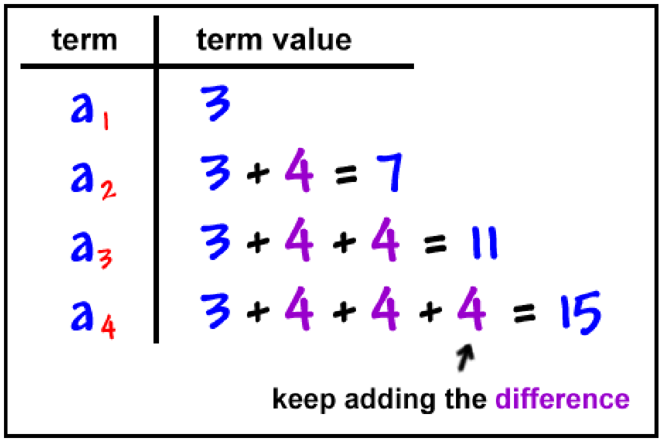
ARITHMETIC SEQUENCES



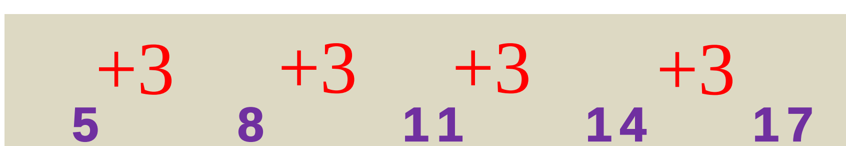
Unit Overview

An important mathematical skill is discovering patterns. In this unit, you will investigate different types of patterns represented in sequences. In this unit, we will look at arithmetic sequences. In the next unit, you will discover geometric sequences.

Arithmetic Sequences

A sequence is a list of numbers in a particular order. Each number in a sequence is called a term. The first term is represented by a_1, the second term is represented by a_2, and so on. a_n represents the nth term of the sequence. One kind of sequence is an **arithmetic sequence**. In an arithmetic sequence each term after the first is found by adding a constant, called the **common difference**, *d*, to the previous term.

An example of an arithmetic sequence



This sequence is arithmetic because to go from one number to the next, you just add or subtract the same amount. It is a simple pattern to add 3 to find the next number in the sequence. The +3 you see in red is just an illustration. The numbers in purple are the sequence.

Mathematically, 3 is known as the common difference. 5 is known as a_1 or the first term in the sequence. $a_2 = 8$, $a_3 = 11$, $a_4 = 14$, $a_5 = 17$ and so on. What is a_6? Well, take a_5 and add 3. a_6 (the 6th term of the sequence = 17 + 3 = 20).

Example #1: Find the next three terms in the following arithmetic sequence.

–4, 5, 14, 23 …

Step #1: Determine the common difference between terms. To do this find the difference between each pair of numbers. It should be the same.

$5-(-4)=9$

$14-5=9$

$23-14=9$

Therefore, $d=9$.

Step #2: Add the common difference to the last term, then to the result, then again and as often as required.

$23+9=32$

$32+9=41$

$41+9=50$

Therefore, the next three terms are 32, 41, 50.

Being able to find numbers in a sequence from numbers already given is known as a **recursive** process. In our first example, we were able take the last term and add 3 to get the next term. However, sometimes, it may be difficult or tedious to do this. For example, what if you were asked for the 200th term of the sequence from the first example? You could work this out recursively by adding 3 until you got to the 200th term, but this could take a long time. Another option is to find the **explicit formula** for the sequence.

In general, we could write an arithmetic sequence like this:

$$
\begin{aligned}
&a_1 \\
&a_2 = a_1 + d \\
&a_3 = a_1 + d + d = a_1 + 2d \\
&a_4 = a_1 + d + d + d = a_1 + 3d \\
&a_5 = a_1 + d + d + d + d = a_1 + 4d \\
&\ldots
\end{aligned}
$$

We can generalize the expression for a_n if we realize that the common difference in a sequence of n terms is added to the terms of the sequence $n-1$ times. For instance, in the sequence 2, 7, 12, 17, there are 4 terms in the sequence therefore $n=4$ and the common difference is $d=5$. Starting with the first term and ending with the last, $d=5$ is added to the terms a total of 3 times.

2 + 5 = 7   7 + 5 = 12   12 + 5 = 17

For a sequence with n terms, the common difference *d* is added to the terms in a sequence $n-1$ times. From this, we obtain the following result:

nth Term of an Arithmetic Sequence

The general term a_n, of an arithmetic sequence whose first term is a_1 and whose common difference is *d*, is given by the formula

$$a_n = a_1 + (n-1)d$$

This is known as an explicit formula. We will use this to find terms that are more difficult like that 200th term mentioned earlier.

Example #2: Find the 20th, 53rd and 100th terms in the following sequence.

15, 23, 31, 39,. . .

Step #1: (a) Identify d, (the common difference).

23 – 15 = 8   31 – 23 = 8   39 – 31 = 8

$$\Rightarrow \quad d = 8$$

(b) Identify n to be used in the formula:

For the 20th term, $n = 20$.

For the 53th term, $n = 53$.

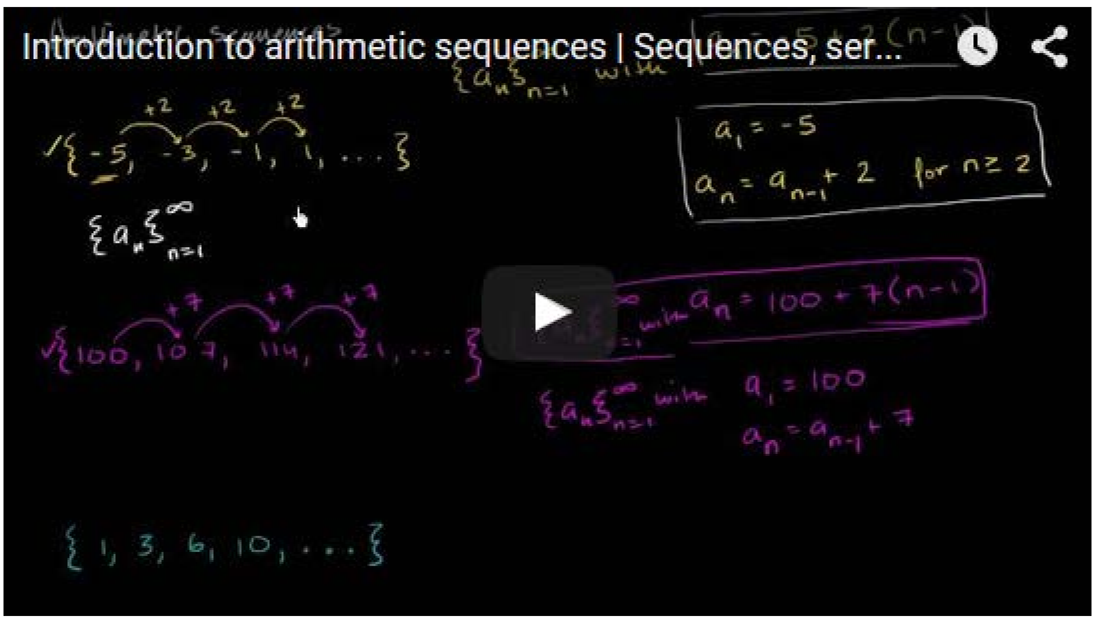
For the 100rd term $n = 100$.

Step #2: Substitute value for n and d into the formula for the n^{th} term of an arithmetic sequence along with $a_1 = 15$, the 1st term in the sequence.

20^{th} term:	**53^{th} term**:	**100^{th} term**:
$a_n = a_1 + (n-1)d$	$a_n = a_1 + (n-1)d$	$a_n = a_1 + (n-1)d$
$a_{20} = 15 + 19 \cdot 8$	$a_{53} = 15 + 52 \cdot 8$	$a_{100} = 15 + 99 \cdot 8$
$a_{20} = 167$	$a_{53} = 431$	$a_{100} = 807$

Example #3: Find the explicit formula for the sequence in example #2. First, define a_1 and *d*. a_1 is the first term so a_1 = 15. d is the common difference which is 8. Plug these into the formula to give $a_n = 15 + (n-1) \cdot 8$.

Click on the link to watch the video "Introduction to arithmetic sequences" or click on the video.



QuickTime **Arithmetic Sequence -- Pyramids (02:13)**

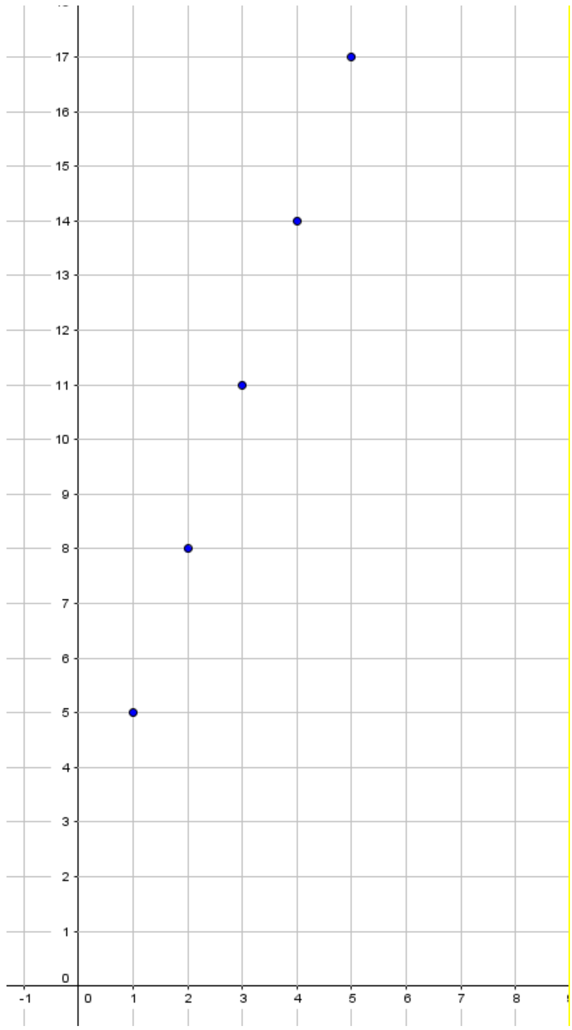
Example #4: Find the explicit formula for the very first example. The one in yellow with a pattern of +3. In this example, we had the numbers 5, 8, 11, 14, 17. Define a_1 and d. a_1 is the first term so a_1 = 5. d is the common difference which is 3. Plug these into the formula to give $a_1 = 5 + (n-1) \cdot 3$.

Notice that our explicit formula shows a linear relationship. Written in the more familiar slope-intercept form, $y = mx + b$ may help you see this. Let's take another look at our very first example (in yellow at beginning.) In this example, we had the numbers 5, 8, 11, 14, 17 with the pattern of adding 3. Let's make a table of values for this sequence. The term # refers to "it's place in line." Term #1 is the first term in the sequence; in this case 5. Term #2 is 8, and so on. Fill in the chart.

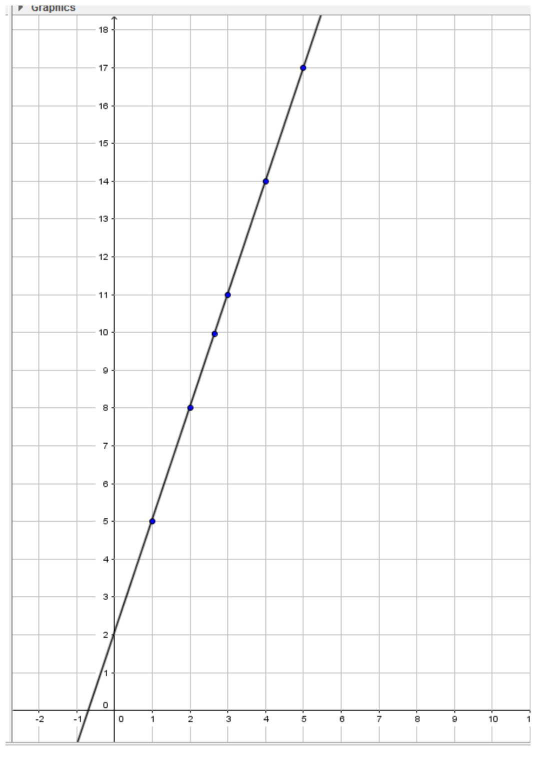
Term #	Value
1	5
2	8
3	11
4	14
5	17

Next, plot these values on a coordinate plane. Let the term # be x and the value be y. So, we will plot (1, 5), (2, 8), (3, 11), (4, 14), and (5, 17).

It should look something like this:



Notice that there is a linear pattern. We could easily draw a straight line through the points.



Knowing this, we should be able to create a linear equation for this line. Remember how to find slope? Take 2 points from your table and find the slope. Using (1, 5) and (2, 8) gives us:

$$\frac{8-5}{2-1}=\frac{3}{1}=3$$

Notice that our slope is the common difference!

The explicit formula for an arithmetic sequence is given as $a_n = a_1 + (n-1)\cdot d$. a_1 is similar to our *b* or *y*-intercept. Finding the linear equation would give us *y* = 3*x* + 2. (Notice the 2 intercept from the graph.) This equation is slightly different from our formula as the graph crosses 2 at *x* = 0, but our explicit formula shows what happens when *x* = 1 (term #1) so we have +5 in our explicit formula. The explicit formula for this pattern is given by $a_n = a_1 + (n-1)\cdot d$ Fill in what we know: a_1= 5 and *d* = 3.

$$a_n = 5 + (n-1)\cdot 3.$$

Notice that this is similar to the linear equation we found. The difference is that the graph line shows all values including values that come before 5 that are not part of our pattern. But it finds the same values!

Our explicit formula can also be written as a function. Remember, function notation is written as *f* (*x*). So, to put our formula for this example in function notation, we simply write:

$$f(n) = 5 + (n-1)\cdot 3.$$

Try this! Use the table of values to find the next value in the arithmetic sequence.

Term #	Value
1	85
2	80
3	75
4	70
5	65
6	?

Click here to check answer

60

Now, use the table to find the explicit formula for this sequence.

Click here to check answer

$a_n = 85 + (n - 1) \cdot -5$

Explanation

The first term, $a_1 = 85$.

The common difference is –5 (notice the numbers are decreasing).

Try this! A sequence begins with 17 and increases by 2.2 with each term. Find the explicit formula for this sequence.

Click here to check answer

$a_n = 17 + (n - 1) \cdot 2.2$

Explanation

The first term, $a_1 = 17$. The common difference is 2.2.

Try this! Now, write your explicit formula in function notation.

Click here to check answer

$f(n) = 17 + (n - 1) \cdot 2.2$

Now, use the formula that you just found (use either the explicit formula or the one in function notation) to find the 14th term of the sequence.

Click here to check answer

$$a_{14} = 17 + (14 - 1) \cdot 2.2$$
$$17 + (13) \cdot 2.2$$
$$17 + 28.6$$
$$45.6$$

Word Problems on Arithmetic Sequences

Example #1: Justin sells cars at a local dealership and earns \$150.00 commission on the first car he sells each month. For each additional car Justin sells during the month, he receives \$10.00 more on his commission. How much commission will Justin earn on the 30th car?

Step #1: Determine whether the situation represents an arithmetic sequence or geometric sequence (Geometric sequences are in the next unit but you will need to know the difference. We will discuss this further in the next unit.)

Commission on 1st car - \$150
Commission on 2nd car - \$160
Commission on 3rd car - \$170
And so on.

The 150, 160, 170, … represents an arithmetic sequence. The question asks for the amount of money earned on the 30th car so use the formula for an arithmetic sequence.

Step #2: Identify the variables.

$$a_1 = \$150.00, \quad d = 10, \quad n = 30$$

Step #3 Substitute and evaluate.

The formula for an arithmetic sequence is $a_n = a_1 + (n-1) \cdot d$. In this example, a_{30} is unknown. First use the formula for an arithmetic sequence to determine the value of a_{30}.

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$a_{30} = 150 + (30 - 1) \cdot 10 = 150 + 29 \cdot 10 = 150 + 290 = \$440$

The commission earned on the 30th car of the month is \$440.

Stop! **Go to Questions #1-31 to complete this unit.**