SOLVING SYSTEMS WITH MATRIX EQUATIONS

$$\begin{bmatrix} 1 & 1 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$$

Unit Overview

In this unit you will explore square matrices, identity matrices and inverse of matrices. You will then learn how to apply these various features of matrices to solve systems of equations.

Square Matrices and Identity Matrices

> During World War II, Navaho code talkers, 29 members of the Navaho Nation, developed a code that was used by the United States Armed Forces. Matrices can be used to interpret secret codes.

A matrix can be used to encode a message and another matrix, **it's inverse**, is used to decode a message once it is received.

Square Matrix

A **square matrix** is a matrix that has the same number of rows and columns.

Example #1: $\begin{bmatrix} 2 & 3 \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$ $\begin{bmatrix} 4 & -9 & 5 \\ 7 & -3 & 8 \\ 0 & 2 & -1 \end{bmatrix}$ $\begin{bmatrix} -2 & -8 & 4 & 8 \\ 0 & 1 & -3 & 6 \\ 4 & -4 & 0 & 9 \\ 11 & 32 & 57 & 3 \end{bmatrix}$

2×2 3×3 4×4

What is true about the number of rows and columns in square matrices?

"Click here" to check your answer.

Square matrices have the same number of rows and columns.

The Identity Matrix for Multiplication

> Let A be a square matrix with *n* rows and *n* columns. Let I be a matrix with the same dimensions and with 1's on the main diagonal and 0's elsewhere. Then:
>
> AI = IA = A.

$$I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The product of a real number and 1 is the same number. The product of a square matrix B, and its identity I, is the matrix B.

Example #2: Multiply Matrix B $\begin{bmatrix} -4 & 4 \\ 2 & 8 \end{bmatrix}$ times the identity matrix of a square matrix.

$$\underset{\mathbf{B}}{\begin{bmatrix} -4 & 4 \\ 2 & 8 \end{bmatrix}} \cdot \underset{\mathbf{I}}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} (-4\cdot 1)+(4\cdot 0) & (-4\cdot 0)+(4\cdot 1) \\ (2\cdot 1)+(8\cdot 0) & (2\cdot 0)+(8\cdot 1) \end{bmatrix} = \underset{\mathbf{B}}{\begin{bmatrix} -4 & 4 \\ 2 & 8 \end{bmatrix}}$$

B × I = B

The result of multiplying Matrix B by the identity matrix (Matrix I) is Matrix B. Just as the product of a real number and 1 is the same number, the product of a square matrix times the identity matrix is the same matrix.

If a Matrix C is multiplied by Matrix I (Identity Matrix), what is the result?

"Click here" to check your answer.

Matrix C

Stop! **Go to Questions #1-3 about this section, then return to continue on to the next section.**

The Inverse of a Matrix

Let A be a square matrix with *n* rows and *n* columns. If there is an $n \times n$ matrix B, such that AB = I and BA = I, then A and B are inverses of one another.

The inverse of matrix a is denoted by A^{-1}.

(Note: $A^{-1} \neq \frac{1}{A}$)

The product of a real number and its multiplicative inverse is 1. The product of a square matrix and its inverse is the identity matrix I.

Example #1: Let C = $\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$ and D = $\begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$ Show that C and D are inverses of one another.

To show that Matrix C and Matrix D are inverses of each other, multiply CD and DC. (Note: Remember, Matrix multiplication is not always commutative so both CD and DC must be calculated.)

$$CD = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \cdot \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} (2 \cdot -7) + (5 \cdot 3) & (2 \cdot 5) + (5 \cdot -2) \\ (3 \cdot -7) + (7 \cdot 3) & (3 \cdot 5) + (7 \cdot -2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$DC = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} (-7 \cdot 2) + (5 \cdot 3) & (-7 \cdot 5) + (5 \cdot 7) \\ (3 \cdot 2) + (-2 \cdot 3) & (3 \cdot 5) + (-2 \cdot 7) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The product CD produces the Identity Matrix. The product DC produces the Identity Matrix.

Since the product of these two matrices CD and DC are equal to the identity matrix, they are inverses of each other.

When a matrix is multiplied by its inverse, the result is what kind of matrix?

"Click here" to check your answer.

The Identity Matrix (Matrix I)

Determinant of a 2 × 2 Matrix

Each square matrix can be assigned a real number called the ***determinant*** *of the matrix*.

Let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A, denoted by det(A) or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is defined as:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Matrix A has an inverse, if and only if, $\det(A) \neq 0$.

Example #2: Find the determinant of matrix A, and then determine if matrix A has an inverse.

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$A = \begin{bmatrix} 7 & 3 \\ 2 & -1 \end{bmatrix}$$

$\det(A) = ad - bc$

$\det(A) = 7(-1) - 3(2)$

$\det(A) = -13$

Since $\det(A) \neq 0$, matrix A has an inverse.

What is true about the determinant of a matrix if the matrix does NOT have an inverse?

"Click here" to check your answer.

The determinant will equal zero.

To use the determinant to find the inverse:

1.) Find the difference of the cross products.

2.) Put this number under 1 and multiply it with the matrix using the following changes:
   a.) change the location of *a* and *d* in the matrix
   b.) change the signs of *b* and *c* in the matrix

General Formula: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example #3: Find the inverse of Matrix B, if it exists.

$B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ $\qquad B^{-1} = \dfrac{1}{5-6}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$

Note: $ad - bc \neq 0$, so matrix B has an inverse.

$$B^{-1} = \frac{1}{-1}\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = -1 \cdot \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ is $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

Let's check to see if Matrix A = $\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$ and Matrix B = $\begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$ are inverses of each other.

What matrix is the result of A × B?

"Click here" to check your answer.

AB = the Identity Matrix (I).

What matrix is the result of B × A?

"Click here" to check your answer.

BA = the Identity Matrix (I).

Are matrices A and B inverses of each other?

"Click here" to check your answer.

Yes

Explain how to determine if matrices A and B are inverses of each other.

"Click here" to check your answer.

Both AB and BA will result in the identity matrix (I).

Stop! **Go to Questions #4-11 about this section, then return to continue on to the next section.**

Solving Systems with Matrix Equations

A system of linear equations can be written as a matrix equation.

Example #1: $2a + 4b = -3$
$a - b = 9$

coefficient matrix, A	variable matrix, *X*	constant matrix, *B*
$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$	$X = \begin{bmatrix} a \\ b \end{bmatrix}$	$B = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

Let's examine how inverses can help solve linear systems written in matrix form. The following procedure is a general procedure to follow when solving systems of equations using matrices.

$AX = B$ *Represent the system as a matrix.
$A^{-1}AX = A^{-1}B$ *Multiply both sides by the inverse matrix.
$IX = A^{-1}B$ * $A^{-1}A = I$ (the Identity Matrix)
$X = A^{-1}B$ *$IX = X$

General Formula: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

To solve:

1.) find the inverse of the **coefficient matrix**

$A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$ $A^{-1} = \frac{1}{-2-4}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}$

$= -\frac{1}{6}\begin{bmatrix} -1 & -4 \\ -1 & 2 \end{bmatrix}$

2.) multiply both sides of the equation by A^{-1}

$$\overset{A^{-1}}{-\frac{1}{6}\begin{bmatrix}-1 & -4\\ -1 & 2\end{bmatrix}} \cdot \overset{A}{\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix}} \cdot \overset{X}{\begin{bmatrix}a\\ b\end{bmatrix}} = \overset{A^{-1}}{-\frac{1}{6}\begin{bmatrix}-1 & -4\\ -1 & 2\end{bmatrix}} \cdot \overset{B}{\begin{bmatrix}-3\\ 9\end{bmatrix}}$$

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \cdot \begin{bmatrix}a\\ b\end{bmatrix} = -\frac{1}{6}\begin{bmatrix}-33\\ 21\end{bmatrix} \qquad (A^{-1}\times A = I)$$

$$\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}\frac{11}{2}\\ -\frac{7}{2}\end{bmatrix} \qquad \left(I\times\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}a\\ b\end{bmatrix} \right)$$

$$a = \frac{11}{2} = 5.5 \quad \text{and} \quad b = -\frac{7}{2} = -3.5$$

Notice, that on the right side of the equation, we found the solution of the system of equations. To simplify the procedure, just multiply the Inverse of Matrix A (the coefficient matrix) times Matrix B (the constant matrix).

QuickTime Inverse Matrix -- Soccer (03:07)

Let's take a look at another example where we use the simplified method.

Example #2: Solve the following system of equations using matrices: $\begin{aligned}5x-4y&=4\\ 3x-2y&=3\end{aligned}$

1.) Determine the coefficient matrix, the variable matrix, and the constant matrix.

coefficient matrix, A	variable matrix, X	constant matrix, B
$A=\begin{bmatrix}5 & -4\\ 3 & -2\end{bmatrix}$	$X=\begin{bmatrix}x\\ y\end{bmatrix}$	$B=\begin{bmatrix}4\\ -3\end{bmatrix}$

Recall that after representing the system as a matrix equation, it can then be solved by multiplying both sides by the inverse matrix to find the solution as shown below algebraically.

$AX = B$	*Represent the system as a matrix.
$A^{-1}AX = A^{-1}B$	*Multiply both sides by the inverse matrix.
$IX = A^{-1}B$	* $A^{-1}A = I$ (the Identity Matrix)
$X = A^{-1}B$	*IX = X

We will proceed on determining X = A^{-1}B.

2.) Find the inverse of the **coefficient matrix**.

General Formula: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{1}{-10-(-12)}\begin{bmatrix} -2 & 4 \\ -3 & 5 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2 & 4 \\ -3 & 5 \end{bmatrix}$$

3.) Multiply the inverse of the coefficient matrix (A^{-1}) times the constant matrix (Matrix B).

X = A^{-1}B

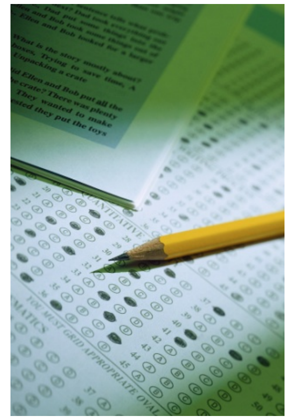
$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} -2 & 4 \\ -3 & 5 \end{bmatrix}}_{A^{-1}} \cdot \underbrace{\begin{bmatrix} 4 \\ 3 \end{bmatrix}}_{B}$$

$$= \frac{1}{2}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{3}{2} \end{bmatrix}$$

$x = 2$ and $y = 1.5$

Application of Inverse Matrices



Example #3: Mr. Shriver prepares a 20 question test for his history class. The test has true/false questions worth 3 points each and multiple-choice questions worth 11 points each for a total of 100 points. Set up a system of equations and use matrices to find the number of each type of question.

1.) Define variables to represent the two unknowns.

Let x = number of true/false questions.
Let y = number of multiple-choice questions.

2.) Write a system of equations for the problem.

$x + y = 20$	There are x amount of true/false questions and y amount of multiple-choice questions with a total of 20 questions on the test.
$3x + 11y = 100$	The value of the true/false questions is $3x$ and the value of the multiple choice questions is $11x$ with a total value of 100 points on the test.

The system of equations is:

$$x + y = 20$$

*The coefficients of x and y are "understood" to be 1. ($1x + 1y = 20$)

$$3x + 11y = 100$$

3.) Write the system of equations as a matrix equation. (Note: Matrix A is the coefficient matrix and Matrix B is the constant matrix.)

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 11 \end{bmatrix} \qquad B = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$$

The Matrix Equation is $\begin{bmatrix} 1 & 1 \\ 3 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$.

4.) Find the inverse of the coefficient matrix (Matrix A) where $a = 1$, $b = 1$, $c = 3$, and $d = 11$.

General Formula: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{1}{1(11) - 1(3)}\begin{bmatrix} 11 & -1 \\ -3 & 1 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 11 & -1 \\ -3 & 1 \end{bmatrix}$$

5.) Solve the matrix equation by multiplying the inverse of Matrix A times the constant matrix, Matrix B.

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \bullet B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 11 & -1 \\ -3 & 1 \end{bmatrix} \bullet \begin{bmatrix} 20 \\ 100 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 11(20) - 1(100) \\ -3(20) + 1(100) \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 120 \\ 40 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$$

There are **15** true/false questions and **5** multiple-choice questions on the test.

Stop! **Go to Questions #12-25 to complete this unit.**