OPERATIONS WITH NUMBERS AND EXPONENTS

Unit Overview

This unit begins with a review of real numbers, their properties, and the order of operations. In addition, the various properties of integer exponents are reviewed and extended to include rational exponents. Using the property of exponents and rational exponents, expressions in radical form will be rewritten in exponential form and vice versa.

Operations with Numbers

Types of Numbers

natural numbers: 1, 2, 3, ….

whole numbers: 0, 1, 2, 3, …

integers: …–2, –1, 0, 1, 2, …

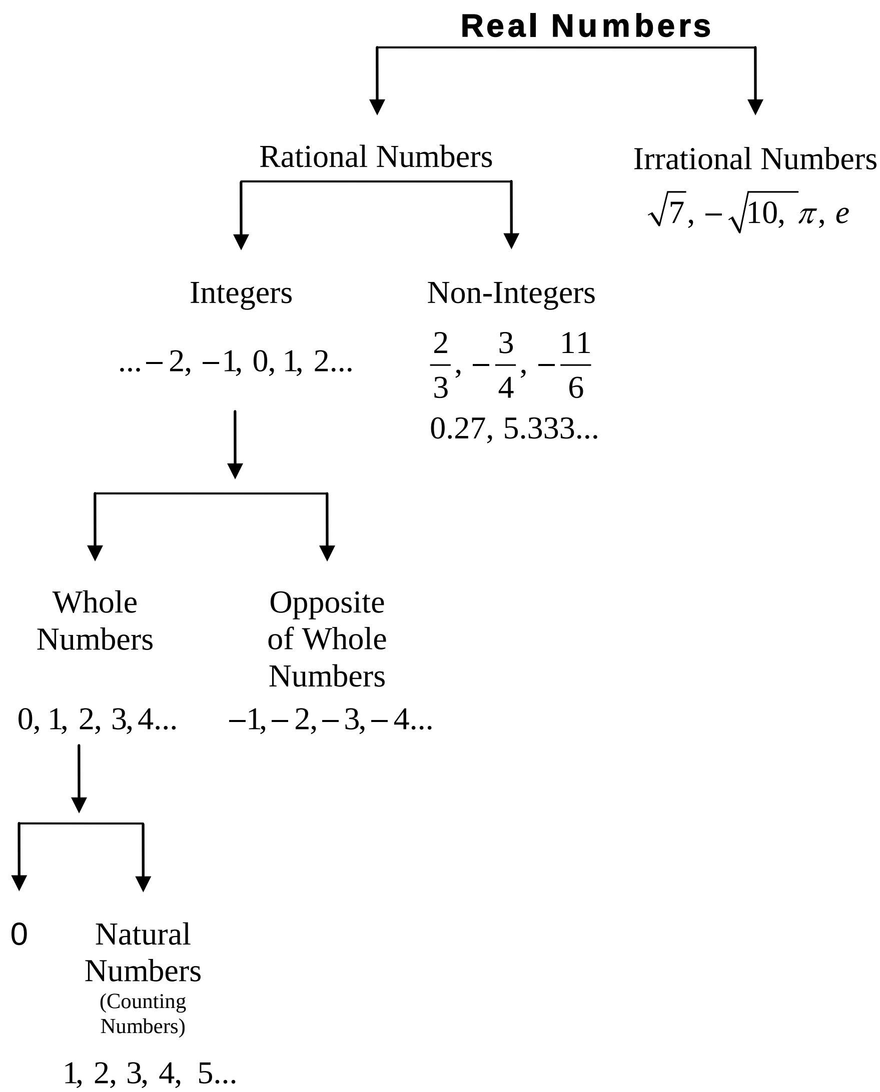
rational numbers: $\frac{p}{q}$ where p and q are integers and $q \neq 0$

QuickTime Rational Numbers -- Recipes (02:54)

irrational numbers: numbers whose decimal part does not terminate or repeat

real numbers: all rational and all irrational numbers

QuickTime Irrational Numbers -- Travel (02:22)


Real Numbers
Rational Numbers
Irrational Numbers
$\sqrt{7}, -\sqrt{10}, \pi, e$
Integers
$\ldots -2, -1, 0, 1, 2\ldots$
Non-Integers
$\frac{2}{3}, -\frac{3}{4}, -\frac{11}{6}$
$0.27, 5.333\ldots$
Whole Numbers
$0, 1, 2, 3, 4\ldots$
Opposite of Whole Numbers
$-1, -2, -3, -4\ldots$
0
Natural Numbers
(Counting Numbers)
$1, 2, 3, 4, 5\ldots$

Properties of Real Numbers

Property	Addition	Multiplication
Closure	$a + b$ is a real number	ab is a real number
Commutative	$a + b = b + a$	$ab = ba$
Associative	$a + (b + c) = (a + b) + c$	$(ab)c = a(bc)$
Identity	$a + 0 = a$	$a \cdot 1 = a$
Inverse	$a + (-a) = 0$	$a\left(\frac{1}{a}\right) = 1$
Distributive	$a(b + c) = ab + ac$	

Order of Operations

To simplify algebraic expressions you must use an order of operations.

Parentheses, **E**xponents, **M**ultiply, **D**ivide, **A**dd, **S**ubtract

"**P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally"

*If multiplication and division are the only two operations, work the problem from left to right

*If addition and subtraction are the only two operations, work the problem from left to right.

QuickTime Introduction (02:08)

Example #1: Evaluate.

$$6 \div 3 \cdot 2$$
$$2 \cdot 2$$
$$4$$

QuickTime Simple Orders--Roller Coaster Capacity (02:33)

Example #2: Evaluate.

$$30 - 3 \cdot 2 + 6 \div 3$$
$$30 - 6 + 2$$
$$24 + 2$$
$$26$$

More Orders--Revenue (03:06)

Example #3: Evaluate.

$7(12-(8-3)\times 6^2)$	-parenthesis
$7(12-5\times 6^2)$	-exponents
$7(12-5\times 36)$	-multiplication with parenthesis
$7(12-180)$	-subtract within parenthesis
$7(-168)$	-multiply
-1176	

Exponents--Around the Loop (03:02)

Stop! **Go to Questions #1-4 about this section, then return to continue on to the next section.**

Properties of Exponents

Product property: $a^m \cdot a^n = a^{m+n}$ (add exponents)

Quotient property: $\dfrac{a^m}{a^n} = a^{m-n}$ (subtract exponents)

Power of a power: $(a^m)^n = a^{mn}$ (multiply exponents)

Power of a product: $(ab)^n = a^n b^n$ (exponents to each term)

Power of a quotient: $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ (exponents to each term)

$* a^0 = 1$ **any number to the zero power is equal to one**

$a^{-1} = \dfrac{1}{a}$ all negative exponents have to be changed to positive exponents to make an expression completely simplified.

Example #1: Simplify.

$c^{-9}c^{14} = c^5$ -add exponents

QuickTime Multiplying with Like Bases (01:19)

Example #2: Simplify.

$(d^{-4})^6$ -multiply exponents

$d^{-24} = \dfrac{1}{d^{24}}$

Example #3: Simplify.

$\dfrac{t^{11}}{t^5} = t^6$ -subtract exponents

Example #4: Simplify.

$$\frac{z^6}{z^{-2}} = z^{6-(-2)} = z^8$$ -subtract exponents

QuickTime Dividing with Like Bases (01:28)

Example #5: Simplify.

$$(3x^2y^3)(-2x^3y) = -6x^5y^4$$ -add exponents on like bases

QuickTime Multiplying Expressions with Like Bases (01:53)

Example #6: Simplify.

$$\frac{-3b^2c^5}{b^7c^2} = \frac{-3c^3}{b^5}$$ -subtract exponents on like bases

QuickTime Dividing Expressions with Like Bases (01:56)

Example #7: Simplify.

$$(-2a^5b^{-3})^2 = 4a^{10}b^{-6} = \frac{4a^{10}}{b^6}$$ -multiply exponents, negative exponent becomes positive if put in the denominator

QuickTime Raising a Power to a Power (02:01)

Example #8: Simplify.

$$\left(\frac{2x^4y^2}{3x^2y^7}\right)^{-3}$$ -raise each term to a power of -3 (power of a quotient)

$$\frac{(2x^4y^2)^{-3}}{(3x^2y^7)^{-3}}$$ -raise each number and variable within parenthesis to a power of -3 (power of a product)

$$\frac{2^{-3}(x^4)^{-3}(y^2)^{-3}}{3^{-3}(x^2)^{-3}(y^7)^{-3}}$$ -multiply exponents (power of a power)

$$\frac{2^{-3}(x^{-12})(y^{-6})}{3^{-3}(x^{-6})(y^{-21})}$$ -subtract exponents

$$\frac{2^{-3}(x^{-6})(y^{15})}{3^{-3}}$$ -write all negative exponents as positive exponents

$$\frac{3^3(y^{15})}{2^3(x^6)}$$ -simplify

$$\frac{27y^{15}}{8x^6}$$

QuickTime Raising a Power to a Power in Rational Expressions (02:54)

Example #9: Simplify. (Another way to do the previous problem)

$\left(\frac{2x^4y^2}{3x^2y^7}\right)^{-3}$ -invert the fraction and raise to a positive power

(Note: A term raised to a negative power equals the term's reciprocal.)

$\left(\frac{3x^2y^7}{2x^4y^2}\right)^{3}$ -raise each term to a power of 3

$\frac{(3x^2y^7)^3}{(2x^4y^2)^3}$ -multipy exponents

$\frac{27x^6y^{21}}{8x^{12}y^6}$ -subtract exponents

$\frac{27y^{15}}{8x^6}$

Stop! **Go to Questions #5-12 about this section, then return to continue on to the next section.**

Rational Exponents

Rational exponents are exponents that are fractions.

Rational exponents are an alternate way to express roots and can be very useful when dealing with more complicated expressions.

First, let's review the terms associated with radicals.

Base	**Index**	**Exponent**
$\sqrt[n]{a^m}$	$\sqrt[n]{a^m}$	$\sqrt[n]{a^m}$
The base is denoted as a.	The index is denoted as n.	The exponent of the base is denoted as m.

Now, let's take a look at what a rational exponent is.

$a^{\frac{1}{n}} = \sqrt[n]{a}$ *The denominator of the rational exponent is the root.

$\sqrt{36} = 36^{\frac{1}{2}}$ $\sqrt[3]{64} = 64^{\frac{1}{3}}$ $\sqrt[4]{16} = 16^{\frac{1}{4}}$

*When the index is 2 as in $\sqrt{36}$, it is understood and not written.

($\sqrt{36}$ means $\sqrt[2]{36} = 36^{\frac{1}{2}}$)

$a^{\frac{m}{n}} = \sqrt[n]{a^m}$ *The numerator of the rational exponent is the exponent of the base.

Example #1: Write $125^{\frac{1}{3}}$ as a radical expression, and then simplify:

$$125^{\frac{1}{3}} = \sqrt[3]{125^1} = \sqrt[3]{125} = 5$$

Example #2: Write $8^{\frac{2}{3}}$ as a radical expression, and then simplify.

$$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

Another way to solve this problem is:

$$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{8 \cdot 8}$$
$$= \sqrt[3]{8} \cdot \sqrt[3]{8} \qquad\qquad *\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$
$$= 2 \cdot 2 = 4$$

Example #3: Write $x^{\frac{1}{5}}$ in radical form.

$$x^{\frac{1}{5}} = \sqrt[5]{x^1} = \sqrt[5]{x}$$

Practice: Write $x^{\frac{3}{7}}$ in radical form, and then answer the following questions.

What is the index of the radical?

"Click here" to check your answer.

The index is 7.

What is the power of x under the radical?

"Click here" to check your answer.

The power of x is 3.

Solution: $x^{\frac{3}{7}} = \sqrt[7]{x^3}$

Example #4: Write $\sqrt[5]{n^4}$ in exponential form.

$$\sqrt[5]{n^4} = n^{\frac{4}{5}}$$

Practice: Write $\sqrt{5y}$ in exponential form, and then answer the following questions.

What is base of the expression?

"Click here" to check your answer.

The base is 5*y*.

What is rational exponent of the expression?

"Click here" to check your answer.

The exponent is 1/2.

Solution: $\sqrt{5y} = \sqrt{(5y)^1} = (5y)^{\frac{1}{2}}$ (The index of the radical is understood to be 2.)

Example #5: Write the expression $\left(\sqrt[3]{x}\right)^2$ in exponential form and simplify.

$$\left(\sqrt[3]{x}\right)^2 = \left(x^{\frac{1}{3}}\right)^2 \qquad *\sqrt[3]{x} = \sqrt[3]{x^1} = x^{\frac{1}{3}}$$

$$= x^{(\frac{1}{3})(2)} \qquad \text{*Power to a Power (multiply exponents)}$$

$$= x^{\frac{2}{3}}$$

Example #6: Simplify the expression: $\left(8^{\frac{1}{3}}\right)\left(8^{\frac{2}{3}}\right)$

$$\left(8^{\frac{1}{3}}\right)\left(8^{\frac{2}{3}}\right) = 8^{\frac{1}{3}+\frac{2}{3}}$$ *Product Property (add exponents)

$$= 8^{\frac{3}{3}} = 8^{1} = 8$$

Practice: Simplify $x^{\frac{1}{7}}x^{\frac{6}{7}}$, and then answer the following questions.

What do you do with the exponents?

"Click here" to check your answer.

Add the exponents.

What is the simplified expression?

"Click here" to check your answer.

Solution: $x^{1} = x$

Example #7: Simplify the expression: $\left(3b^{\frac{1}{2}}\right)\left(2b^{\frac{4}{3}}\right)$

*Notice that only the *b*'s are raised to the powers.

$$\left(3b^{\frac{1}{2}}\right)\left(2b^{\frac{4}{3}}\right) = (3)\left(b^{\frac{1}{2}}\right)(2)\left(b^{\frac{4}{3}}\right)$$

$$= (3)(2)\left(b^{\frac{1}{2}}\right)\left(b^{\frac{4}{3}}\right)$$ *Commutative Property $\left[\left(b^{\frac{1}{2}}\right)(2) = (2)\left(b^{\frac{1}{2}}\right)\right]$

$$= 6\left(b^{\frac{1}{2}+\frac{4}{3}}\right)$$ *Multiply the coefficients $(3)(2) = 6$.

$$= 6\left(b^{\frac{3}{6}+\frac{8}{6}}\right)$$ *Product Property (add exponents)

$\left[\text{LCD of 2 and 3 is 6. } \left(\frac{1}{2}=\frac{3}{6}\right), \left(\frac{4}{3}=\frac{8}{6}\right)\right]$

$$= 6b^{\frac{11}{6}}$$

Example #8: Simplify the expression: $x^{\frac{3}{5}} \div x^{\frac{1}{10}}$.

$$x^{\frac{4}{5}} \div x^{\frac{7}{10}} = x^{\frac{8}{10}-\frac{7}{10}}$$ *Quotient Property (subtract exponents)

$\left[\text{LCD of 5 and 10 is 10. } \left(\frac{4}{5}=\frac{8}{10}\right)\right]$

$$= x^{\frac{1}{10}}$$

Practice: Simplify $\frac{n^{\frac{9}{8}}}{n^{\frac{5}{8}}}$, and then answer the following questions.

What do you do with the exponents?

"Click here" to check your answer.

Subtract the exponents.

What is the simplified expression?

"Click here" to check your answer.

Solution: $\boldsymbol{n^{(4//8)} = n^{(1/2)}}$

Example #9: Simplify the expression: $\frac{4x^2}{2x^{\frac{1}{2}}}$.

$$\frac{4x^2}{2x^{\frac{1}{2}}} = \frac{(\cancel{4})^2(x^2)}{(\cancel{2})^1(x^{\frac{1}{2}})}$$

*Cancel the coefficients.

$$= 2 \cdot \frac{x^2}{x^{\frac{1}{2}}}$$

$$= 2 \cdot x^{2 - \frac{1}{2}}$$

*Quotient Property (subtract exponets)

$$= 2x^{\frac{3}{2}}$$

$*\left(2 - \frac{1}{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}\right)$

Example #10: Simplify the expression: $(n^4)^{\frac{3}{2}}$

$$(n^4)^{\frac{3}{2}} = n^{(4)\,(\frac{3}{2})}$$ *Power to a Power (multiply exponents.)

$$= n^6$$ $$*\left(4(\frac{3}{2}) = \frac{\cancel{4}^{\,2}}{1} \times \frac{3}{\cancel{2}_{\,1}} = \frac{6}{1} = 6\right)$$

Practice: Simplify $\left(b^4\right)^{\frac{3}{4}}$, and then answer the following questions.

What do you do with the exponents?

"Click here" to check your answer.

Multiply the exponents.

What is the simplified expression?

"Click here" to check your answer.

Solution: $b^{12/4} = b^3$

Example #11: Simplify the expression: $\left(x^{-\frac{3}{5}}\right)^5$

$$\left(x^{-\frac{3}{5}}\right)^5 = x^{(-\frac{3}{5})(5)}$$ *Power to a Power (multiply exponents.)

$$= x^{-3}$$ $$*\left(-\frac{3}{5}(5) = -\frac{3}{\cancel{5}_{\,1}} \times \frac{\cancel{5}^{\,1}}{1} = -\frac{3}{1} = -3\right)$$

$$= \frac{1}{x^3}$$ $*a^{-n} = \frac{1}{a^n}$ (Write the exponent as a positive number.)

Example #12: Simplify the expression: $\left(x^{-\frac{1}{5}}y^{\frac{3}{2}}\right)^{-10}$

$$\left(x^{-\frac{1}{5}}y^{\frac{3}{2}}\right)^{-10} = \left(x^{-\frac{1}{5}}\right)^{-10}\left(y^{\frac{3}{2}}\right)^{-10}$$

*$(ab)^n = a^n b^n$ (exponents to each term)

$$= x^2 y^{-15}$$

* Power to a Power (multiply exponents)

$$\left(\begin{array}{l} -\frac{1}{5}(-10) = +\frac{1}{\cancel{5}^1}\times\frac{\cancel{10}^2}{1} = \frac{2}{1} = 2 \\ \frac{3}{2}(-10) = -\frac{3}{\cancel{2}^1}\times\frac{\cancel{10}^5}{1} = -\frac{15}{1} = -15 \end{array}\right)$$

$$= x^2 \cdot \frac{1}{y^{15}}$$

*$a^{-n} = \frac{1}{a^n}$

(Write the exponent as a positive number.)

$$= \frac{x^2}{y^{15}}$$

* Simplify

$$\left(\frac{x^2}{1}\cdot\frac{1}{y^{15}} = \frac{x^2}{y^{15}}\right)$$

*Note: The negative exponent in the numerator becomes positive when put in the denominator of the fraction.

QuickTime Simplifying Expressions with Negative Exponents (08:13)

Practice: Simplify $\left(8k^3m^{-6}\right)^{\frac{1}{3}}$, and then answer the following questions.

What is the coefficient of the expression in the numerator?

"Click here" to check your answer.

8 to the (1/3) means cube root of 8 which equals 2.

What is the entire expression in the numerator of the solution?

"Click here" to check your answer.

$2k^{(3/3)} = 2k^1 = 2k$

What is the entire expression in the denominator of the solution?

"Click here" to check your answer.

$m^{(6/3)} = m^2$

*Remember: The negative exponent in the numerator becomes positive when put in the denominator of the fraction.

Example #13: Simplify the expression: $(27p^9)^{\frac{4}{3}}$

$(27p^9)^{\frac{4}{3}} = (27)^{\frac{4}{3}}(p^9)^{\frac{4}{3}}$ *$(ab)^n = a^n b^n$ (exponents to each term)

$= (3^3)^{\frac{4}{3}}(p^9)^{\frac{4}{3}}$ *$27 = 3^3$

$= 3^4 p^{12}$ *Power to a power (multiply exponents)

$$\left(3^{3(\frac{4}{3}) = \frac{3}{1}\times\frac{4}{3} = \frac{12}{3} = 4} = 3^4\right)$$

$$\left(p^{9(\frac{4}{3}) = \frac{9}{1}\times\frac{4}{3} = \frac{36}{3} = 12} = p^{12}\right)$$

Practice: Simplify $\left(2y^{\frac{2}{3}}\right)^3$, and then answer the following questions.

What is the coefficient of the solution?

"Click here" to check your answer.

$2^3 = 8$

What is the exponent of *y* in the solution?

"Click here" to check your answer.

The exponent of *y* is (2/3)(3/1) = 6/3 = 2

What is the solution to the expression?

"Click here" to check your answer.

$8y^2$

Fractional Exponents (06:21)

Example #14: Simplify the expression: $\left(\dfrac{x^{\frac{1}{3}}}{y^{\frac{-3}{2}}}\right)^6$

$$\left(\dfrac{x^{\frac{1}{3}}}{y^{\frac{-3}{2}}}\right)^6 = \dfrac{(x^{\frac{1}{3}})^6}{(y^{\frac{-3}{2}})^6}$$

$*\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ (exponents to each term)

$$= \dfrac{x^2}{y^{-9}}$$

*Power to a power (multiply exponents)

$$\left(x^{\frac{1}{3}(6)=\frac{1}{3}\times\frac{6}{1}=\frac{6}{3}=2} = x^2 \right)$$

$$\left(y^{\frac{-3}{2}(6)=\frac{-3}{2}\times\frac{6}{1}=\frac{-18}{2}=-9} = y^{-9} \right)$$

$$= \dfrac{x^2}{1}\cdot\dfrac{1}{y^{-9}}$$

$$\dfrac{x^2}{y^{-9}} = \dfrac{x^2(1)}{(1)y^{-9}} = \dfrac{x^2}{1}\cdot\dfrac{1}{y^{-9}}$$

$$= \dfrac{x^2}{1}\cdot\dfrac{y^9}{1}$$

$*a^{-n} = \dfrac{1}{a^n}$

(Write the exponent as a positive number.)

$$\dfrac{1}{y^{-9}} = 1 \div y^{-9} = 1 \div \dfrac{1}{y^9} = 1 \times \dfrac{y^9}{1}$$

$$= \dfrac{x^2y^9}{1}$$

$$= x^2y^9$$

Did you notice? In the second step $\frac{x^2}{y^{-9}}$, the shortcut is to just move y to the numerator as a positive power $\frac{x^2y^9}{1}$, but remember that a 1 is left in the denominator; thus, the fraction simplifies to x^2y^9 .

Practice: Simplify $\frac{4x^{-2}}{3y^{-3}}$, and then answer the following questions. Use the shortcut to simplify your work.

What is the numerator of the solution?

"Click here" to check your answer.

$4y^3$ (The *y* is moved to the numerator with a positive power.)

What is the denominator of the solution?

"Click here" to check your answer.

$3x^2$ (The *x* is moved to the denominator with a positive power.)

Example #15: Simplify the expression: $\left(\frac{x^{-3}}{x^5}\right)^{\frac{1}{2}}$

$\left(\frac{x^{-3}}{x^5}\right)^{\frac{1}{2}} = (x^{-8})^{\frac{1}{2}}$ *Quotient Property (subtract the exponents)

$(-3-5=-8)$

$= x^{-4}$ *Power to a power (multiply exponents)

$\left(x^{-8(\frac{1}{2})=\frac{-8}{1}\times\frac{1}{2}=\frac{-8}{2}=-4} = x^{-4}\right)$

$= \frac{1}{x^4}$ $*a^{-n} = \frac{1}{a^n}$ (Write the exponent as a positive number.)

Stop! **Go to Questions #13-29 to complete this unit.**