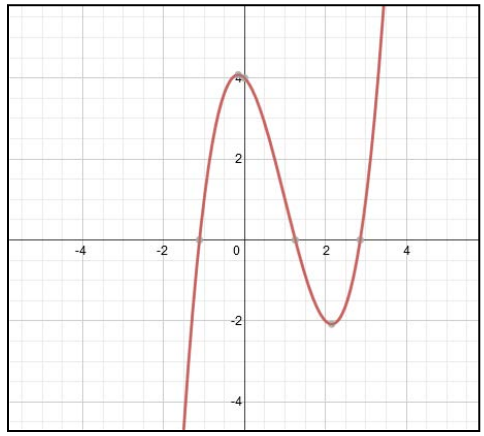
POLYNOMIAL FUNCTIONS



Polynomial functions are important in real-world situations. For example, a polynomial function can model the relationship between the dimensions of a rectangular box and its volume. Polynomial functions can also model the volume of irregularly shaped buildings. In this unit you will identify, evaluate, add, subtract, and classify polynomials. You will also multiply and divide polynomials using long division and synthetic division. At the conclusion of the unit the Factor Theorem will be used to solve problems.

Introduction to Polynomial Functions

In studying polynomials, we will first become familiar with the vocabulary associated with polynomial functions.

Basic Terminology: Functions and Polynomials (05:34)

monomial: a numeral, a variable, or a product of a numeral and one or more variables.

Examples: 5, x, $-3xy$

term: an expression that is a number, a variable, or the product of a number and one or more variables

Example: $2x + 3y + 7$ has 3 terms.

Note: Terms are separated with plus signs.

constant: a monomial with no variable, basically any number that has a constant value.

Examples: 6, –9

coefficient: the numerical factor of a monomial.

Examples:

- In $3xy$, the coefficient is 3.
- In $\frac{x}{6}$ (also written as $\frac{1}{6}x$), the coefficient is $\frac{1}{6}$.
- In x^9 (also written as $1x^9$), the coefficient is "understood" to be 1and is not normally written

degree of a monomial: the sum of the exponents of the variables.

Example: The monomial x^2yz (also written as $x^2y^1z^1$) has a degree of 4.

polynomial: a monomial or a sum of terms that are monomials.

Note*: In a polynomial, there **cannot be any variable exponents or any variables in the denominator of a term.

Binomials and trinomials are polynomials used frequently in algebra.

binomial: a polynomial with two terms.

Example: $3x + 5$

trinomial: a polynomial with three terms.

Example: $-4x^2 + 6x - 8$

degree of a polynomial: the largest monomial degree of the polynomial.

Example #1: $9x^4 + 3x^3 + 4x^2 - x + 1$ has a degree of 4 because the highest exponent is 4.

**Note*: Terms are separated with plus and minus signs.

leading coefficient: the coefficient of the term with the highest degree.

Example #2: In $9x^4 + 3x^3 + 4x^2 - x + 1$, the leading coefficient is 9 because it is the coefficient of x^4 which is the term that has the highest degree in the polynomial.

QuickTime Polynomial Functions (03:33)

Let's review the vocabulary associated with polynomial functions.

How many terms are in this expression? $3x^{10} + x^9 - 3x^6 - 2x^3 - 6x + 4$

Click here" to check your answer.

Six terms

What is the constant term in this expression? $3x^2 + 3x + 4$

Click here" to check your answer.

The constant term is 4.

What is the coefficient in this expression? $6x^3$

Click here" to check your answer.

The coefficient is 6.

What is the "understood" coefficient in this expression? u^5

Click here" to check your answer.

The "understood" coefficient is 1.

What is the degree in this monomial? $3x^4y^2y$

Click here" to check your answer.

The degree is 7. (Recall, y is the same as y^1.)

What is the degree in this polynomial? $-8w^7 + 5w^5 - 2w^3 - 6w + 2$

Click here" to check your answer.

The degree is 7.

What is the leading coefficient in this polynomial? $-8w^7+5w^5-2w^3-6w+2$

Click here" to check your answer.

The leading coefficient is –8.

What name is given to a polynomial with 3 terms?

Click here" to check your answer.

Trinomial

When given values for variables in polynomials, evaluate them by following the rules for Order of Operations.

Example #3: Evaluate the polynomial $-x^2-7x+11$ for $x = 3$.

$-x^2-7x+11$

$-(-3)^2-7(-3)+11$ Replace x with (-3) and simplify.

$-9+21+11$

23

The polynomial evaluates to 23 when $x = 3$.

Example #4: Evaluate the polynomial $5x^3-6x^2-2x$ for $x = -2$

$5x^3-6x^2-2x$

$5(-2)^3-6(-2)^2-2(-2)$ Replace x with -2 and simplify.

$5(-8)-6(4)-2(-2)$

$-40-24+4$

-60

The polynomial evaluates to –60 when $x = -2$.

Stop! **Go to Questions #1-8 about this section, then return to continue on to the next section.**

Adding and Subtracting Polynomials

To add or subtract polynomials combine like terms; this means add or subtract the coefficients of any terms that have the same variable and the same exponent. Make sure the like terms are rewritten next to each other; this will make it easier to add or subtract.

Example #1: Find the sum. $(6x^3+3x^2-4)+(10-3x-5x^2+2x^3)$

$(6x^3+3x^2-4)+(10-3x-5x^2+2x^3)$

$6x^3+3x^2-4+10-3x-5x^2+2x^3$ -Rewrite without parenthesis. Like terms are identified by color.

$6x^3+2x^3+3x^2-5x^2-3x-4+10$ -Arrange the terms in descending powers.

$8x^3-2x^2-3x+6$ -Collect like terms.

The closure properties for polynomials are very similar to those for integers. The sum, difference, or product of any two integers is an integer, and the sum, difference, or product of any two polynomials is a polynomial.

The previous example shows the closure property for addition of polynomials.

Closure Property

a polynomial + a polynomial = a polynomial

$(6x^3+3x^2-4)+(10-3x-5x^2+2x^3)=8x^3-2x^2-3x+6$

When **subtracting polynomials**, make sure to **change all the signs** of the **second quantity** when rewriting the problem.

Example #2: Find the difference. $(5x^2-6x-11)-(-8x^3+x^2+2)$

$(5x^2-6x-11)-(-8x^3+x^2+2)$

$5x^2-6x-11+8x^3-x^2-2$ -Rewrite without parenthesis. *Notice that all the signs have changed on the second quantity.

$8x^3 + 5x^2 - x^2 - 6x - 11 - 2$	-Arrange the terms in descending powers.
$8x^3 + 4x^2 - 6x - 13$	-Collect like terms.

This example shows the closure property for subtraction of polynomials.

Closure Property

a polynomial – a polynomial = a polynomial

$$(5x^2 - 6x - 11) - (-8x^3 + x^2 + 2) = 8x^3 + 4x^2 - 6x - 13$$

Stop! **Go to Questions #9-15 about this section, then return to continue on to the next section.**

Products of Polynomials

To express a function in standard form means to multiply all terms together and put them in descending order (largest to smallest).

Example #1: Express the function in standard form. $f(x)=(x-1)(x+4)(x-3)$

$f(x)=(x-1)(x+4)(x-3)$	
$f(x)=(x^2+3x-4)(x-3)$	FOIL the first two quantities.
$f(x)=x^2(x-3) \ + \ 3x(x-3) \ -4(x-3)$	Use the distributive property to multiply.
$f(x)=x^3-3x^2+3x^2-9x-4x+12$	Simplify
$f(x)=x^3-13x+12$	Combine like terms. *Note*: $\cancel{-3x^2}+\cancel{3x^2}=0$

Closure Property

a polynomial $\times$ a polynomial = a polynomial

$(x^2+3x-4)(x-3) = x^3-13x+12$

QuickTime Multiplying Polynomials (02:02)

Stop! **Go to Questions #16-19 about this section, then return to continue on to the next section.**

Factoring Polynomials

Just as a quadratic expression is factored by expressing it as a product of two factors, a polynomial expression of a degree greater than 2 is factored by expressing it as a product of more than two factors.

Example #1: Factor the polynomial. $x^3 - 10x^2 + 16x$

$x^3 - 10x^2 + 16x$ - Factor out the GCF of *x*.

$x(x^2 - 10x + 16)$ - Factor the trinomial into binomials.

$x(x-8)(x-2)$

Note: It is a good practice to check factoring by using multiplication.

$x(x-8)(x-2) = x(x^2 - 10x + 16) = x^3 - 10x^2 + 16x$ ✓

Example #2: Factor the polynomial. $x^3 + 6x^2 - 5x - 30$

This polynomial can be factored in pairs.

$(x^3 + 6x^2) + (-5x - 30)$ - Group the terms in pairs.

$x^2(x+6) - 5(x+6)$ - Factor out the GCF of each pair to produce a quantity that is common.

$(x+6)(x^2 - 5)$ - Factor $(x+6)$ from each term.

Factoring the Sum and Difference of Cubes

The sum or difference of polynomial cubes have formulas that make the factoring simpler.

Factoring the Sum of Cubes

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

Example #3: Factor $x^3 + 27$.

Think of $x^3 + 27$ as $x^3 + 3^3$ since $3^3 = 27$.

Then let $a = x$ and $b = 3$ and substitute into the formula.

$$x^3 + 3^3 = (x+3)(x^2 - x(3) + 3^2)$$

$$x^3 + 3^3 = (x+3)(x^2 - 3x + 9) \qquad \text{- Simplify}$$

The signs between the terms in the sum of cubes will always be plus (**+)** in the first factor and minus (–), then plus (+), in the second factor.

$$x^3 + 3^3 = (x+3)(x^2 - 3x + 9)$$

The x comes from $\sqrt[3]{x^3} = x$. The 3 comes from $\sqrt[3]{27} = 3$.

The x^2 in the second quantity comes from squaring x.

The $3x$ in the second quantity comes from multiplying 3 and x.

The 9 in the second quantity comes from squaring the 3.

Factoring the Difference of Cubes

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Example #4: Factor $x^3 - 125$.

Think of $x^3 - 125$ as $x^3 - 5^3$ since $5^3 = 125$.

Then let $a = x$ and $b = 5$ and substitute into the formula.

$x^3 - 5^3 = (x - 5)(x^2 + x(5) + 5^2)$

$x^3 - 5^3 = (x - 5)(x^2 + 5x + 25)$ - Simplify

The signs between the terms in the sum of cubes will always be plus (**+)** in the first factor and plus (+) for both terms in the second factor.

$$x^3 - 5^3 = (x - 5)(x^2 + 5x + 25)$$

Stop! **Go to Questions #20-24 about this section, then return to continue on to the next section.**

The Factor Theorem

Factor Theorem: $x - r$ is a factor of the polynomial expression that defines the function P, if and only if, r is a solution of $P(x) = 0$; that is, if and only if $P(r) = 0$.

What this is saying is that a binomial is a factor of a polynomial expression, if and only if, the numerical value r, when replaced for all the x values, yields 0.

Example #1: Is $x - 1$ a factor of $f(x) = x^3 - x^2 - 5x - 3$?

By the Factor Theorem, we will test the value of 1 since $r = 1$ in the factor, $x - 1$.

$f(1) = (1)^3 - (1)^2 - 5(1) - 3$ - Replace all x values with 1.

$= 1 - 1 - 5 - 3$

$= -8$

This means that $x - 1$ is not a factor of the polynomial because $f(1) \neq 0$.

Example #2: Is $x + 2$ a factor of $f(x) = x^3 - 2x^2 - 5x + 6$?

By the Factor Theorem, we express the factor in terms of $x - r$, so $x + 2$ must be written as $x - (-2)$.

We will test the value of –2 in the polynomial function.

To find $f(-2)$ replace all x values with –2.

$f(-2) = (-2)^3 - 2(-2)^2 - 5(-2) + 6$ - Replace all x values with –2.

$= -8 - 2(4) + 10 + 6$

$= -8 - 8 + 10 + 6$

$= -16 + 16$

$= 0$

This means that $x + 2$ is a factor of the polynomial because $f(-2) = 0$.

QuickTime What is the Factor Theorem? (07:32)

Stop! **Go to Questions #25-28 about this section, then return to continue on to the next section.**

Long Division of Polynomials

Long division of polynomials is similar to long division of whole numbers, except quantities are manipulated instead of numbers.

To use long division:

1.) Make sure that all terms are in descending order.

2.) Make sure that all terms are represented! For example, if there is no x^2 term in the polynomial, insert $0x^2$ for the term that is missing. This is algebraically sound because the value of $0x^2$ is zero and does not affect the value of the polynomial; yet, it serves as a placeholder to keep the alignment in order.

Example #1: Divide: $\dfrac{x^3+3x^2-4x-12}{x-2}$

x^2+5x+6	
$x-2\overline{)x^3+3x^2-4x-12}$	How many times will x go into x^3?
$\underline{-(x^3-2x^2)}$	Multiply $x-2$ by x^2 and subtract.
$5x^2-4x$	Bring down the $-4x$.
$\underline{-(5x^2-10x)}$	How many times will x go into $5x^2$?
$6x-12$	Multiply $x-2$ by $5x$ and subtract.
$\underline{6x-12}$	How many times will x go into $6x$?
0	Multiply $x-2$ by 6 and subtract.

QuickTime Factoring Polynomials (05:22)

Example #2: Divide $(x^3-7x-6)\div(x+1)$ Since there is no x^2 term, replace it with $0x^2$ to hold the place value.

x^2-x-6	
$x+1\overline{)x^3+0x^2-7x-6}$	How many times will x go into x^3?
$\underline{-(x^3+x^2)}$	Multiply $x+1$ by x^2 and subtract.
$-x^2-7x$	Bring down the $-7x$.
$\underline{-(-x^2-x)}$	How many times will x go into $-x^2$?
$-6x-6$	Multiply $x+1$ by $-x$ and subtract.
$\underline{-6x-6}$	How many times will x go into $-6x$?
0	Multiply $x+1$ by 6 and subtract.

QuickTime Reviewing Long Division (09:18)

Stop! **Go to Questions #29-31 about this section, then return to continue on to the next section.**

Synthetic Division

Another form of division is called synthetic division. Synthetic division can be used to divide a polynomial only by a linear binomial of the form $x - r$ and only uses the coefficients of each term. (When using nonlinear divisors, use long division as discussed in the previous section.)

Example #1: Use synthetic division to find the quotient for the division problem.

$$(x^3 + 3x^2 - 4x - 12) \div (x - 2)$$

Step #1: Write out the coefficients of the polynomial, and then write the r-value, 2, of the divisor $x - 2$. **Notice that you use the opposite of the divisor sign**. Write the first coefficient, 1, below the line.

$$\begin{array}{c|cccc} 2 & 1 & 3 & -4 & -12 \\ & & & & \\ \hline & 1 & & & \end{array}$$

Step #2: Multiply the r-value, 2, by the number below the line, 1, and write the product, 2, below the next coefficient.

$$\begin{array}{c|cccc} 2 & 1 & 3 & -4 & -12 \\ & & 2 & & \\ \hline & 1 & & & \end{array}$$

Step #3: Write the **sum** (not the difference) of 3 and 2, (5), below the line. Then, multiply 2 by the number below the line, 5, and write the product, 10, below the next coefficient.

$$\begin{array}{c|cccc} 2 & 1 & 3 & -4 & -12 \\ & & 2 & 10 & \\ \hline & 1 & 5 & & \end{array}$$

Step #4: Write the sum of –4 and 10, (6), below the line. Multiply 2 by the number below the line, 6, and write the product, 12, below the next coefficient.

$$\begin{array}{c|cccc} 2 & 1 & 3 & -4 & -12 \\ & & 2 & 10 & 12 \\ \hline & 1 & 5 & 6 & \boxed{0} \end{array}$$

The remainder is 0, and the resulting numbers 1, 5 and 6 are the coefficients of the quotient. The quotient will start with an exponent that is one less than the dividend. $1x^2 + 5x + 6$

The quotient is $x^2 + 5x + 6$.

Thus, $(x^3 + 3x^2 - 4x - 12) \div (x - 2) = x^2 + 5x + 6$.

Now let's try one that has a remainder.

Example #2: Use synthetic division to find the quotient and remainder for the division problem.

Remember to bring down the coefficient of the first term of the polynomial that is being divided (1 in this case) and continue on by multiplying by the divisor (*r*-value of the linear divisor) and adding.

$$(x^3 - 2x^2 - 22x + 40) \div (x - 4)$$

$$\begin{array}{r|rrrr} 4 & 1 & -2 & -22 & 40 \\ & & 4 & 8 & -56 \\ \hline & 1 & 2 & -14 & \boxed{-16} \end{array}$$

Since there is a remainder in this problem, the answer is written using a fraction with the divisor, $x - 4$, as the denominator. Notice that the sign between the last term and the fraction is the same as the sign of the remainder.

$$1x^2 + 2x - 14 - \frac{16}{x-4}$$

Thus, $(x^3 - 2x^2 - 22x + 40) \div (x - 4) = x^2 + 2x - 14 - \dfrac{16}{x-4}$.

Division of two polynomials is not always a polynomial as is shown in this example. Therefore polynomials are **not closed** under the operation of division. In the fourth term of the answer, the fraction has $x - 4$ in the denominator. In a polynomial, there **cannot** be any variables in the denominator of a term.

QuickTime Synthetic Division (17:20)

Let's work through a division problem using synthetic division. Here is the problem.

$$(3x^3 - 2x^2 + 3x - 4) \div (x - 3)$$

What number is used as the *r*-value to complete the division?

Click here" to check your answer.

***r* = 3**

What are the coefficients of the polynomial?

Click here" to check your answer.

3, –2, 3, –4

What are the missing numbers after multiplying and subtracting?

$$\begin{array}{r|rrrr} 3 & 3 & -2 & 3 & -4 \\ & & ? & 21 & ? \\ \hline & 3 & 7 & ? & 68 \end{array}$$

Click here" to check your answer.

9, 24, 72

What is the quotient (final answer in polynomial form)?

Click here" to check your answer.

$$3x^2 + 7x + 24 + \frac{68}{x-3}$$

Stop! **Go to Questions #32-40 to complete this unit.**