GEOMETRIC SEQUENCES

$$t_n = t_1 (r)^{n-1}$$

$$t_3 = 3 (2)^{3-1}$$

$$\boxed{t_3 = 12}$$

Unit Overview

An important mathematical skill is discovering patterns. In this unit, you will investigate different types of patterns represented in geometric sequences. You will also discern the difference between an arithmetic sequence and a geometric sequence.

Geometric Sequences

Watch this video:

Example 2: Increasing Geometric Sequence--Population (02:30)

An example of a geometric sequence

$$5 \overset{\times 2}{\ } 10 \overset{\times 2}{\ } 20 \overset{\times 2}{\ } 40 \overset{\times 2}{\ } 80$$

A geometric sequence is one in which each number is multiplied by a constant ratio to get the next number in the sequence. In the example above, notice that each term is multiplied by 2 to get the next term.

The following is another example of a **Geometric Sequence** that still follows the pattern of multiplying by 2.

$$2,\ 2^2,\ 2^3,\ 2^4,\ 2^5 = 2,\ 4,\ 8,\ 16,\ 32$$

Another way to look at this sequence is:

$$a_1 = 2,\ \ a_2 = a_1 \cdot 2,\ \ a_3 = a_2 \cdot 2,\ \ a_4 = a_3 \cdot 2,\ \ a_5 = a_4 \cdot 2 \ldots$$

In each case, the next term in the sequence is the product of the previous term and the constant 2. For a geometric sequence, the number that multiplies the present term to give the next result is called the **common ratio**. The effect of the common ratio is perhaps better seen in the next example.

$$4,\ 4 \cdot 3,\ 4 \cdot 3^2,\ 4 \cdot 3^3,\ 4 \cdot 3^4 =$$

$$4,\ 12,\ 36,\ 108,\ 324,\ 972 =$$

$$a_1,\ a_1 \cdot 3,\ a_2 \cdot 3,\ a_3 \cdot 3,\ a_4 \cdot 3$$

In this example, the common ratio is 3 and is denoted $r = 3$.

Geometric Sequence: A Geometric Sequence is a sequence where each term after the first term, a_1, is the product of the preceding term and the common ratio, r, where $r \neq 0$ or 1. The terms of a geometric sequence can be represented by;

$$a_1,\ \ a_2 = a_1 \cdot r,\ \ a_3 = a_2 \cdot r,\ \ a_4 = a_3 \cdot r,\ \ a_5 = a_4 \cdot r\ \ldots$$

This formula for the general term is as follows:

nth Term of a Geometric Sequence

The nth term, a_n of a geometric sequence whose first term is a_1 and whose ratio is r is given by the explicit formula

$$a_n = a_1 r^{n-1}, \text{ where } n \geq 1$$

Example #1: Find the next three terms in the following geometric sequence.

5, 15, 45, …

Step #1: Determine the common ratio between terms.

$$\frac{15}{5} = 3$$

$$\frac{45}{5} = 3$$

Step #2: Multiply the last term by the common ratio and then again and as often as required.

$$45 \cdot 3 = 135$$

$$135 \cdot 3 = 405$$

$$405 \cdot 3 = 1215$$

Therefore, the next three terms are 135, 405, 1215.

Example #2: Find the first five terms of the geometric sequence for which $a_1 = 576$ and $r = -\frac{1}{2}$.

$$a_1 = 576$$

$$a_2 = 576 \cdot -\frac{1}{2} = -288$$

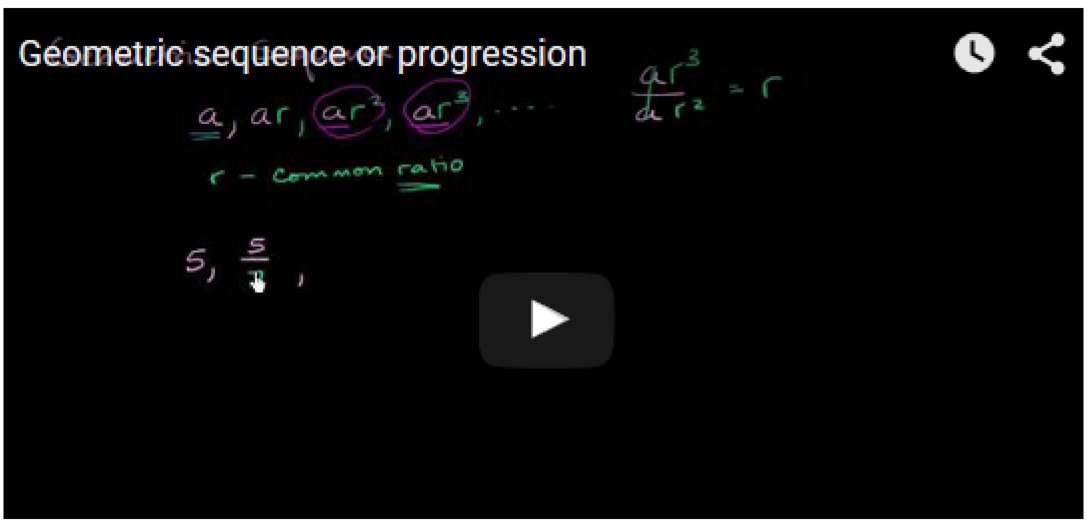
$$a_3 = -288 \cdot -\frac{1}{2} = 144$$

$$a_4 = 144 \cdot -\frac{1}{2} = -72$$

$$a_5 = -72 \cdot -\frac{1}{2} = 36$$

The first five terms of the sequence are 576, –288, 144, –72, 36.

Click on the link to watch the video "[Geometric sequence or progression]" or click on the video.



Example #3: Find the 6th term of the geometric sequence for which $a_1 = 2$ and $r = -3$.

Substitute the values for n and r into the formula for the nth term of an geometric sequence along with $a_1 = 2$.

$$a_n = a_1 r^{n-1}$$
$$a_6 = (2)(-3)^5$$
$$a_6 = (2)(-243)$$
$$a_6 = -486$$

The 6th term of the sequence is –486.

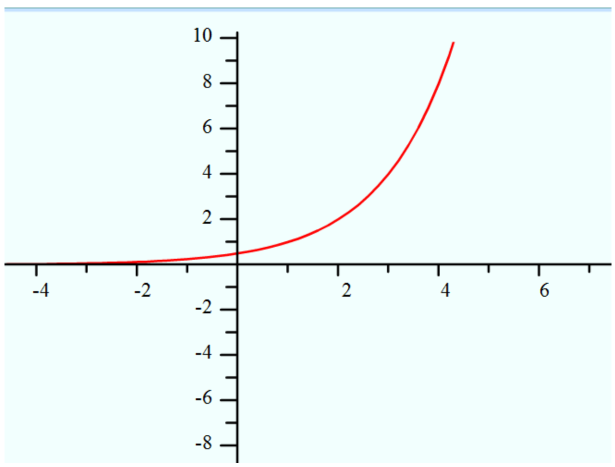
Stop! **Go to Questions #1-7 about this section, then return to continue on to the next section.**

Example #4: Find the explicit formula for the sequence 2, 4, 8, 16, 32, …

First, find the common ratio (r). In this example, 4/2 = 2, 8/4 = 2, 16/8 = 2 and 32/16 = 2 so 2 is our common ratio.

Using our explicit formula , $a_n = a_1 r^{n-1}$, we will plug in 2 for r and 2 for a_1 (the first term). This gives the formula $a_n = 2\times 2^{n-2}$.

Using our exponent rules, we could simplify this further. Remember that when we have like bases, we can add exponents. If we think of this as $a_n = 2^1 \times 2^{n-2}$, we can add the exponents to get $a_n = 2^{1+n-2} = 2^{n-1}$. Notice that this is an exponential function. If graphed, we would get this:



***Geometric sequences are exponential functions.**

Example #5: Using the explicit formula $a_n = 5\times 4^{n-1}$, state a_1 and r. Then, find the first 4 terms of the sequence.

First, using the formula, you can easily identify a_1 and r. $a_1 = 5$ based on its position in the formula. $r = 4$, again based on its position in the formula. Now, use the formula to find the first 4 terms. We already know the first term (a_1) is 5.

2nd term: $a_n = 5\times 4^{2-1} = 5\times 4^1 = 20$

3rd term: $a_n = 5\times 4^{3-1} = 5\times 4^2 = 5\times 16 = 80$

4th term: $a_n = 5\times 4^{4-1} = 5\times 4^3 = 5\times 64 = 320$

Example #6: Use the table of values to write a function for the geometric sequence.

Term #	**Value**
1	–1
2	5
3	–25
4	125
5	–625

First, find the common ratio, *r*.

$$\frac{5}{-1} = -5$$
$$\frac{-25}{5} = -5$$
$$\frac{125}{-5} = -5$$

We see that –5 is the common ratio.

Next, identify the first term a_1. The first term is –1 so $a_1 = -1$.

Now, since we have 2 negative numbers, make sure to be careful and not make mistakes with those negatives!

We will use our explicit formula, $a_n = a_1 r^{n-1}$, but since we are writing in function notation, we will write it as $f(n) = a_1 r^{n-1}$.

$$f(n) = -1(-5)^{n-1}$$

Notice that the –5 is in parenthesis. This is because –5 is *r*. All of *r* is taken to the exponent. Without including the negative in parenthesis, we are making *r* a positive 5. Also, we are **not** taking (–1) × (–5). Exponents first, and this will take care of the negative 5. Whatever you get after working the exponent is what will be multiplied by –1.

To make sure you can use this function properly,

Try this! Using the function just found, find the 6th term of the geometric sequence.

"Click here" to check the answer.

$$\boldsymbol{f(6) = (-1) \times (-5)^{6-1}}$$

$$\boldsymbol{f(6) = (-1) \times (-5)^{5}}$$

$$\boldsymbol{f(6) = (-1)(-3125)}$$

$$\boldsymbol{f(6) = 3125}$$

Stop! **Go to Questions #8-12 about this section, then return to continue on to the next section.**

Arithmetic or Geometric?

In the last unit, you learned about arithmetic sequences. In this unit, you have learned about geometric sequences. If it is not specified, how do you know which type of sequence it is? How do you know which rule to use to create the explicit formula?

Remember that arithmetic sequences had a common difference. A number that you added or subtracted to find the next term. In a geometric sequence, you multiply by a common ratio to find the next term. When given problems that aren't specified, you must discern if you have a common difference or a common ratio.

For the next 4 problems, identify each sequence as arithmetic, geometric, or neither. If the sequence is arithmetic, state the common difference. If the sequence is geometric, state the common ratio.

Identify each sequence. 3, –3, 3, –3, ….

"Click here" to check the answer.

Geometric, $r = -1$

Identify each sequence. 1, 4, 9, 16, …

"Click here" to check the answer.

Neither, the same number is neither added nor multiplied each time to get the next term.

Identify each sequence. 25, 50, 75, 100, …

"Click here" to check the answer.

Arithmetic, $d = 25$

Identify each sequence. 2, 1, 0.5, 0.25

"Click here" to check the answer.

Geometric, *r* = ½ or 0.5

Stop! **Go to Questions #13-17 about this section, then return to continue on to the next section.**

Word Problems on Geometric Sequences

Example#1: A culture of bacteria doubles every 3 hours. If there are 200 bacteria present at the beginning, how many bacteria will there be after 24 hours?

Step #1: Determine whether the situation represents an arithmetic or geometric sequence.

Starting with 200 and doubling produces the following sequence.

200, 400, 800,…

The sequence 200, 400, 800, … is a geometric sequence.

Step #2: Identify the variables.

If the bacteria doubles every 3 hours, it will double 8 times in a 24 hour period. So, we are looking for the 8th term of the sequence.

$$a_1 = 200 \quad r = 2, \quad n = 8$$

Step #3: Substitute and evaluate.

Use the formula for a geometric sequence.

$$a_n = a_1 r^{n-1}$$

$$a_8 = 200(2)^7$$

$$a_8 = 25,600$$

There are 25,600 bacteria present after 24 hours.

Example #2: Mrs. Readworthy is serious when she says she wants her students to study. Each week, she requires them to study 5 times the number of spelling words required the week before. If she starts off the year requiring only 1 spelling word, how many spelling words will students be studying by week 4?

From the problem, we discern that $a_1 = 1$ (1 word the first week). We also see the common ratio is 5. From this, we write the formula $a_n = a_1 r^{n-1}$ and fill in what we know.

$a_n = 1 \times 5^{n-1}$.

We are solving for week 4, so $a_4 = 1 \times 5^{4-1} = 1 \times 5^3 = 1 \times 125 = 125$.

Mrs. Readworthy will require 125 spelling words on week 4.

Stop! **Go to Question #18 about this section, then return to continue on to the next section.**

The Fibonacci Sequence

There is one special sequence worth noting since we are discussing sequences. That is the Fibonacci Sequence. It is a special pattern found by adding the previous 2 numbers in the sequence.

$a_1 = 1$ and $a_2 = 1$. By adding the first 2 terms, $a_3 = 2$.

Continuing the pattern 1, 1, 2, 3, 5, 8, 13, …

Notice that the pattern is to add the previous two terms to get the next term.

Watch this video about the Fibonacci Sequence:

QuickTime Fibonacci Sequence (03:50)

Watch this video to see where the Fibonacci Sequence is seen in nature:

QuickTime The Fibonacci Sequence in Nature (03:00)

What type of sequence is the Fibonacci Sequence?

Is it arithmetic? geometric?

Do we add the same number each time to get the next term? No.

Do we multiply by the same number each time to get the next term? No.

We always add the previous two terms. So, this sequence is not arithmetic or geometric. We can represent it with an explicit formula, but that is beyond the scope of this lesson. However, we can represent it with a recursive formula. Remember, recursive means that it is dependent on knowing the previous term(s). For this sequence, we could write, $f(n + 1) = f(n) + f(n - 1)$ for $n \geq 1$. We must also state that $f(0) = f(1) = 1$. This means that the first two terms are 1. Then the formula basically says that the next term $(n + 1)$ equals the last two terms added together. If you'd like to learn more about this fascinating sequence, take it further and research it. There are many wonderful websites and books on this topic.

Stop! **Go to Questions #19-23 to complete this unit.**