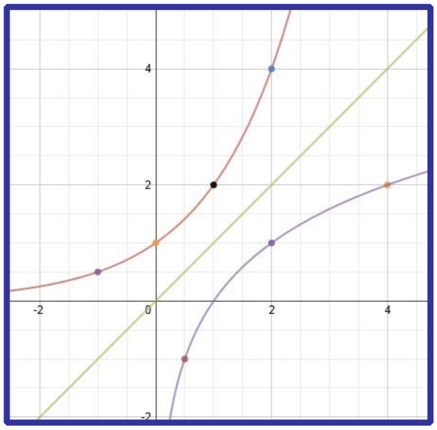
FUNCTIONS AND INVERSES OF FUNCTIONS



Unit Overview

The unit begins with basic function concepts: functions as relations, domain, range, and evaluating functions. The unit continues with finding the inverse of functions, and determining whether the inverse of a function is a function.

Review of Functions

Relation: a relationship between two variables such that each value of the first variable is paired with one or more values of the second variable; **a set of ordered pairs**.

Example #1: {(2, 4), (–4, 5), (2, –7), (0, 9)}

Function: a relationship between two variables such that each value of the first is paired with exactly one value of the second variable; **all domain values (*x*-values) are different.**

Example #2: {(2, 4), (0, 6), (7, 4), (–9, 4)}

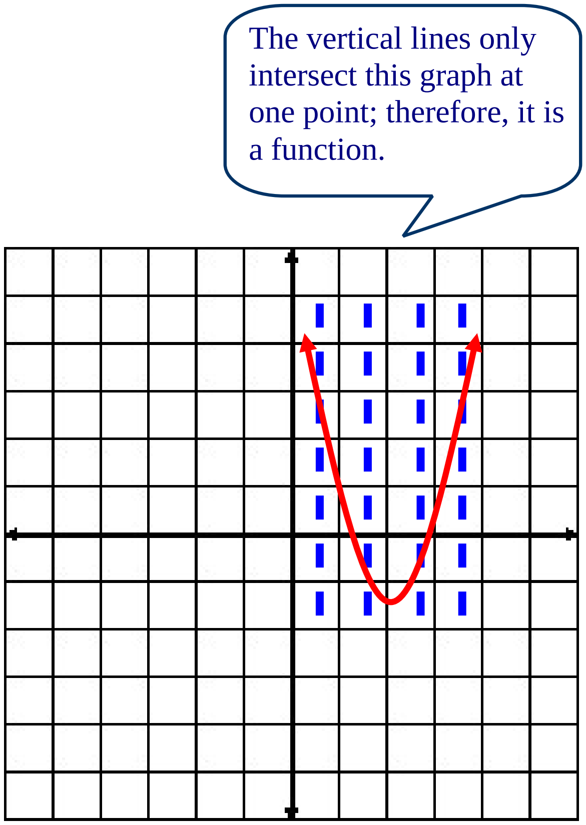
Domain: the set of all possible values of the first variable (all *x*-values)

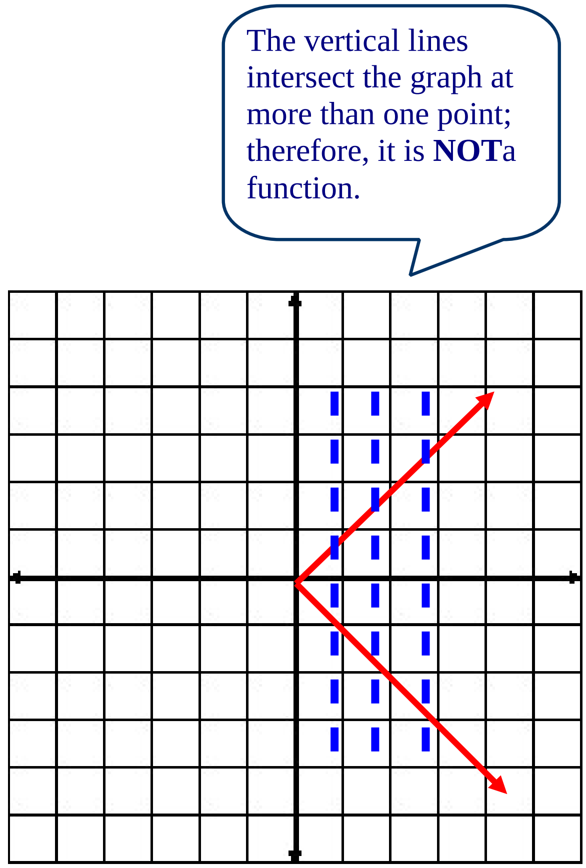
From example #2 above: domain = {2, 0, 7, –9}

Range: the set of all possible values of the second variable (all *y*-values)

From example #2 above: range = {4, 6}

Vertical Line Test: If every vertical line intersects a given graph at no more than one point, then the graph represents a function.





Function Notation: If there is a correspondence between values of the domain, x, and values of the range, y, that is a function; then $y = f(x)$, and (x, y) can be written as $(x, f(x))$.

$f(x)$ is read "f of x". The number represented by $f(x)$ is the value of the function f at x.

The variable $\boldsymbol{x}$ is called the **independent variable** and the variable $\boldsymbol{y}$, **or** $\boldsymbol{f(x)}$, is called the **dependent variable**.

To evaluate a function for a specific variable, replace x with the given value and solve.

Example #3: Evaluate $f(x) = -1.2x^2 + -4x - 3$ for $x = 1$.

$$f(x) = -1.2x^2 + -4x - 3$$

$$f(1) = -1.2(1)^2 + 4(1) - 3$$

$$f(1) = -1.2 + 4 - 3$$

$$f(1) = -0.2$$

When $x = 1$, the value of $f(x) = -0.2$

Example #4: Evaluate $g(x) = 3x^2 - x + 1$ for $x = 4$.

*Note: Other letters may be used when denoting functions.

$$g(4) = 3(4)^2 - 4 + 1$$

$$g(4) = 3(16) - 4 + 1$$

$$g(4) = 48 - 4 + 1$$

$$g(4) = 45$$

When $x = 4$, the value of $g(x) = 45$.

Stop! **Go to Questions #1-8 about this section, then return to continue on to the next section.**

Inverses of Functions

The **inverse of a relation** consisting of the ordered pairs (*x*, *y*) is the set of all ordered pairs (*y*, *x*). (switch the *x* and *y*) An inverse is just a matter of reversing the *x* and *y* coordinates.

Consider the relation {(1, 2), (4, –2), (3, 2)}.

- The domain of the relation is {1, 4, 3} and the range of the relation is {–2, 2}.
- The relation is a function because each domain value is paired with exactly one range value.

To find the **inverse** of the relation, switch the *x* any *y* values.

- The point (1,2) becomes the point (2,1).
- The point (4,-2) becomes the point (-2,4).
- The point (3,2) becomes the point (2,3).
- The inverse is {(2, 1), (–2, 4), (2, 3)}.
- The domain of the inverse is {2, –2}.
- The range of the inverse is {1, 4, 3}.

*The relation is a function but the inverse is NOT a function because the domain value 2 is paired with two range values. {(2, 1), (–2, 4), (2, 3)}.

The range of a relation is the domain of the inverse. The domain of a relation is the range of the inverse. Domain and range for the inverse just switch what they were from the original function. The inverse of a function may or may not be a function.

Let's consider the points in the table.

x	*y*
0	0
1	1
2	4
3	9

Is the relation a function?

Answer: The relation IS a function since each domain value (x) is paired with exactly one range value (y).

$0 \longrightarrow 0$

$1 \longrightarrow 1$

$2 \longrightarrow 4$

$3 \longrightarrow 9$

We denote the relation in function notation as $f(x)$ since it is a function.

x	$f(x)$
0	0
1	1
2	4
3	9

Is the inverse of $f(x)$ a function?

x	$f(x)$
0	0
1	1
2	4
3	9

Switch the domain and range. $\longrightarrow$

x	$f^{-1}(x)$
0	0
1	1
4	2
9	3

* The inverse of a function f is denoted by f^{-1}.
This is read as "f inverse" or "the inverse of f."

Answer: The inverse of the function IS a function also because each domain value is paired with exactly one range value.

Domain and Range of the Inverse

If a function $f(x)$ has an inverse $g(x)$, all of the domain values (x-values) in $f(x)$ are the range values (y-values) in $g(x)$ and all of the range values (y-values) in $f(x)$ are the domain values (x-values in $g(x)$).

For example, given the function $f(x) = x + 2$, notice that point (1, 3) in $f(x)$ becomes point (3, 1) on $g(x)$. We get this by $f(1) = 1 + 2 = 3$; therefore, (1, 3) is a point for $f(x)$. To get the inverse, we swap the x and y values to get (3, 1).

What point on $g(x)$ is the inverse of point (0, 1) on $f(x)$?

"Click here" to check your answer.

The reflected point is (1, 0).

In general, any point (x, y) in $f(x)$ becomes what ordered pair in its inverse, $g(x)$?

"Click here" to check your answer.

(y, x)

Stop! **Go to Questions #9-16 about this section, then return to continue on to the next section.**

If a function is defined by an equation, then the inverse of the function is found by switching the *x* and *y* in the equation, and then solving the new equation for *y*. To find the inverse equation, simply switch the variables *x* and *y* around as in example #1 below.

Example #1: $y = 3x - 2$

$x = 3y - 2$ Switch the *x* and *y*.

$x + 2 = 3y$ Add 2 to each side.

$\frac{x+2}{3} = y$ Divide each side by 3.

$\frac{1}{3}x + \frac{2}{3} = y$ Write $\frac{(1)x+2}{3}$ as two fractions.

The inverse of $y = 3x - 2$ is $y = \frac{1}{3}x + \frac{2}{3}$.

Sometimes, the equation is written in function notation as in the following two examples. To keep the computations simple, we remember that $f(x)$ can be written as *y* and vice versa.

Example #2: Find the inverse $f(x) = \frac{x+8}{4}$

$f(x) = \frac{x+8}{4}$

$y = \frac{x+8}{4}$ $f(x)$ can be written as *y*.

$x = \frac{y+8}{4}$ Switch the *x* and *y*.

$4x = y + 8$ Multiply each side by 4 $\left(\frac{y+8}{\cancel{4}_1} \cdot \frac{\cancel{4}^1}{1} = \frac{y+8}{1} = y+8\right)$.

$4x - 8 = y$ Subtract 8 from each side.

The inverse function of $f(x) = \frac{x+8}{4}$ is $y = 4x - 8$.

*The inverse of a function f is denoted by f^{-1}. This is read as "f inverse" or "the inverse of f."

Thus, we can state that $f^{-1}(x) = 4x - 8$ is the inverse of $f(x) = \frac{x+8}{4}$.

Example #3a: Find the inverse of $f(x) = x + 5$.

$f(x) = x + 5$

$y = x + 5$ $f(x)$ can be written as y.

$x = y + 5$ Switch the x and y.

$x - 5 = y$ Subtract 5 from both sides.

The inverse function of $f(x) = x + 5$ is $y = x - 5$.

*The inverse of a function f is denoted by f^{-1}.

Therefore, $f^{-1}(x) = x - 5$ is the inverse of $f(x) = x + 5$.

Example #3b: Using $f(x) = x + 5$ from *example #3a*, find $f(9)$ and the inverse point.

Plug 9 into $f(x)$. $f(9) = 9 + 5 = 14$. This yields the point (9, 14).

To find the inverse, just switch x and y. (9, 14) becomes (14, 9).

You can check this by plugging this point into the inverse function $f^{-1}(x) = x - 5$.

$f(14) = 14 - 5 = 9$.

Try this! Find the inverse of the point (2, –6).

"Click here" to check your answer.

Answer: (–6, 2)

Try this! Find the inverse of $f(x) = x - 19$.

"Click here" to check your answer.

Answer: $y = x + 19$
Switch x and $y \rightarrow x = y - 19$
Solve for y. $x + 19 = y$

Find $f^{-1}(8)$ for the previous function.

"Click here" to check your answer.

Answer: 27
The inverse function was $y = x + 19$
$8 + 19 = 27$

Try this! Find the inverse of $f(x) = \dfrac{2x-6}{8}$.

"Click here" to check your answer.

Answer: $f^{-1}(x) = 4x + 3$

Solution: $y = \dfrac{2x-6}{8}$

Switch x and y: $x = \dfrac{2y-6}{8}$

Solve for y: Multiply both sides by 8. $8x = 2y - 6$

Add 6 to both sides. $8x + 6 = 2y$

Divide both sides by 2 (each term). $4x + 3 = y$

So, $y = 4x + 3$ or $f^{-1}(x) = 4x + 3$

Example #4: Find the inverse of $y=(x+2)^2-5$.

First, switch the *x* and *y* variables. $x=(y+2)^2-5$

Next, solve for *y*. Add 5 to both sides.
$x+5=(y+2)^2$

Now, take the square root of both sides.
$\pm\sqrt{x+5}=\sqrt{(y+2)^2}$
$\pm\sqrt{x+5}=y+2$

Now, subtract 2 from both sides to isolate *y* by itself.
$\pm\sqrt{x+5}-2=y$

This could also be written as $y=-2\pm\sqrt{x+5}$.

Example #5: Are $y = 2x + 4$ and $y=\frac{x}{2}-2$ inverses of each other?

To find this out, we need to find the inverse of the original function and see if it matches the one given. We start with $y = 2x + 4$. First, find the inverse. Remember, just switch the *x* and *y* variables.

$x = 2y + 4$

Now, solve for *y*. First, subtract 4 from both sides.
$x - 4 = 2y$

Now, divide everything by 2.
$\frac{x}{2}-\frac{4}{2}=\frac{2y}{2}$

Simplify. $\frac{x}{2}-2=y$

Does this yield the same inverse given in the problem? Yes. Therefore, $y = 2x + 4$ and $y=\frac{1}{2}x-2$ are inverses.

Stop! **Go to Questions #17-30 to complete this unit.**