step Functions -- Grocery Store (02:39)

Let's take a look at how a step function can be applied.

Example #3: A downtown parking garage charges $4 for the first hour. The charge is $2 for each additional hour *or part of an hour*. The maximum charge to park in the garage is $10. The maximum time to leave the car in the garage is 12 hours.

(a) Write a function and graph the step function that shows the fee (f) in terms of the number of hours (t).

A table can be useful to organize the given information.

	Fee	Number of Hours
First hour	$f(t) = 4$	if $0 < t \leq 1$
Between 1 and 3 hours*	$f(t) = 2[t] + 4$	if $1 < t \leq 3$
More than 3 hours	$f(t) = 10$	if $3 < t \leq 12$

*During the hours between 1 and 3, the fee is based on usage of an hour or part of an hour; thus, we use the integer function.

The graph is a step function because each hour is paired in intervals with a single fee.

t	$f(t)$
1	4
$1 < t \leq 2$	6
$2 < t \leq 3$	8
$3 < t \leq 12$	10
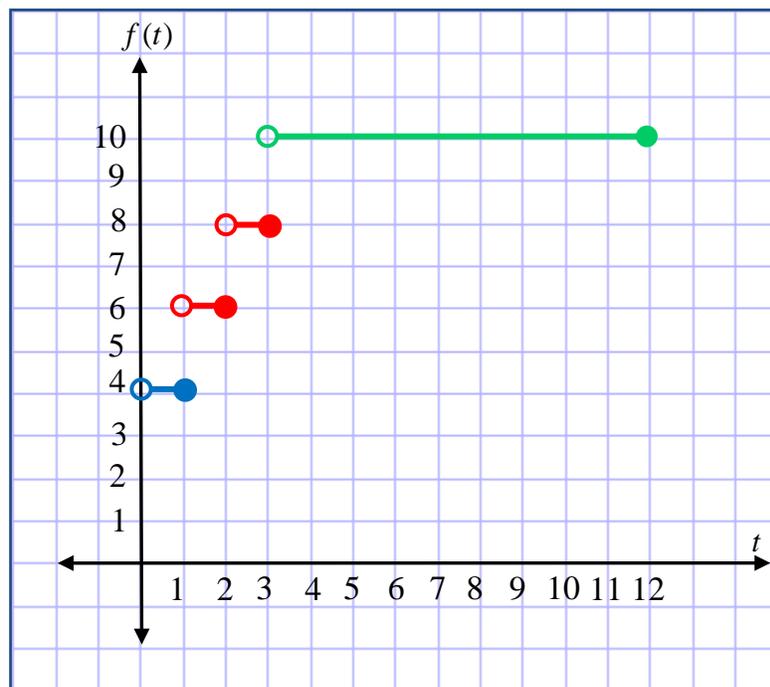
*Notice on this graph the open circle is on the left of the line segment and the closed circle is on the right.

(b) Use the graph to find the cost of parking in the garage for 2 1/2 hours.

What is the cost of parking for 2 1/2 hours?

"Click here" to check your answer.

The cost for 2 1/2 hours is $8.

(c) State the domain and the range of the function.

What is the domain of the function?

"Click here" to check your answer.

The domain is $0 < t \leq 12$ hours

What is the range of the function?

"Click here" to check your answer.

The range is {4, 6, 8, 10}.

The domain is $0 < t \leq 12$ since the limit for parking in the garage is no more than 12 hours.

The range is {4, 6, 8, 10} as these are the only different fees that can be charged.

The graphs of step functions are often used to model real-world problems such as fees for cell phones, postage rates and taxes.
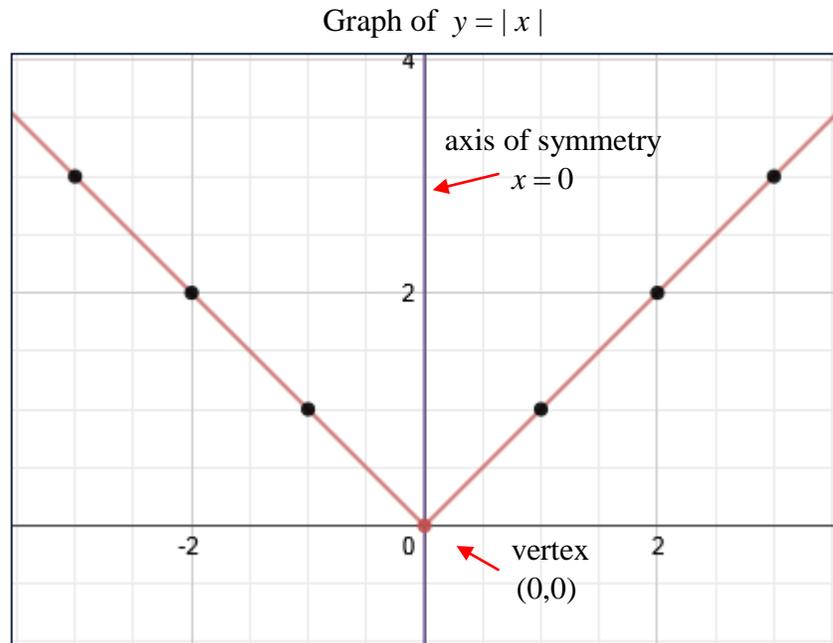
Absolute Value Functions

The **absolute value function** is another piecewise function.

The simplest example of an absolute value function is $f(x) = |x|$.

   *Recall that absolute value is the positive value of the number or expression within the bars.

To graph the absolute value function, make a table of values, and then graph.

Graph of $y = |x|$

x	$y = |x|$
–3	3
–2	2
–1	1
0	0
1	1
2	2
3	3

axis of symmetry
$x = 0$

vertex
(0,0)

The graph of an absolute value function is V-shaped and symmetric about a vertical line called the axis of symmetry. The graph has either a maximum point (highest) or a minimum (lowest) point called the vertex.

For $f(x) = |x|$ the vertex is the point (0, 0). It is a minimum point since it is the lowest point on the graph. The axis of symmetry is the line $x = 0$, the y-axis. The axis of symmetry divides the graph into two mirror images.

The domain of $f(x) = |x|$ is all real numbers.

The range of $f(x) = |x|$ is $y \geq 0$ (in other words, zero and all positive numbers).

Stop! **Go to Questions #1-5 about this section, then return to continue on to the next section.**

Transformations of Absolute Value Functions

A **transformation** of a function is an alteration of the function that results in an alteration of its graph. The original graph is referred to as the "parent function".
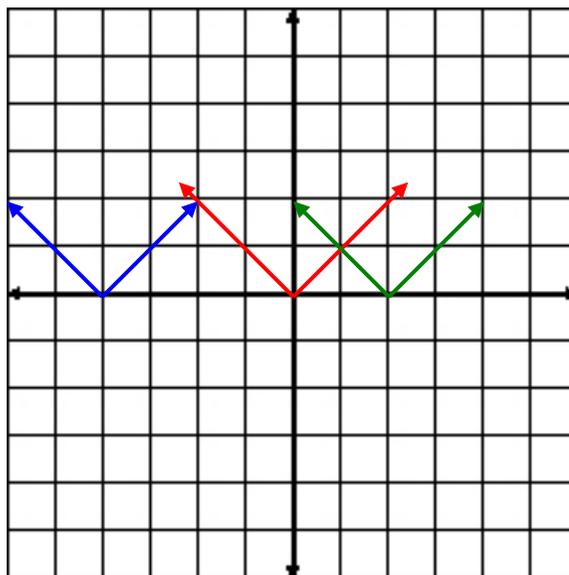
Horizontal Transformations

Parent function → $f(x) = |x|$

$f(x) = |x+4|$ moved 4 units left

$f(x) = |x-2|$ moved 2 units right

*The constant is contained with the variable; therefore, it transforms the function horizontally in the **opposite** direction of the sign.

How will $f(x) = |x - 7|$ be transformed from the parent function $f(x) = |x|$?

"Click here" to check your answer.

The function will be moved 7 units to the right.

How will $f(x) = |x + 5|$ be transformed from the parent function $f(x) = |x|$?

"Click here" to check your answer.

The function will be moved 5 units to the left.

*Note: In both previous questions, the constant is contained with the variable; thus, each one transforms horizontally in the **opposite** direction of the sign
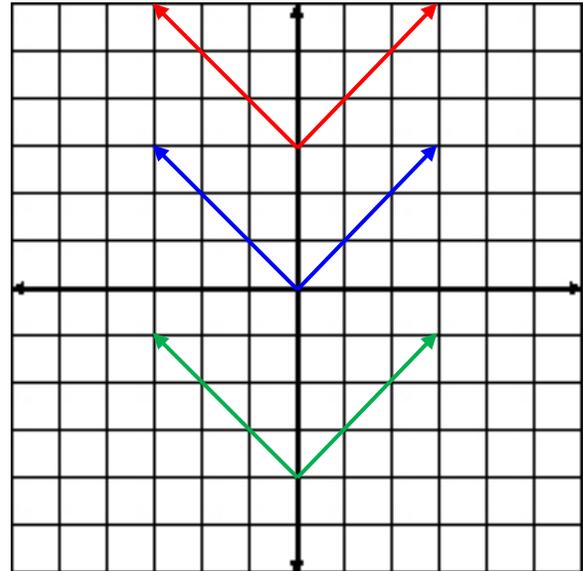
Vertical Transformations

Parent function $\rightarrow$ $f(x) = |x|$

$f(x) = |x| + 3$ the graph is moved up 3 units

$f(x) = |x| - 4$ the graph is moved 4 units down

*The graphs are moved up or down as indicated by the constant in the function.

How will $f(x) = |x| + 2$ be transformed from the parent function $f(x) = |x|$?

"Click here" to check your answer.

The function will be moved up 2 units.

How will $f(x) = |x| - 10$ be transformed from the parent function $f(x) = |x|$?

"Click here" to check your answer.

The function will be moved down 10 units.
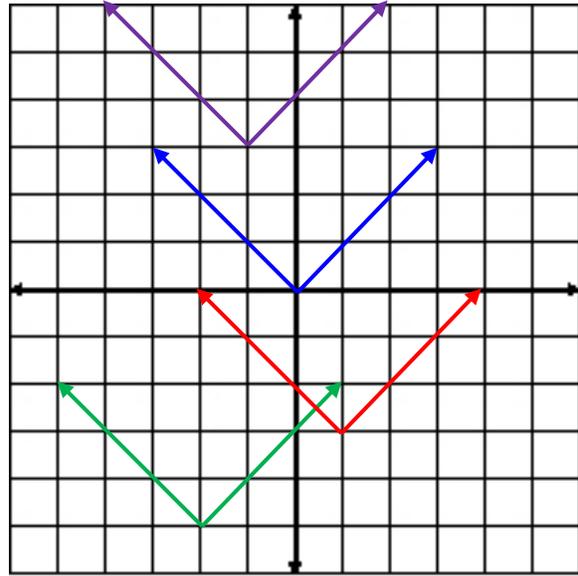
Vertical and Horizontal Transformations

Parent function → $f(x) = |x|$

$f(x) = |x-1| - 3$ the graph is moved right 1 unit and up 3 units

$f(x) = |x+2| - 5$ the graph is moved left 2 units and 5 units down

$f(x) = |x+1| + 3$ the graph is moved left 1 unit and up 3 units

*The graphs are moved horizontally based on the number contained with the variable and vertically based on the constant that is added or subtracted.

How will $f(x) = |x - 7| + 12$ be transformed from the parent function $f(x) = |x|$?

"Click here" to check your answer.

The function is translated 7 units to the right and 12 units up.

What is the axis of symmetry of $f(x) = |x - 7| + 12$?

"Click here" to check your answer.

$x = 7$

What is the vertex of $f(x) = |x - 7| + 12$?

"Click here" to check your answer.

vertex = (7, 12)

*The axis of symmetry ($x = 7$) is the vertical line that divides the graph into two mirror images. The vertex (7, 12) is the lowest point of the graph.

Example #1: For the function, $f(x) = |x - 4| + 3$ describe the transformation in terms of the parent graph $f(x) = |x|$. State the vertex and the axis of symmetry.

axis of symmetry
$x = 4$

$f(x) = |x - 4| + 3$

Solution: The graph is translated 4 units to the right and 3 units up. The vertex is (4, 3) and the axis of symmetry is the line $x = 4$. The axis of symmetry is always a vertical line that passes through the vertex.

vertex
$(4, 3)$
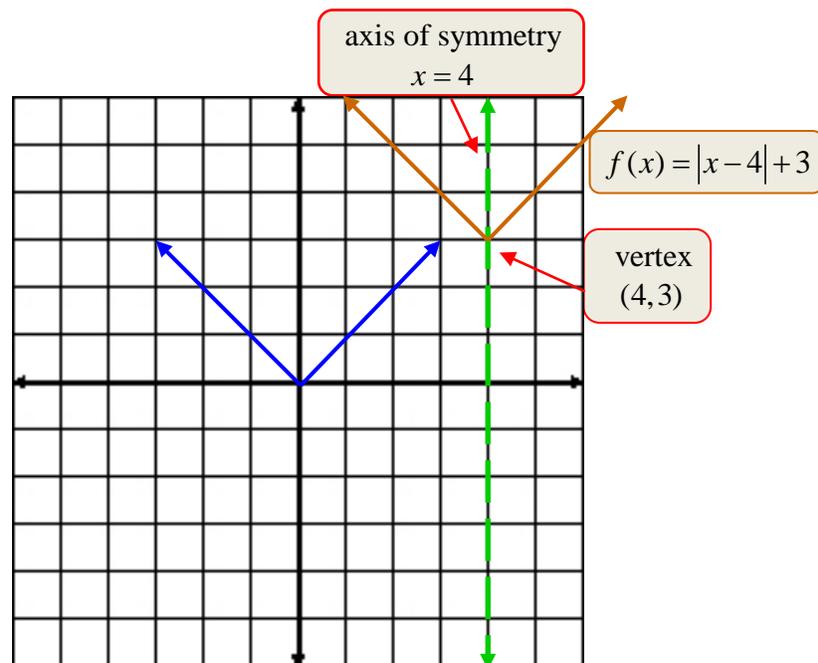
Vertical Stretch and Compression

In the graph of $f(x) = |x|$, the right branch has a slope of $+1$ and the left branch has a slope of -1.

The graph of $f(x) = a|x|$ is a stretch or compression of the graph of $f(x) = |x|$. The right branch has a slope of a and the left branch has a slope of $-a$.

Parent function → $f(x) = |x|$

$f(x) = 5|x|$ The graph is narrower. The graph is a vertical stretch of the parent graph by a factor of 5. The right branch has a slope of $+5$ and the left branch has a slope of -5.

$f(x) = \dfrac{1}{4}|x|$ The graph is wider. The graph is a vertical compression of the parent graph by a factor of 1/4. The right branch has a slope of $+1/4$ and the left branch has a slope of $-1/4$.
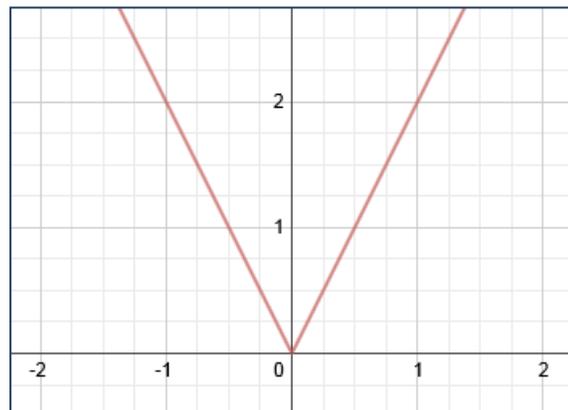
$y = 5|x|$

$y = |x|$

$y = \dfrac{1}{4}|x|$
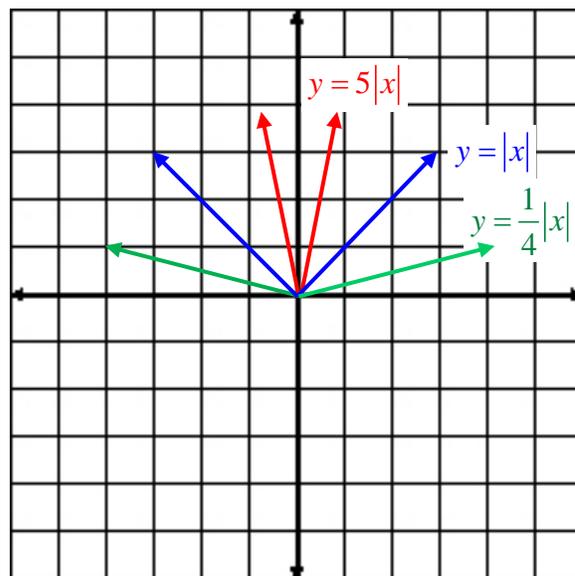
Example #2: Describe the graph of $f(x) = 2|x|$ as it relates to the parent function $f(x) = |x|$.

Solution:

The graph is a vertical stretch of the graph of $f(x) = |x|$ by a factor of 2.

The vertex is $(0, 0)$. The vertex remains at $(0, 0)$ since there is no horizontal or vertical translation. The vertex is a minimum (lowest) point of the graph.

To graph the right branch, start at $(0, 0)$ and graph the line $y = 2x$. To graph the left branch start at $(0, 0)$ and graph the line $y = -2x$. The axis of symmetry is $x = 0$.
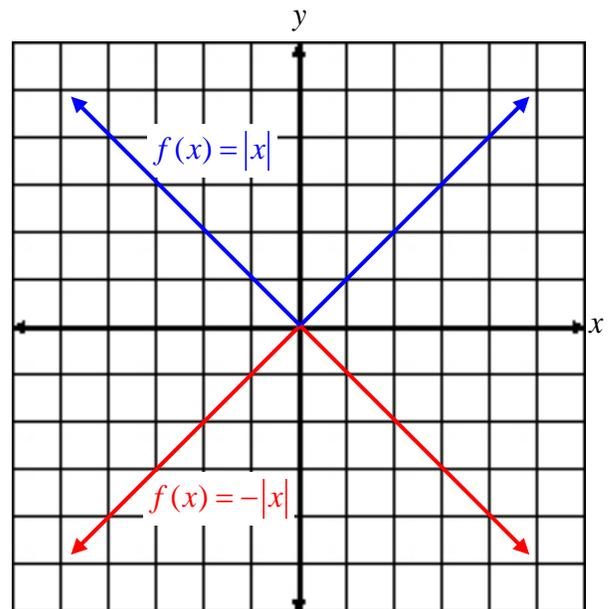
Reflection

$f(x) = |x|$

$f(x) = -|x|$

If $y = f(x)$, then $y = -f(x)$ gives a reflection of the graph across the *x*-axis.

*Note: One graph appears to be a mirror image of the other graph.

Example #3: Describe the graph of $f(x) = -\dfrac{2}{3}|x|$ as it relates to the parent function $f(x) = |x|$.

Solution:

The graph is a vertical compression of the graph of $f(x) = |x|$ by a factor of 2/3.

The vertex is (0, 0). The vertex remains at (0, 0) since there is no horizontal or vertical translation.

The negative sign in front of the fraction causes the graph to reflect over the *x*-axis.
Thus, the vertex is a maximum (highest) point of the graph.
To graph the right branch, start at (0, 0) and graph the line $y = -\dfrac{2}{3}x$. To graph the left branch start at

(0, 0) and graph the line $y = \dfrac{2}{3}x$. The axis of symmetry is $x = 0$.
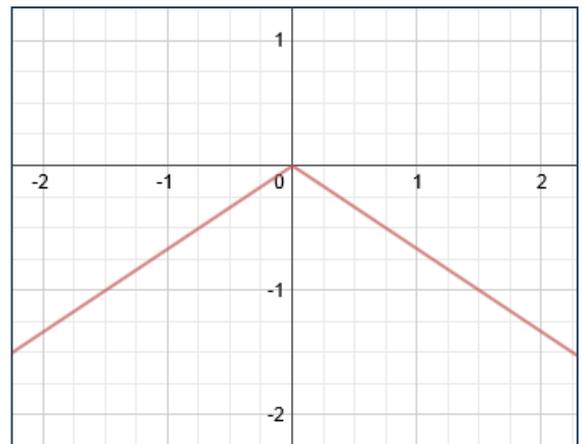
Summary of Absolute Value Transformations

The **general form** of an absolute value function is:

$$f(x) = a|x-h| + k \quad \text{where:}$$

a is the stretch or compression

h is the horizontal translation

k is the vertical translation

The transformations that occur on the parent function $f(x) = |x|$ are:

(1) Horizontal Translation

$f(x) = |x+h|$ translates the graph to the left h units

$f(x) = |x-h|$ translates the graph to the right h units

*Note: Since the minus sign is part of the general form, when the equation states $|x+h|$ this is actually $|x-(-h)|$, thus the graph translates to the left.

(2) Vertical Translation

$f(x) = |x| + k$ translates the graph up k units

$f(x) = |x| - k$ translates the graph down k units

(3) Vertical Stretch and Compression

$f(x) = a|x|$ results in a vertical stretch by a factor of a, if $a > 0$

$f(x) = a|x|$ results in a vertical compression by a factor of a, if $0 < a < 1$ (in other words, if a is a fraction between 0 and 1).

(4) Reflection

$f(x) = -|x|$ reflects the graph over the x-axis

*Note: $f(x) = |-x|$ reflects the graph over the y-axis

**Note the graph of $f(x) = |2x|$, $f(x) = |-2x|$, and the graph of $f(x) = 2|x|$ are identical.

***The graphs of $f(x) = |-3x|$ and $f(x) = -|3x|$ are not identical; one is the reflection across the x-axis of the other.

Stop! **Go to Questions #6-30 to complete this unit.**