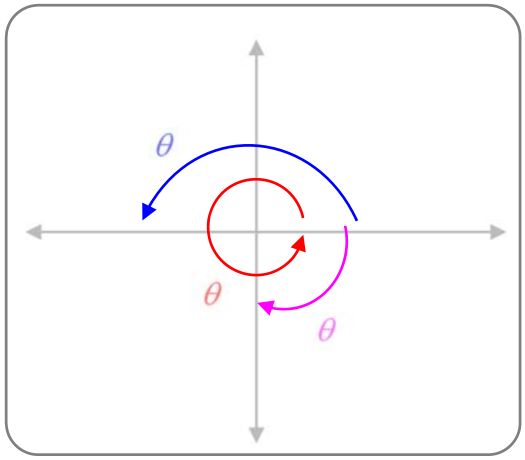
CIRCULAR TRIGONOMETRY



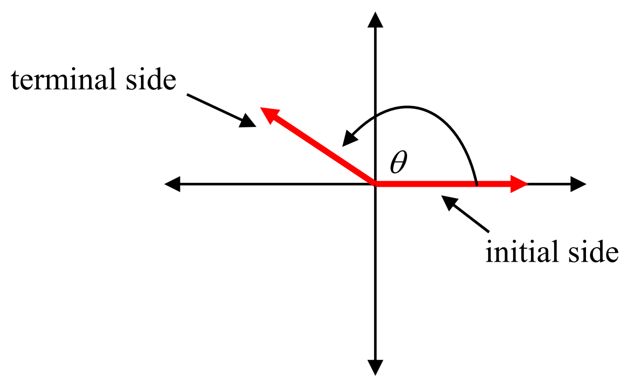
Unit Overview

This unit reviews angles and angle measurement. You will review special right triangles and apply them to the study of circular trigonometry.

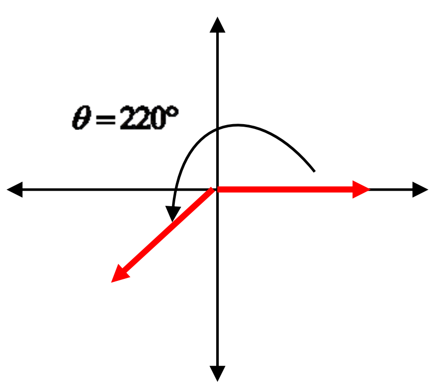
Angle Measurement

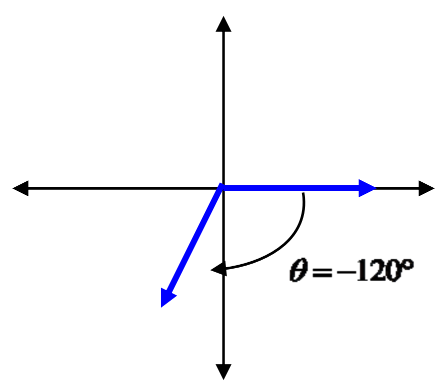
In a plane, an angle is formed by rotating a ray, called the initial side of the angle, around its endpoint until it coincides with another ray called the terminal side. An angle is in *standard position* if the vertex is at the origin and its initial side is along the positive *x*-axis.

Standard Position



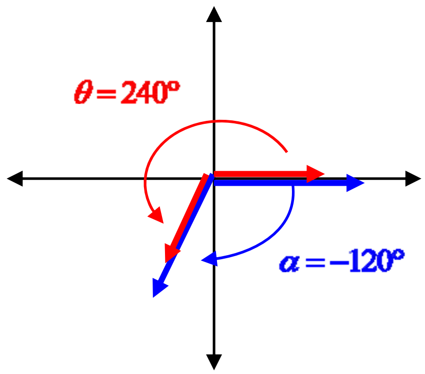
The measure of an angle in standard position is the amount of rotation from the initial side to the terminal side. If an angle rotates counter clockwise, the measure is *positive*. If the angle rotates clockwise, the measure is *negative*. *One rotation is equal to 360º.





Two angles in standard position are **coterminal angles** if they have the same terminal side.

Example #1:



The two angles shown $\alpha = -120$ and $\theta = 240°$ are considered coterminal because they share a terminal side.

To find coterminal angles to a given angle such that $-360° < \theta < 360°$, add or subtract 360° to the given angle until the coterminal angles satisfy the given condition of $-360° < \theta < 360°$.

Example #2: Find all coterminal angles such that $-360° < \theta < 360°$ for 560°.

a.) since $560°$ is already greater than $360°$, we can not add $360°$ but we need to start subtracting $360°$.

b.) $560° - 360° = 200°$

since 200 is within our domain try to subtract $360°$ again

$200° - 360° = -160°$

again $-160°$ is within our domain so subtract again

$-160° - 360° = -520°$

since this is not within our domain, the only coterminal angles of $560°$ are $200°$ and $-160°$.

Stop! **Go to Questions #1-5 about this section, then return to continue on to the next section.**

Reference Angles

For any angle θ in standard position, the reference angle θ_{ref} is the positive acute angle formed by the terminal side of θ and the nearest part of the *x*-axis.

The reference angle of any angle can be found using the following:

If the terminal side of θ is in quadrant I, then $\theta_{ref} = \theta$.

If the terminal side of θ is in quadrant II, then $\theta_{ref} = |180° - \theta|$.

If the terminal side of θ is in quadrant III, then The reference angle of any angle can be found using the following:

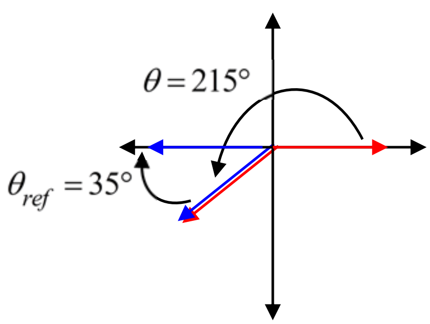
If the terminal side of θ is in quadrant I, then $\theta_{ref} = \theta$.

If the terminal side of θ is in quadrant II, then $\theta_{ref} = |180° - \theta|$.

If the terminal side of θ is in quadrant III, then $\theta_{ref} = |180° - \theta|$.

If the terminal side of θ is in quadrant IV, then $\theta_{ref} = |360° - \theta|$.

Example #1:



The reference angle for the red angle $\theta = 215°$ is the blue angle $\theta_{ref} = 35°$

Since the angle 215° is located in quadrant III, use $\theta_{ref} = |180° - \theta|$.

$$|180° - \theta| = |180° - 215| = |-35°| = 35°.$$

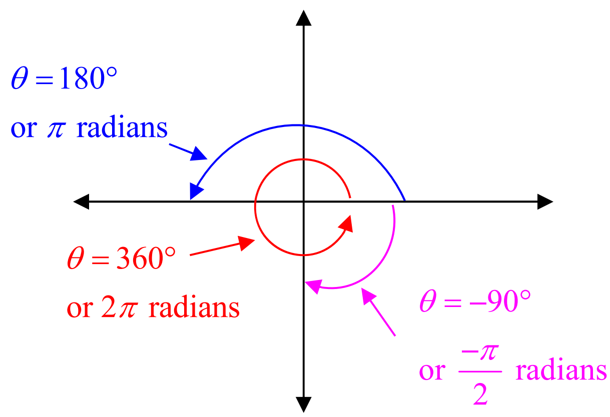
QuickTime Trigonometric Values (07:08)

Stop! Go to Questions #6-9 about this section, then return to continue on to the next section.

Degree and Radian Measures

Degree measure of angles is used extensively in engineering, surveying and navigation. Another unit of angle measure is called the **radian**, which is better suited for certain mathematical development, scientific work and engineering applications.

The radian measure of an angle is equal to the length of the arc on the unit circle (a circle centered at the origin with a radius of 1) that is intercepted by the angle in standard position. The circumference of any circle is $2\pi r$, where r is the radius of the circle. So the circumference of a unit circle is $2\pi(1)$ or 2π radians. Therefore an angle representing one complete revolution of the circle measures 2π radians or 360°.Thus, an angle of 180° = π radians and 90° = $\frac{\pi}{2}$ radians.



You can convert from degrees to radians and visa versa.

Converting Angle Measures	
Degrees to radians	Radians to degrees
Multiply by $\frac{\pi \text{ radians}}{180°}$	Multiply by $\frac{180°}{\pi \text{ radians}}$

Example #1: Convert $40°$ from degrees to radians and radians to degrees.

Multiply by $\frac{\pi \text{ radians}}{180°}$

$$\frac{40°}{1} \times \frac{\pi \text{ radians}}{180°} = \frac{40\pi \text{ radians}}{180} = \frac{2\pi}{9} \text{ radians}$$

An angle of $40°$ measures $\frac{2\pi}{9}$ radians.

Example #2 Convert $\frac{\pi}{6}$ radians to degrees.

Multiply by $\frac{180°}{\pi \text{ radians}}$

$$\frac{\pi \text{ radians}}{6} \times \frac{180°}{\pi \text{ radians}} = \frac{180^{\circ}}{6} = 30^{\circ}$$

An angle of $\frac{\pi}{6}$ radians measures 30°.

***Stop!* Go to Questions #10-17 about this section, then return to continue on to the next section.**

Circular Trigonometry

Trigonometry is the study of angles and triangles. The word trigonometry comes from the Greek words for “triangle measure

Angles whose measures are multiples of 30° and 45° are commonly used in trigonometry. These angle measures are equivalent to radian measures of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, respectively.

The critical values in Quadrant I are as follows:

$$0° = 0^r \qquad 30° = \frac{\pi}{6}^r \qquad 45° = \frac{\pi}{4}^r \qquad 60° = \frac{\pi}{3}^r \qquad 90° = \frac{\pi}{2}^r$$

If you remember that 180° = π radians, it is easy to remember the other angles.

90° is half of 180°, so 90° equals half of π, or $\frac{\pi}{2}$ radians.

60° is a third of 180°, so 60° equals a third of π, or $\frac{\pi}{3}$ radians.

45° is a fourth of 180°, so 45° equals a fourth of π, or $\frac{\pi}{4}$ radians.

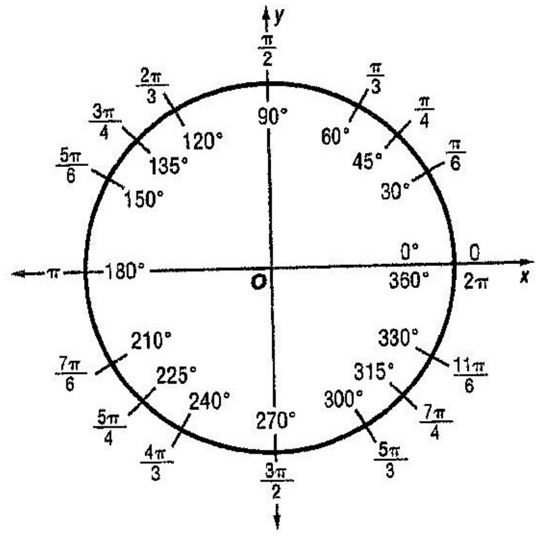
30° is a sixth of 180°, so 30° equals a sixth of π, or $\frac{\pi}{6}$ radians.

In addition all whole number multiples will also be critical values for all angles ≤ 360°.

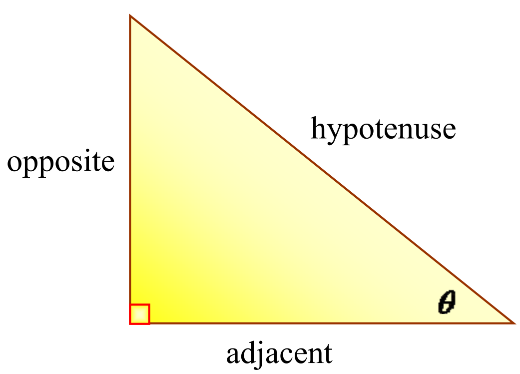
For example:

$$7(30°) = 210° \Rightarrow 7\left(\frac{\pi}{6}\right) = \frac{7\pi}{6} \quad \textit{and} \quad 5(45°) = 225° \Rightarrow 5\left(\frac{\pi}{4}\right) = \frac{5\pi}{4}$$

The unit circle with critical values labeled in radian and degrees angles is shown below:



The trig(short for trigonometry) ratios sine(sin), cosine (cos) and tangent (tan) are based on properties of right triangles. These three ratios are the most common trig ratios.



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

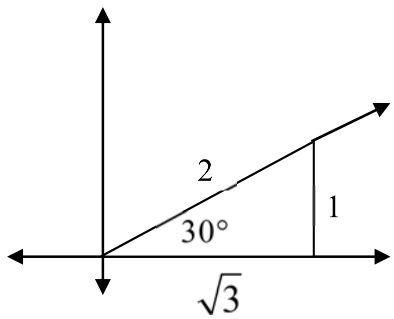
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

There are certain angles whose exact trig functions can be found without a calculator. These angles are 30°, 45°, 60°, and any angle having any of these three as reference angles. The ratios of the lengths of the sides of each triangle is shown below. Throughout this unit, it will be helpful for you to learn the exact values of sin, cos, and tan of these angles.

Special Right Triangles

30° Triangle

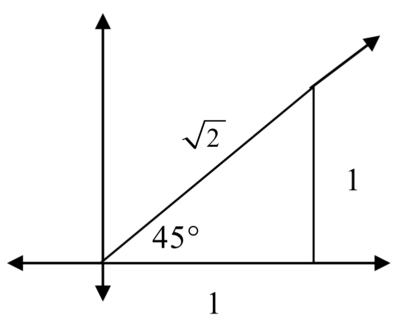


$$\sin 30 = \frac{1}{2}$$

$$\cos 30 = \frac{\sqrt{3}}{2}$$

$$\tan 30 = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

45° Triangle

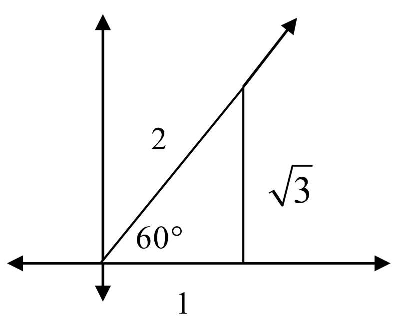


$$\sin 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45 = 1$$

60° Triangle



$$\sin 60 = \frac{\sqrt{3}}{2}$$

$$\cos 60 = \frac{1}{2}$$

$$\tan 60 = \sqrt{3}$$

Six Trig Functions (02:33)

Six Trig Functions and More (12:13)

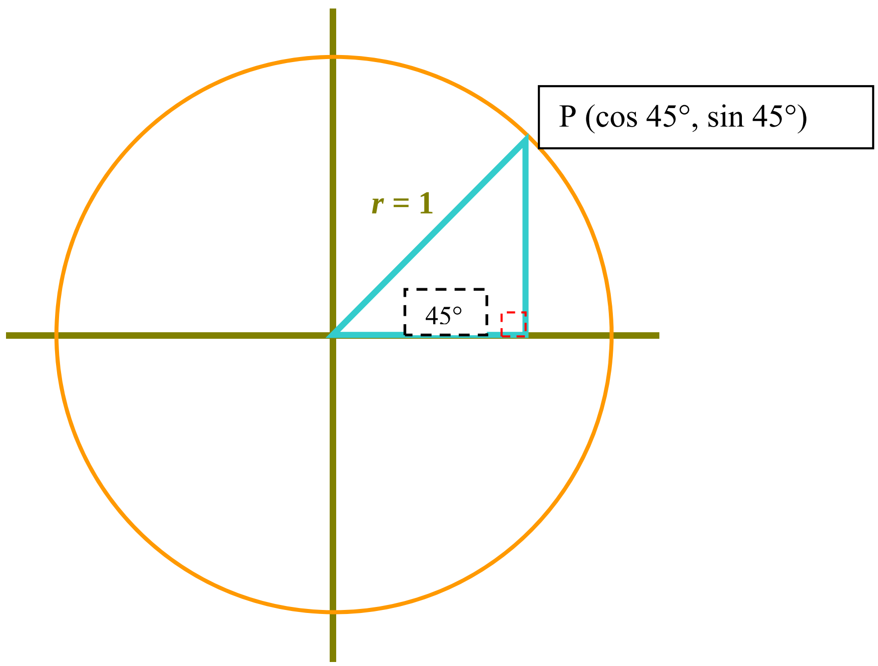
Solving Right Triangles (04:15)

Special Right Triangles (04:25)

Polar Coordinates: Radar Blips (05:12)

Right Triangles in Quadrant 1

In the circle diagram below, a right triangle can be inscribed in the circle in the following manner. For an angle in standard position, the terminal side of an angle in the unit circle hits at a point whose *x*-coordinate is the angle's cosine and whose *y*-coordinate is the angle's sine, that is cos $\theta = x$ and sin $\theta = y$.



In the above diagram, the hypotenuse of the triangle is also the radius of the circle. If we allow $r = 1$ and select one of the special right triangles to be inscribed in each quadrant of the circle, the trigonometric values can be established for each critical value from 0^r to $2\pi^r$.

Example #1: Find the exact values of the sin $\frac{\pi}{4}$, cos $\frac{\pi}{4}$ and tan $\frac{\pi}{4}$.

For this angle, we inscribe a 45-45-90 right triangle in the circle with the vertex of the $45°\left(\frac{\pi}{4}\right)$ angle at the origin.

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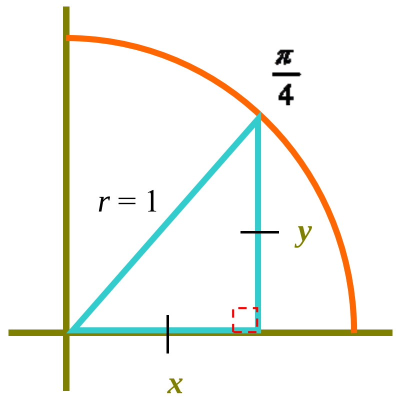
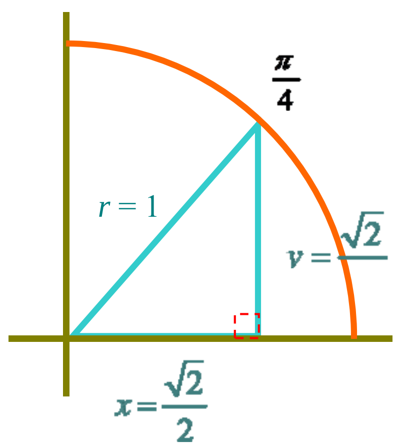


Using the values from the table Special Right Triangles, the inscribed 45-45-90 triangle is now completely labeled as:

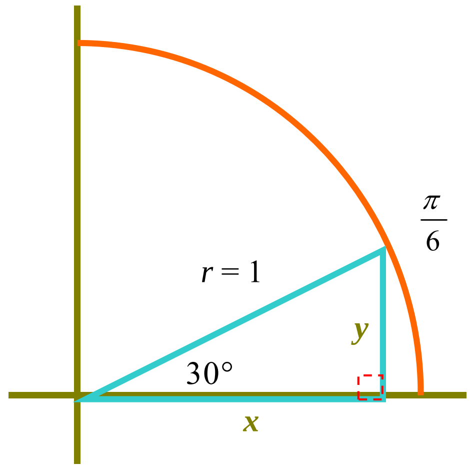


The exact values of the sin $\frac{\pi}{4}$, cos $\frac{\pi}{4}$ and the tan $\frac{\pi}{4}$:

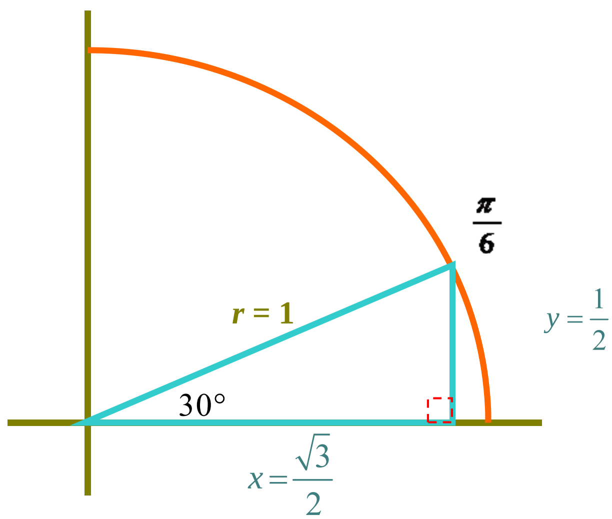
$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \tan\frac{\pi}{4} = 1$$

Example #2: Find the exact value of sin $\frac{\pi}{6}$, cos $\frac{\pi}{6}$ and tan $\frac{\pi}{6}$.

For this angle, we inscribe a 30-60-90 right triangle in the circle with the vertex of the $30°\left(\frac{\pi}{6}\right)$ angle at the origin.



Using the values for the 30-60-90 triangle, the inscribed triangle for $\frac{\pi}{6}$ is now labeled as:



The exact values for the three trigonometric ratios for $\frac{\pi}{6}$ are:

$$\sin\frac{\pi}{6}=\frac{1}{2}\ ,\qquad \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2},\qquad \tan\frac{\pi}{6}=\frac{\sqrt{3}}{3}$$

QuickTime Polar Coordinates: Radar Blips (05:12)

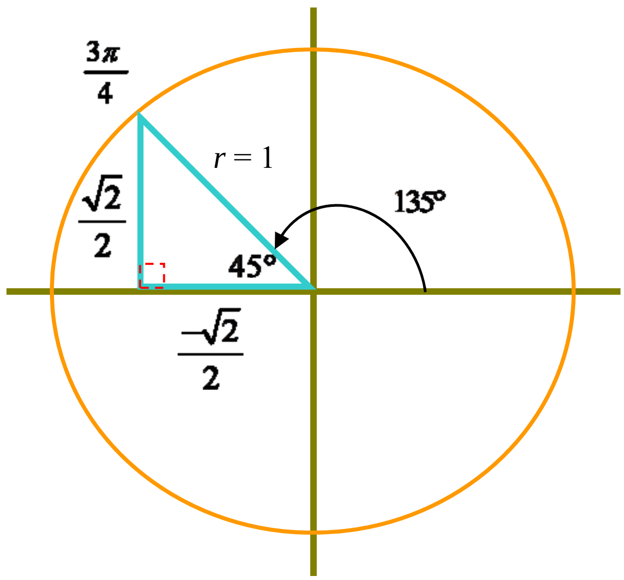
Stop! **Go to Questions #18-20 about this section, then return to continue on to the next section.**

Right Triangles and Reference Angles in Quadrants #2, 3, 4

As the special right triangles are inscribed in quadrants #2, 3, and 4, the legs of the triangles take on values to reflect the **direction** taken in order for the radius (hypotenuse) to intersect the critical value on the unit circle.

Example #1: Find the exact value of sin $\frac{3\pi}{4}$, cos $\frac{3\pi}{4}$ tan $\frac{3\pi}{4}$.

Note $\frac{3\pi}{4}$ radians = 135°. The terminal side of the angle is in quadrant II. Therefore the reference angle is |180° –135°|= 45°.



Notice that values of the legs of the inscribed triangle are still $\frac{\sqrt{2}}{2}$. However, the horizontal leg is labeled as negative. This reflects the fact that the triangle is inscribed in Quadrant #2. Using the values for the trigonometric ratios for a 45-45- 90 right triangle, the values for $\frac{3\pi}{4}$ are as follows:

$$\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}, \quad \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}, \quad \tan\frac{3\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$$

The chart below summarizes the signs of the trigonometric ratios for each quadrant.

Trig Ratio	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
sin θ	positive	positive	negative	negative
cos θ	positive	negative	negative	positive
tan θ	positive	negative	positive	negative

Example #2: Find the exact value of sin $\frac{4\pi}{3}$, cos $\frac{4\pi}{3}$ tan $\frac{4\pi}{3}$.

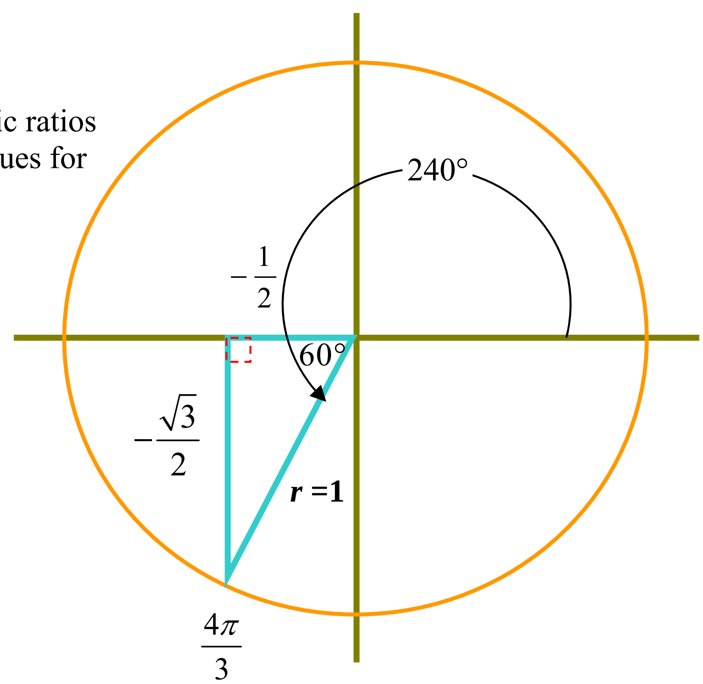
Note $\frac{4\pi}{3} = 240°$. The terminal side of the angle is in quadrant III. Therefore the reference angle is |180° – 240°|= |–60°| = 60°. Therefore, the inscribed triangle for $\frac{4\pi}{3}$ is 30-60-90 with the 60^0 used as our reference angle. The values of the legs of this triangle are known to equal $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$. This time, both these values are labeled as negative to reflect the triangle's position in Quadrant #3.

Using the values for the trigonometric ratios for a 30-60-90 right triangle, the values for $\frac{4\pi}{3}$ are as follows:

$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$

$\cos\frac{4\pi}{3} = -\frac{1}{2}$

$\tan\frac{4\pi}{3} = \sqrt{3}$



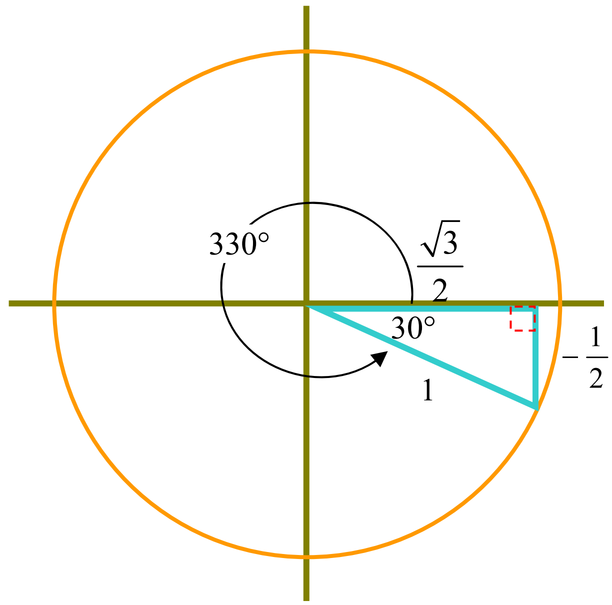
Example #3: Find the exact value of $\sin\frac{11\pi}{6}$, $\cos\frac{11\pi}{6}$ and $\tan\frac{11\pi}{6}$.

Note $\frac{11\pi}{6} = 330°$. Continuing with the same reasoning in the previous examples, the following inscribed triangle and ratios are obtained.

$$\sin\frac{11\pi}{6} = -\frac{1}{2}$$

$$\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan\frac{11\pi}{6} = -\frac{\sqrt{3}}{3}$$

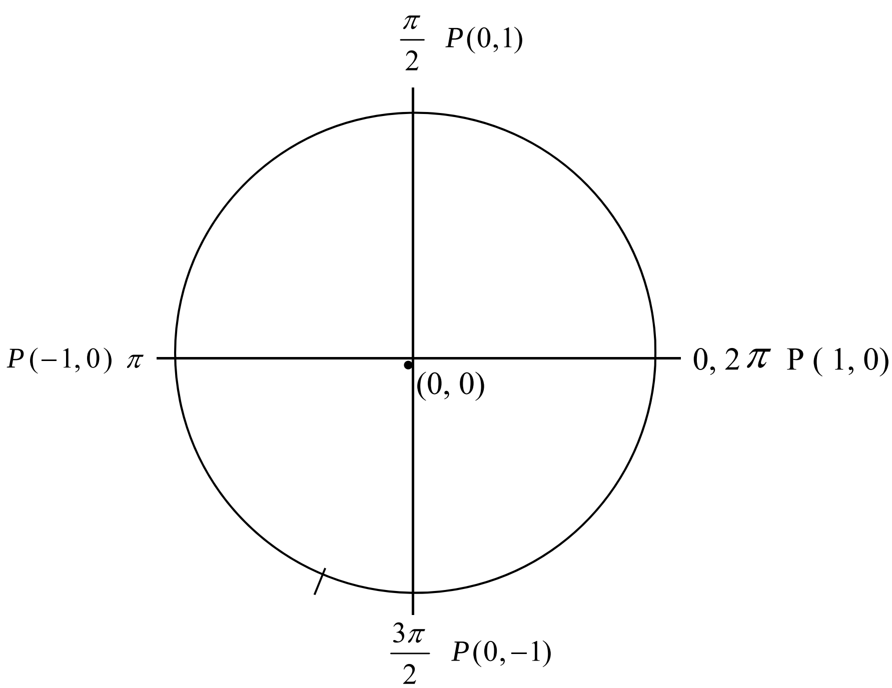


The Unit Circle (07:07)

Stop! **Go to Questions #21-38 about this section, then return to continue on to the next section.**

Trigonometric Ratios for Quadrantal Angles

If the terminal side of an angle in standard position coincides with one of the axes it is called a quadrantal angle. In the previous discussion, inscribed triangles were used to evaluate the trigonometric ratios of the angles found in each quadrant of the unit circle. However, the angle measures of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π (0°, 90°, 180°, 270°, 360°) are not found "inside" any of the Quadrants # 1, 2, 3, or 4. Each of those angle's position on the unit circle is found on either the *x* or *y*-axis which are the boundary lines between the four quadrants.



It is easy to find the trigonometric ratios for these angles since the terminal sides lie along an axis.

Example #1: Find the exact value of sin $\frac{3\pi}{2}$, cos $\frac{3\pi}{2}$, tan $\frac{3\pi}{2}$

Note $\frac{3\pi}{2} = 270°$ Locate the degree measure of the unit circle. The angle intersects the unit circle at P(0, –1) The *x* coordinate represents the angles cosine and the *y*-coordinate represents the angles sine. The angle's cosine also represents the adjacent side of an inscribed angle and the angle's sine represents the opposite side.

Therefore the exact value of sin $\frac{3\pi}{2}$, cos $\frac{3\pi}{2}$, tan $\frac{3\pi}{2}$ are as follows:

$$\sin\frac{3\pi}{2} = -1, \quad \cos\frac{3\pi}{2} = 0, \quad \tan\frac{3\pi}{2} = \frac{-1}{0} = \text{undefined.}$$

Stop! **Go to Questions #39-50 to complete this unit.**