

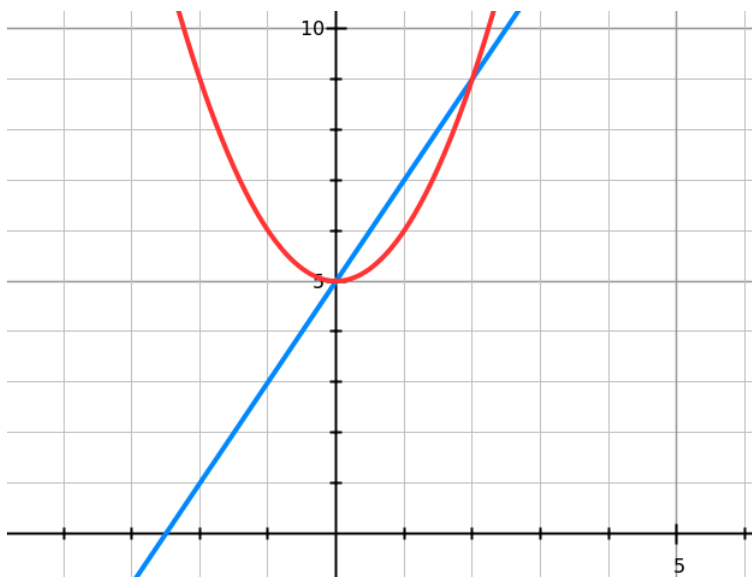
FUNCTIONS – INTERSECTIONS, INTERPRETATIONS, AND DEEPER UNDERSTANDING

Unit Overview

In this unit, you will study graphs and function behavior including linear, quadratic, exponential, absolute value and trigonometric functions. You will look at intersection points, intercepts, intervals where the function is positive or negative, intervals where the function increases or decreases, and end behavior. You will learn to find the midline, amplitude, and period for trigonometric graphs. You will finish by comparing functions through word problems.

Intersection Points

First, we are going to look at intersecting graphs. This is where two graphs cross, or intersect each other. For example, look at the graph below:



Notice that there is one line and one parabola. These two functions cross at two points. Reading the graph, you see that they intersect at $(0,5)$ and $(2,9)$.

In this section, you will find intersections of two functions by looking at graphs, tables, and equations.

Recall: What do the points on a graph represent?

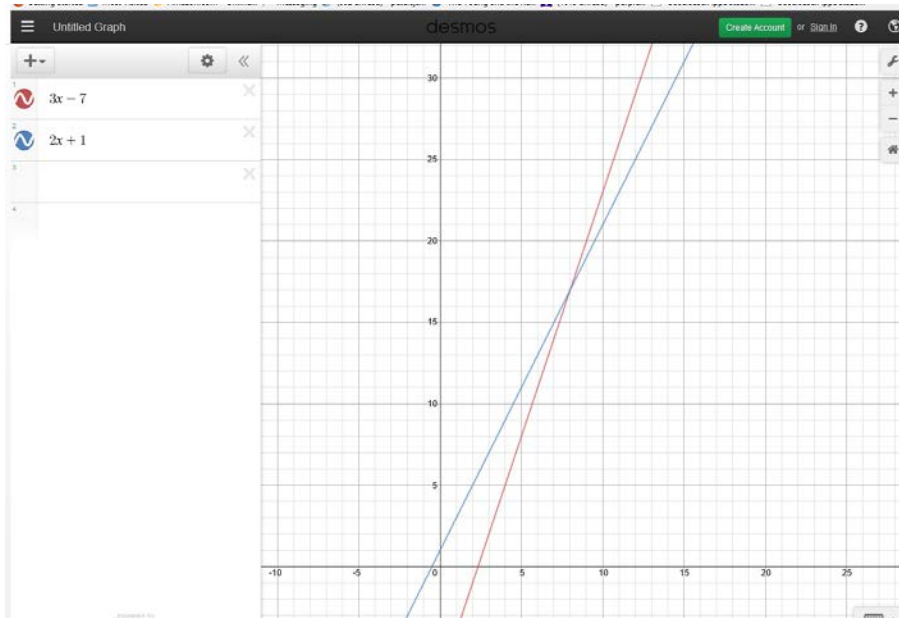
Answer: All the points along the graph represent solutions to the equation (or function).

Recalling this, what do the intersection points of two graphs represent?

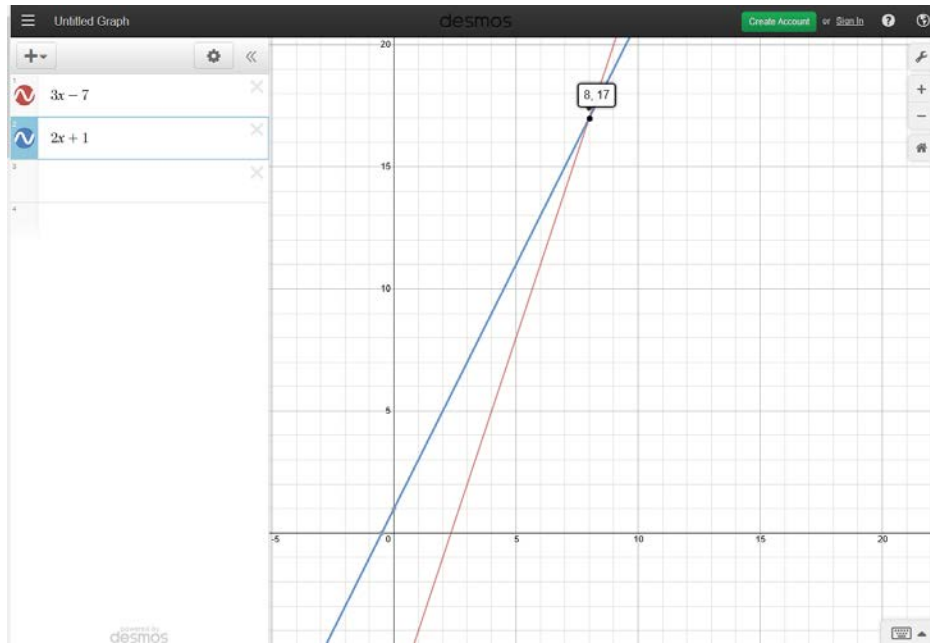
Answer: The intersection points represent solutions to both equations. In other words, an intersection point is a solution to both equations. If more than one intersection point exists, then there is more than one solution that fits both equations.

Example #1: Find the intersection points for the following functions, $f(x) = 3x - 7$ and $f(x) = 2x + 1$.

Remember from an earlier unit that you can solve a system of equations by graphing, by substitution, or by elimination. We are going to use the graphing method in this unit. However, we are going to expand that by looking at equations that are non-linear as well. Let's start with this linear example as a review. You will need to either graph by hand, with a graphing calculator, or use an online graphing utility. I am going to demonstrate using <https://www.desmos.com/calculator>. Go to this site and enter the two equations. When you enter the equations, it will look like this:

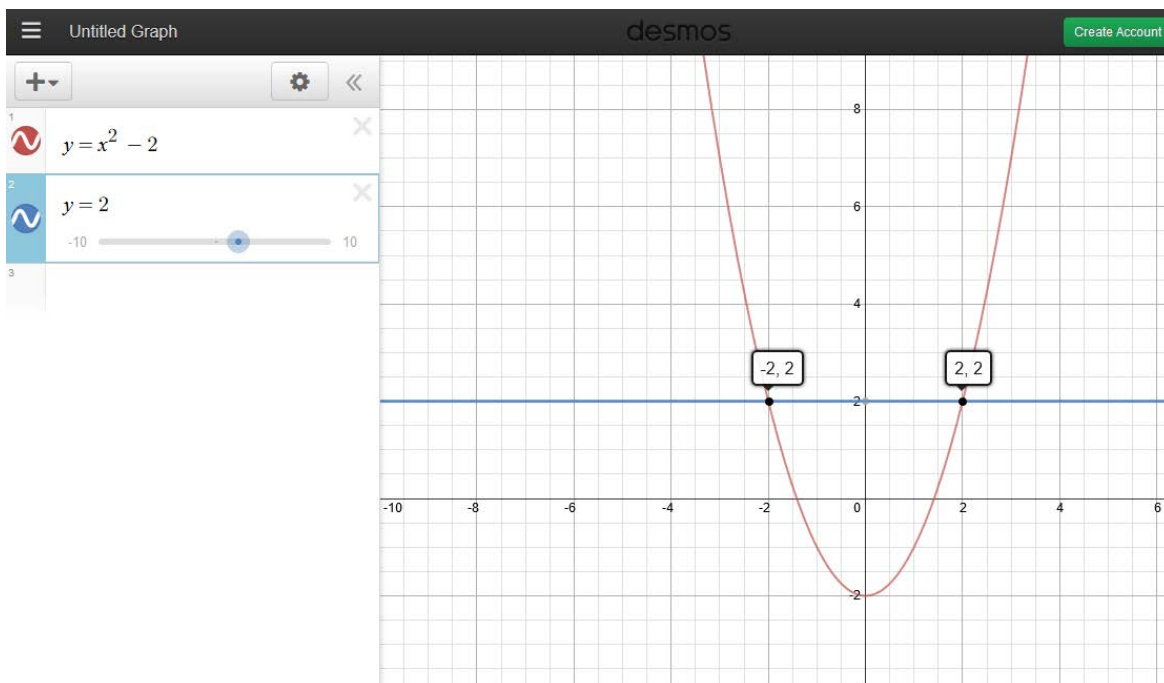


Enter the equations on the left. You do not have to type the $f(x) =$ portion. Just type " $3x - 7$ " in the first rectangle and " $2x + 1$ " in the second rectangle. You may need to click and "pull" the graph down by moving your mouse in order to see the intersection point. You can also zoom in and out by clicking on the + and - on the right hand side of the screen. In this example, you may be able to easily read the intersection point from the graph. Find (8, 17) and you should see the lines intersect. However, should a point not cross at a whole number or be difficult to tell for sure, you can click on the intersection point and the point will then show for you. Like this:



The intersection point for $y = 3x - 7$ and $y = 2x + 1$ is $(8, 17)$.

Example #2: Find the intersection point(s) for the equations $y = x^2 - 2$ and $y = 2$. First, clear the equations you already used. Just use the backspace key. Now, type the new equations into <https://www.desmos.com/calculator> to get the graphs. To type an exponent, use the ^ symbol. This is above the 6 key. You will type $y = x^2 - 2$ for the first equation. For the second, type $y = 2$. Notice that to the left of the equation, you see a color. This will indicate which graph is which when you see it. It is likely that you will be able to easily identify the points of intersection, but if not, click on them and they will show. It should look like this:



The intersection points are $(-2, 2)$ and $(2, 2)$.

You should also know how to find the intersection points algebraically. This is the same as finding the solutions to a system of equations in units 10-11 as this is a system of equations. Only this time, one is linear and one is quadratic. This system is pretty simple as you already know $y = 2$. Simply substitute $y = 2$ into the first equation like this: $2 = x^2 - 2$. Now solve for x .

$$2 = x^2 - 2$$

$$4 = x^2 \quad \text{Add 2 to both sides.}$$

$$\sqrt{4} = \sqrt{x^2} \quad \text{Take square root of both sides.}$$

$$x = \pm 2$$

Therefore, the points of intersection are $(-2, 2)$ and $(2, 2)$.

If given a more complicated system such as $x^2 + y^2 = 25$ and $y = x - 5$, you use the same steps you've learned for solving a system of equations. You just may have additional steps. It would look like this:

1st: Substitute $x - 5$ in for y into the first equation.

$$x^2 + (x - 5)^2 = 25$$

2nd: Solve for x .

$$x^2 + x^2 - 10x + 25 = 25$$

Multiply out the $(x - 5)^2$.

$$2x^2 - 10x = 0$$

Combine like terms and subtract 25 from both sides.

$$x(2x - 10) = 0$$

Factor.

$$x = 0 \quad \text{and} \quad 2x - 10 = 0$$

Set each factor to zero and solve.

$$2x = 10$$

$$x = 5$$

3rd: Substitute the x -values in and solve for y .

$$y = x - 5$$

$$y = 0 - 5 = -5$$

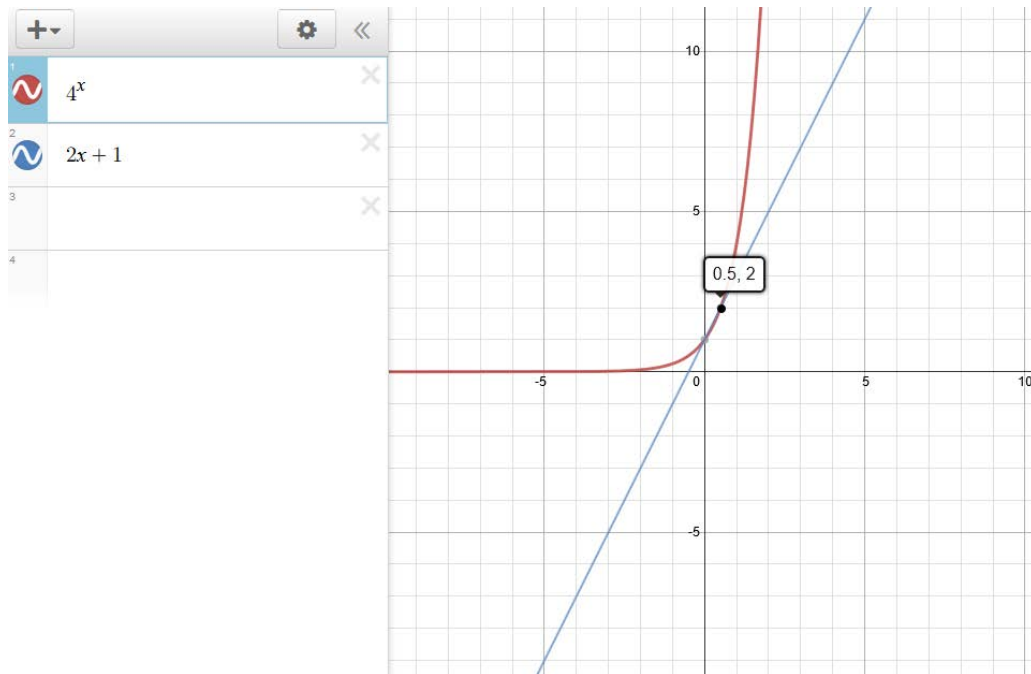
Giving the ordered pair $(0, -5)$.

$$y = 5 - 5 = 0$$

Giving the ordered pair $(5, 0)$.

You can check your solutions by graphing.

Example #3: Find the intersection points of $y = 4^x$ and $y = 2x + 1$. (Remember to clear the old equations first!) Again, you can do this by hand by making a table of values, or for linear equations, use slope to help you graph. You can use a graphing calculator or continue to use the graphing utility at <https://www.desmos.com/calculator>. Remember to use ^ when entering exponents. So, for this problem, you will enter $y = 4^x$. Click on the intersection point. It should look like this:



The intersection point for these equations is (0.5, 2).

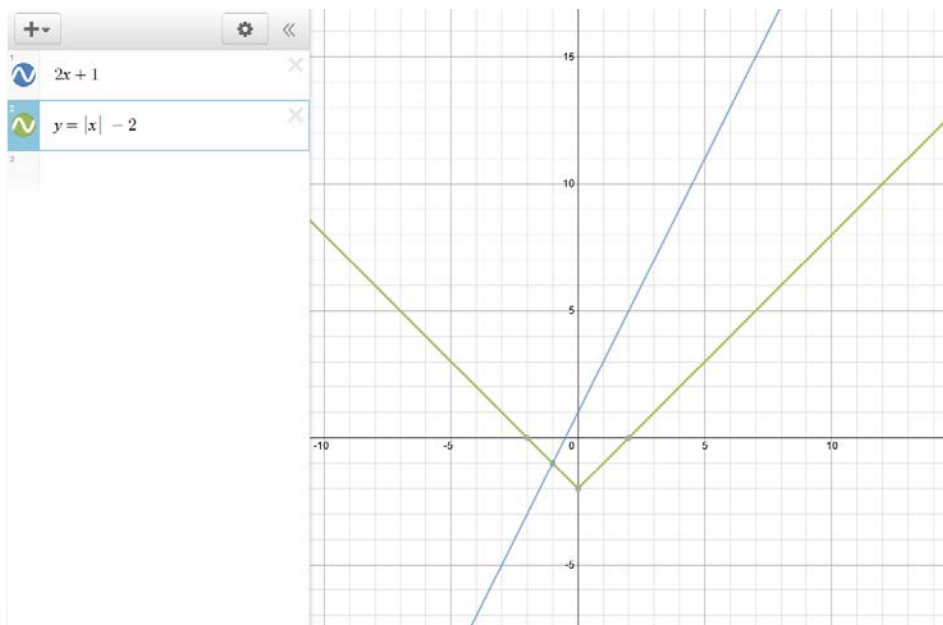
Example #4: Find the intersection point for $y = |x| - 2$ and $y = 2x + 1$. The first equation uses the absolute value symbol. To type this, hold the shift key while pushing the | key (underneath the backspace key and also with the \ key). Type these equations into <https://www.desmos.com/calculator>. You could also try making a table of values to see where the graphs intersect. Let's try this. Start with a simple table with x -values and 1 column for each equation's y -value:

x	$y = x - 2$	$y = 2x + 1$
-2		
-1		
0		
1		
2		

Now fill it in:

x	$y = x - 2$	$y = 2x + 1$
-2	$y = -2 - 2 = 2 - 2 = 0$	$y = 2 \times -2 + 1 = -4 + 1 = -3$
-1	$y = -1 - 2 = 1 - 2 = -1$	$y = 2 \times -1 + 1 = -2 + 1 = -1$
0	$y = 0 - 2 = 0 - 2 = -2$	$y = 2 \times 0 + 1 = 0 + 1 = 1$
1	$y = 1 - 2 = 1 - 2 = -1$	$y = 2 \times 1 + 1 = 2 + 1 = 3$
2	$y = 2 - 2 = 2 - 2 = 0$	$y = 2 \times 2 + 1 = 4 + 1 = 5$

Notice that you get the same y -value for $x = -1$. This means you have an intersection point at $x = -1$ and $y = -1$ or $(-1, -1)$. To check this, we will also try the graphing method. It should look like this:

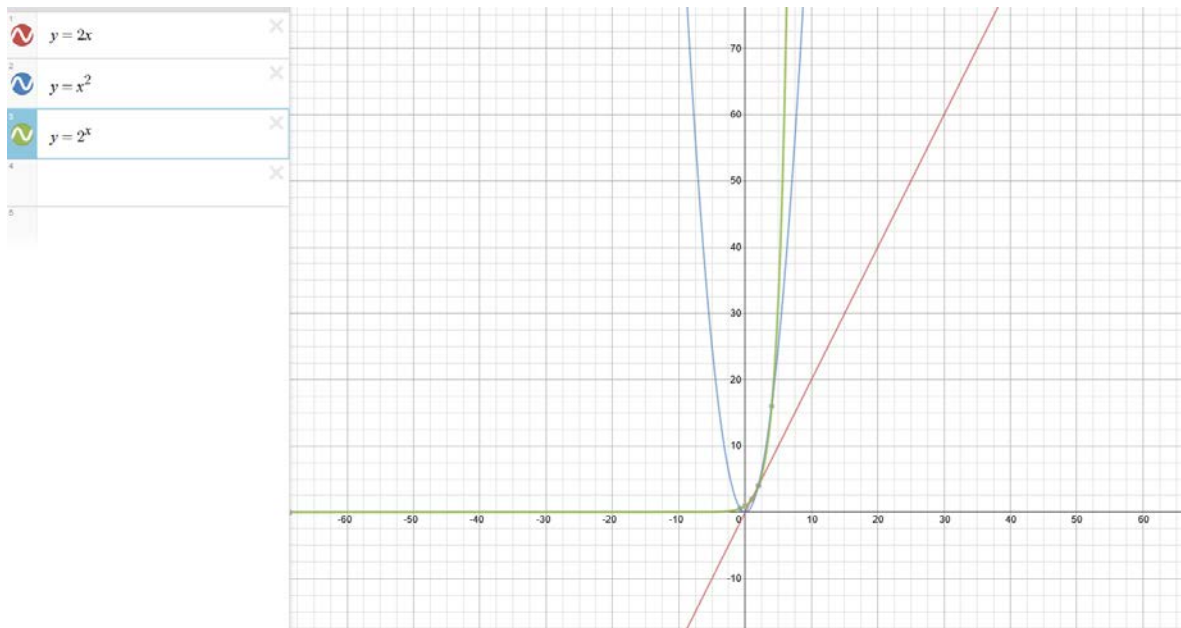


You can see from the graph that $(-1, -1)$ is an intersection point. Remember that you can also click the intersection point and the graphing utility will list it for you.

Stop! Go to Questions #1-7 about this section, then return to continue on to the next section.

Linear vs. Quadratic vs. Exponential Functions

Now that we've graphed many equations and found intersection points, let's look at the behavior of these graphs. Let's put 3 of our functions together. Using <https://www.desmos.com/calculator>, first clear any equations you have previously typed. For this problem, graph $y = 2x$, $y = 2^x$, and $y = x^2$. We are not looking for intersection points, but at graph behavior. The first thing to notice is this: which graph increases the fastest? Look at the linear graph (the straight line). How fast does it increase? Now, look at the quadratic graph (x^2). Now look at the exponential graph (2^x). Notice that the exponential graph begins increasing much faster than the others. The quadratic and exponential graph quickly overtake the linear graph, but the exponential graph begins increasing even faster than the quadratic graph. In other words, for each x -value, the y -value is getting much larger for the exponential graph than for the linear or quadratic graph.



Click on the link to watch the video "[Comparing exponential and quadratic models](#)" or click on the video.

Comparing exponential and quadratic models | Algebr...

will be shipped to Japan during the next t months is modeled by the function $j(t) = 2^t$. The number of cars that will be shipped to Vietnam during the next t months is modeled by the function $v(t) = 2t^2$.

Which country had received more cars from the Cozy Car Company after 5 months?

$j(5) = 2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ (Vietnam)

$v(5) = 2(5)^2 = 50$

Which country had received more cars from the Cozy Car Company after 7 months?

$j(7) = 2^7 = 2^5 \times 2 \times 2 = 128$ Japan

$v(7) = 2(7)^2 = 2 \cdot 49 = 98$

Will the country which received more cars from the Cozy Car Company after 7 months continue to receive more cars than the other country in future months?



Do practice here: [Compare the growth rates of exponentials and polynomials](#)

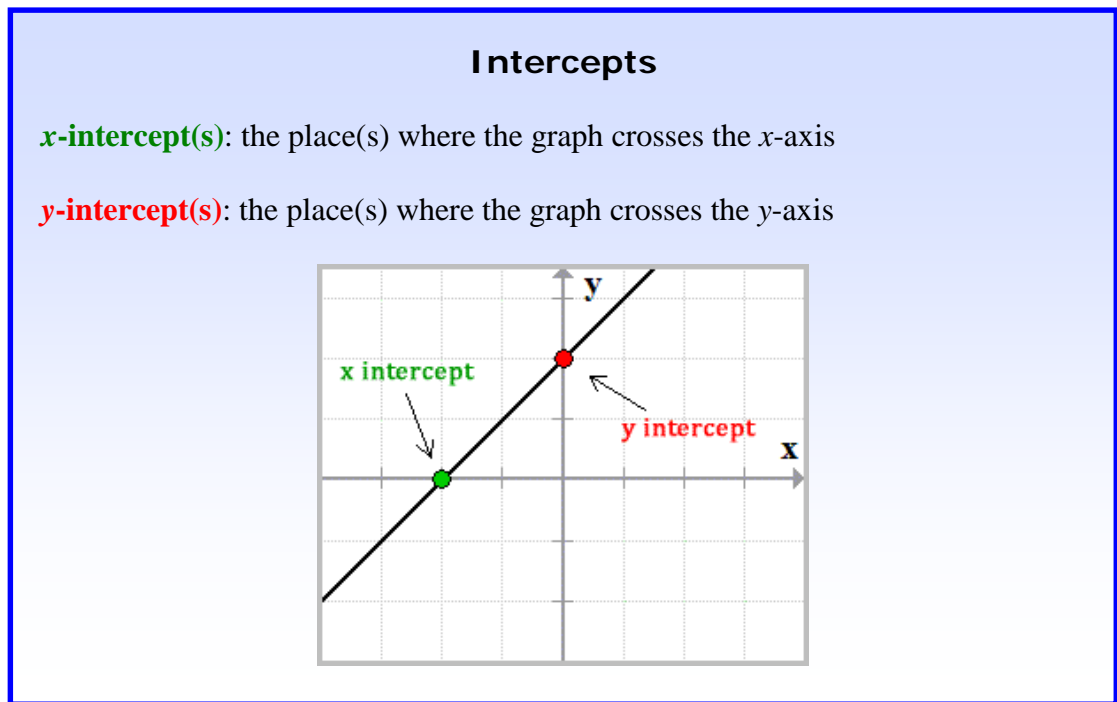
Stop! Go to Questions #8-14 about this section, then return to continue on to the next section.

Function Features

In this section, you will identify key features of a function including intercepts, intervals where the function is increasing or decreasing, intervals where it is positive or negative, maximums and minimums, and end behavior. You will also look at features of trigonometric functions including amplitude, midline, and periodicity.

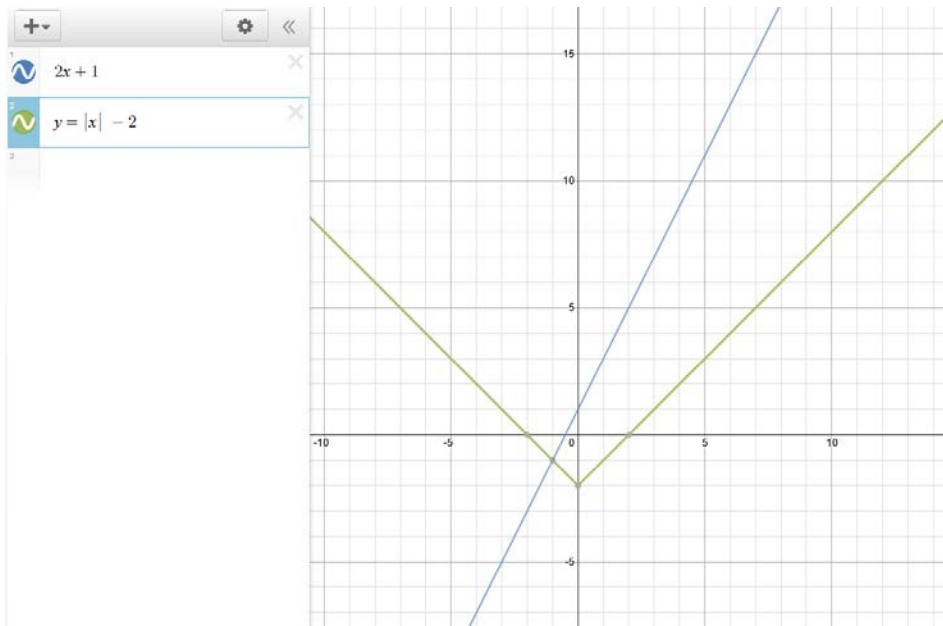
Finding x-intercepts and y-intercepts

You may recall from an earlier unit, finding x - and y - intercepts for a graph. Remember that if you have an equation in slope-intercept form ($y = mx + b$), you can easily read the y -intercept (b) from the equation. However, if given the graph, you should also be able to read the intercepts from the graph.

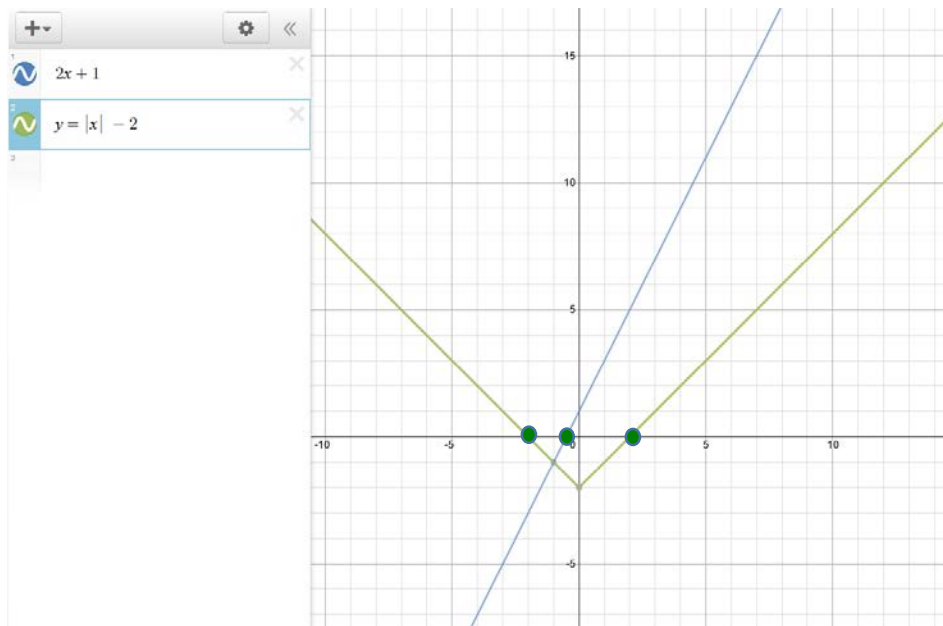


Example #1: Recall the equations and graphs from earlier in the lesson, $y = 2x + 1$ and $y = |x| - 2$.

Their graphs are below.



The blue graph is $y = 2x + 1$ and the green graph is $y = |x| - 2$. What are the x -intercepts for each function? Remember the x -intercepts are where the graph crosses the x -axis. Because the x -axis is where $y = 0$, you could solve algebraically by substituting 0 for y and then solving for x . Since we are given the graph, if it crosses at nice, integer values, we can read these directly from the graphs.



The x -intercepts have been marked with green dots. Notice the x -intercepts for the absolute value graph does cross at integer values and is easy to read from the graph. The x -intercepts are $(-2, 0)$ and $(2, 0)$. The linear equation (in blue) does cross between 0 and -1 . To solve this, we can substitute 0 for y and solve algebraically.

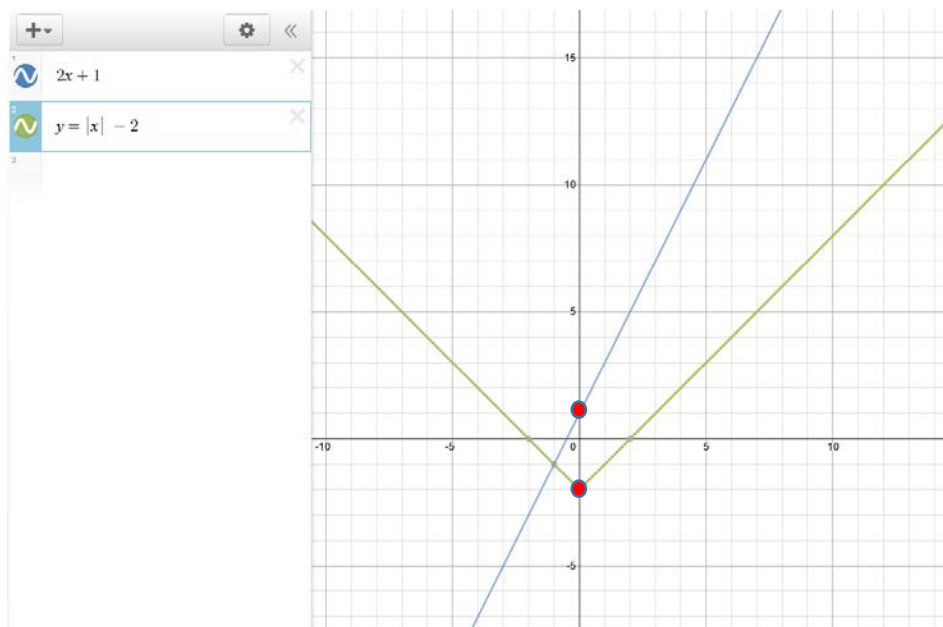
$$0 = 2x + 1$$

$$-1 = 2x \quad \text{Subtract 1 from both sides.}$$

$$-\frac{1}{2} = x \quad \text{Divide both sides by 2.}$$

$$\left(-\frac{1}{2}, 0\right) \quad x\text{-intercept.}$$

The y-intercepts are found by looking at where the graphs cross the y-axis.



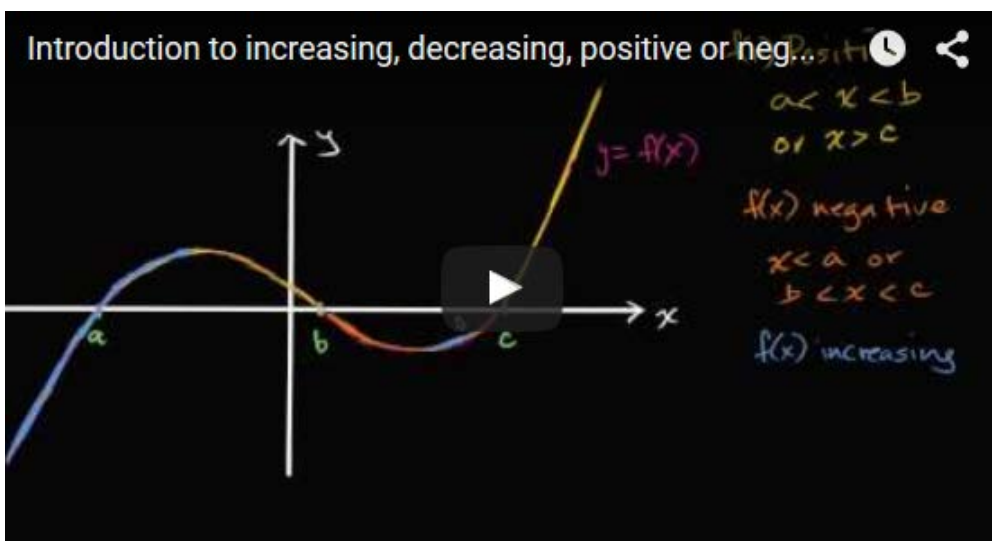
The red dots above show where the two graphs intersect (or cross) the y-axis. You can read these points from the graph. For $y = 2x + 1$, the y-intercept is $(0, 1)$. For $y = |x| - 2$, the y-intercept is $(0, -2)$. Remember the y-axis is where $x = 0$, so if the point cannot be easily read from the graph, you can solve algebraically by substituting $x = 0$ into the equation and solving for y .

Stop! Go to Questions #15-17 about this section, then return to continue on to the next section.

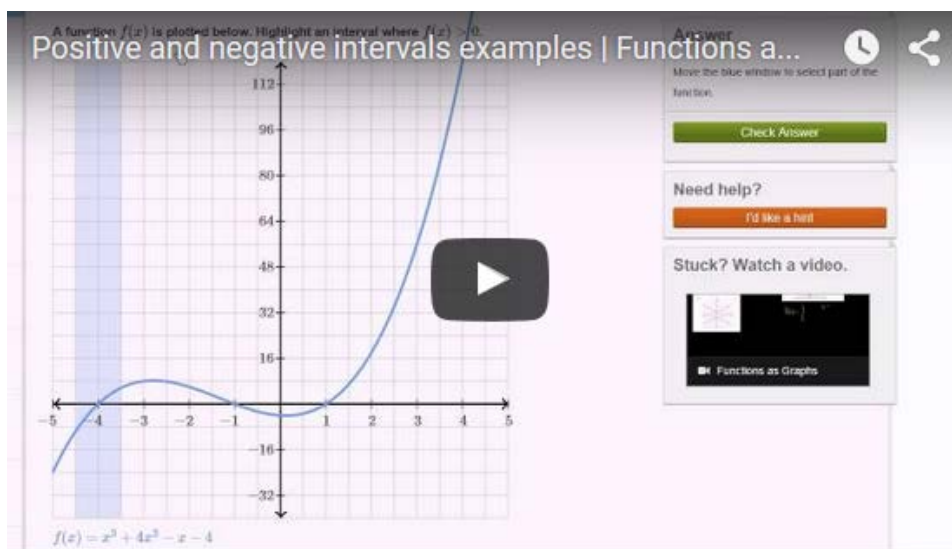
Increasing/Decreasing and Positive/Negative Intervals

Watch this video that explains when a function is increasing or decreasing as well as intervals where the function is positive and where the function is negative.

Click on the link to watch the video "[Introduction to increasing, decreasing, positive or negative intervals](#)" or click on the video.



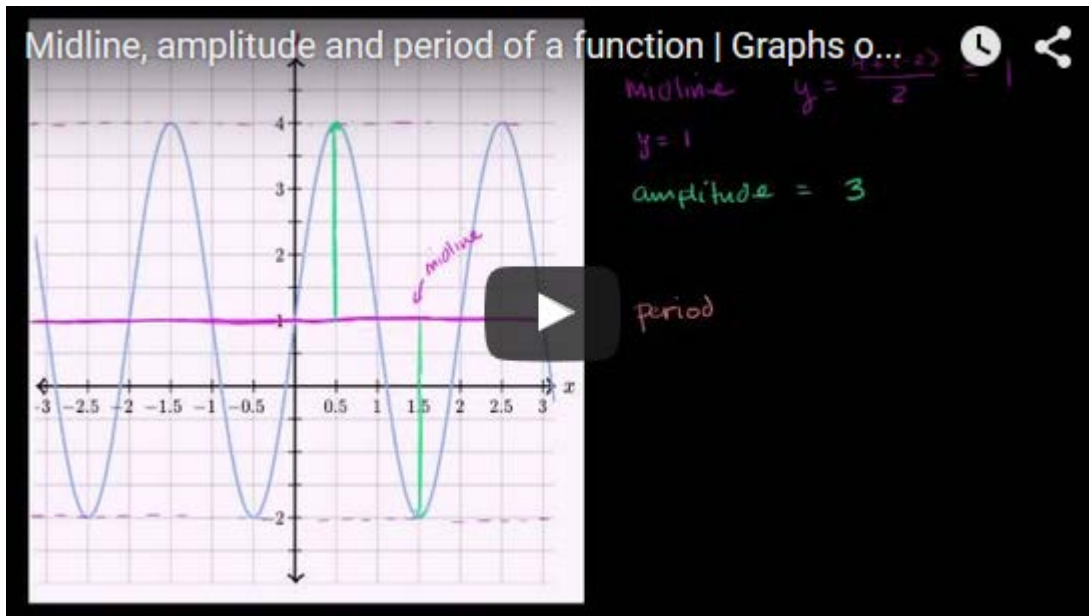
Click on the link to watch the video "[Positive and negative intervals examples | Functions and their graphs](#)" or click on the video.



Stop! Go to Questions #18-26 about this section, then return to continue on to the next section.

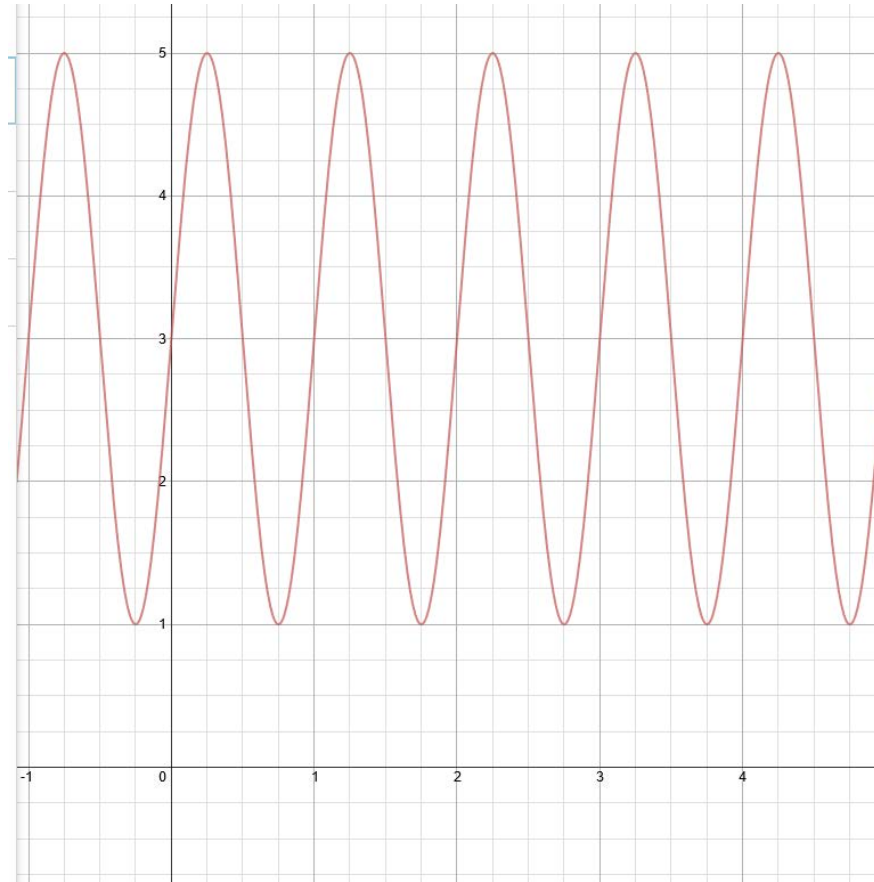
Trigonometric Functions

Click on the link to watch the video "[Midline, amplitude and period of a function | Graphs of trig functions | Trigonometry](#)" or click on the video.





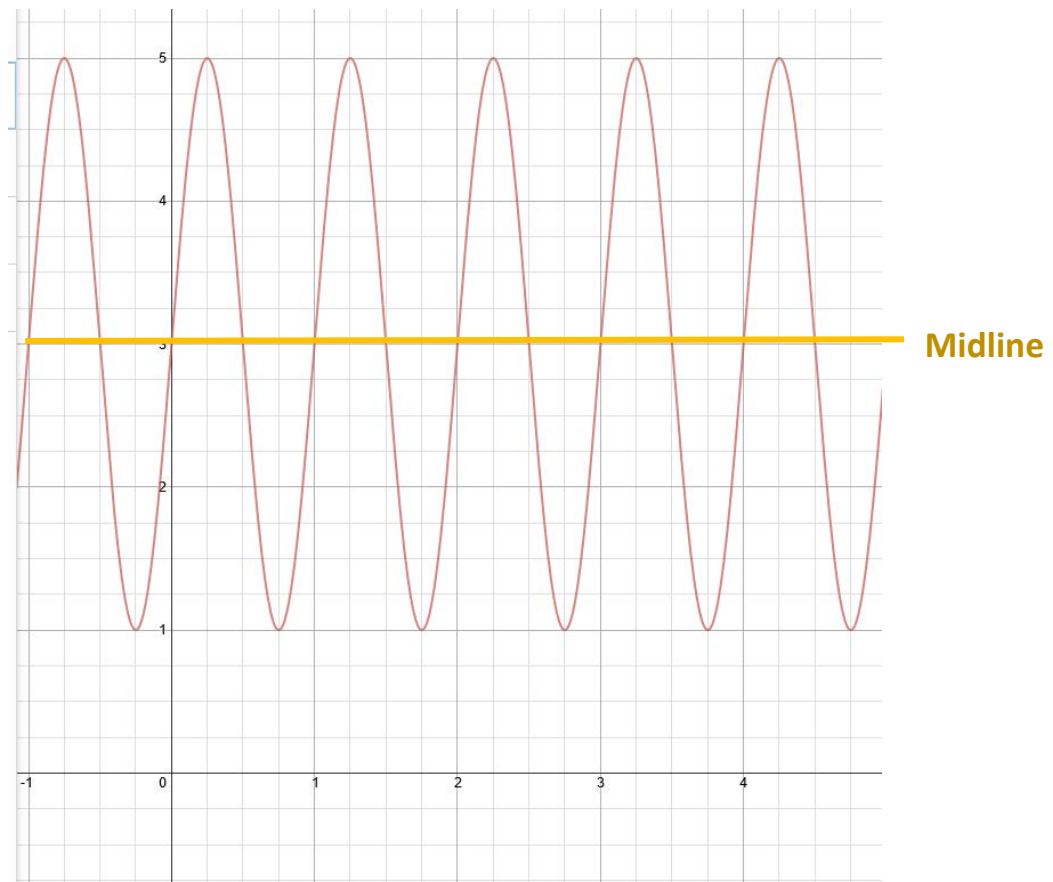
Try this! Find the midline, amplitude, and period of the following graph.



Click [here](#) to check your answer for midline.

$$\text{Midline: } y = \frac{1+5}{2} = \frac{6}{2} = 3$$

*Notice that the graph repeats between $y = 1$ and $y = 5$.
Halfway between is 3, therefore the midline is $y = 3$.*

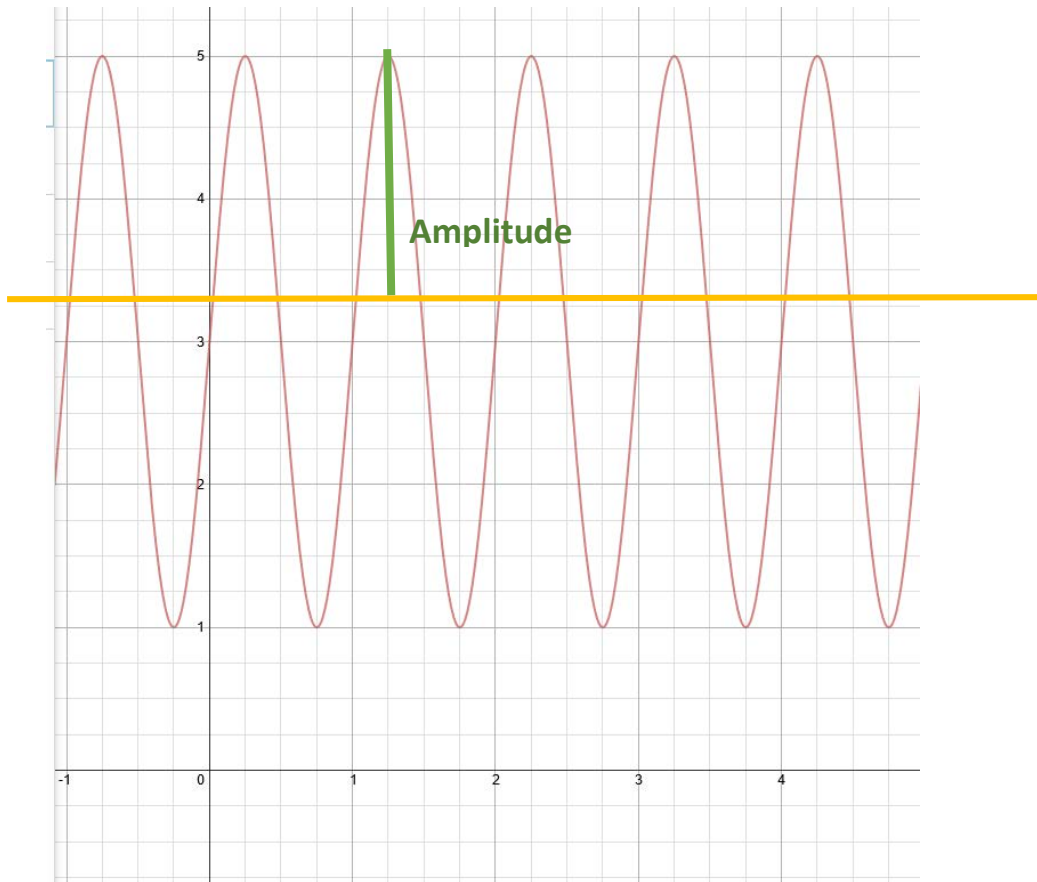


Click [here](#) to check your answer for amplitude.

The amplitude is how much the graph varies from the midline.

The graph goes from 3 to 5 or 3 to 1. The distance is 2 so the amplitude = 2.

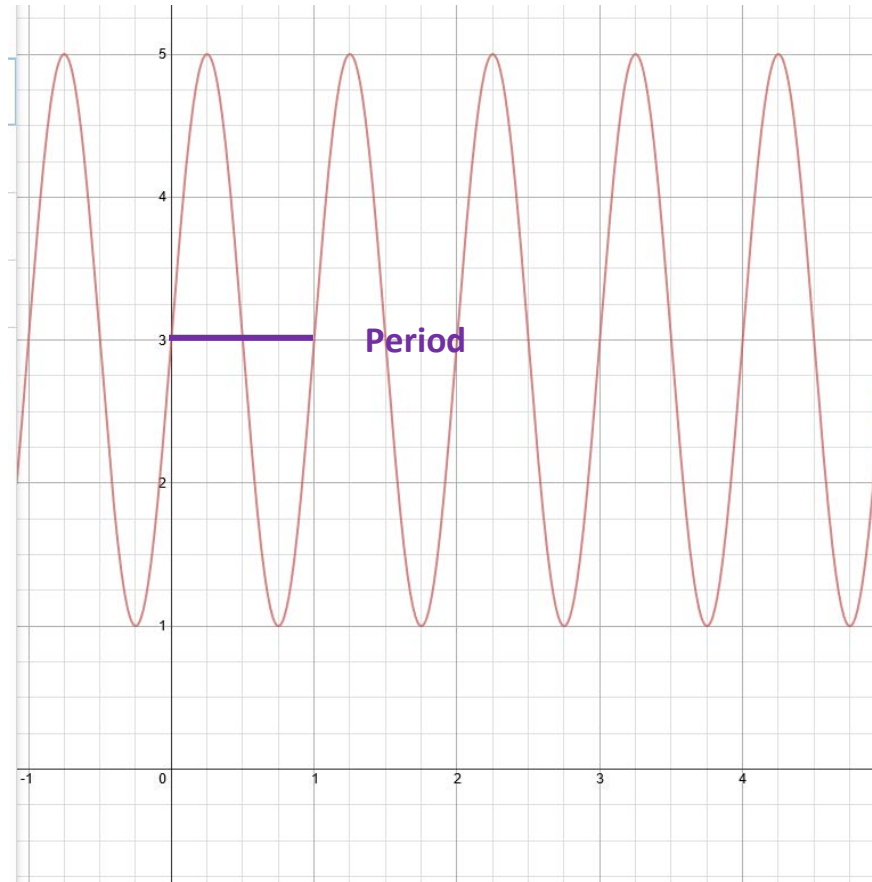
You can calculate this by taking the maximum value – the midline value or $5 - 3 = 2$.



Click [here](#) to check your answer for period.

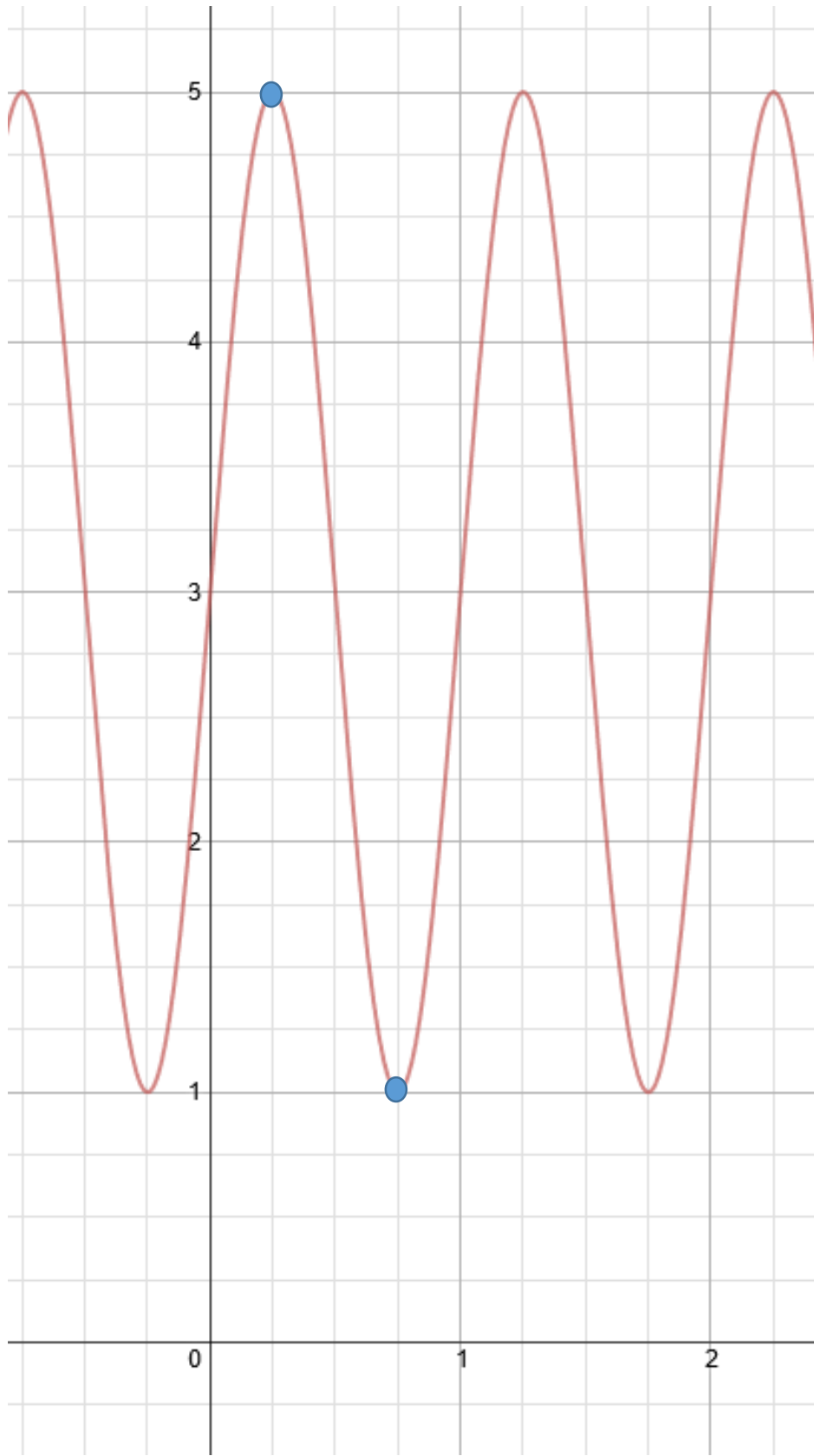
Period = 1. If you start at the point (0, 3), the graph completes a cycle at (1, 3).

The distance from left to right is 1 or $1 - 0 = 1$.



You have already learned how to find the maximum and minimum points on a parabola. This is the vertex point and the highest or lowest point on a graph. However, when we have a graph that oscillates, or repeats a pattern like the trigonometric function above, we also look for relative maximums and relative minimums. This is like the vertex point. It is the highest or lowest point for a particular section of the graph. If given a domain of $0 < x < 1$, you may be asked for the relative max or relative min point. To do this, look at the graph between 0 and 1 on the x -axis. Find the lowest (min) or highest (max) point depending on what you are asked. For the graph in the previous problem, find the relative max and relative min for the domain of $0 < x < 1$.

Answer: The maximum point on the graph when x is between 0 and 1 is (0.25, 5). Because the increments on the x -axis are broken up into fourths, you can easily see the x -value is $\frac{1}{4}$ or 0.25. The y -value is 5. For the minimum point, look for the lowest point on the graph at $(\frac{3}{4}, 1)$ or (0.75,1). See the points highlighted below:



There are other local maximums and minimums as well if given a different domain or interval for x .

Stop! Go to Questions #27-36 about this section, then return to continue on to the next section.

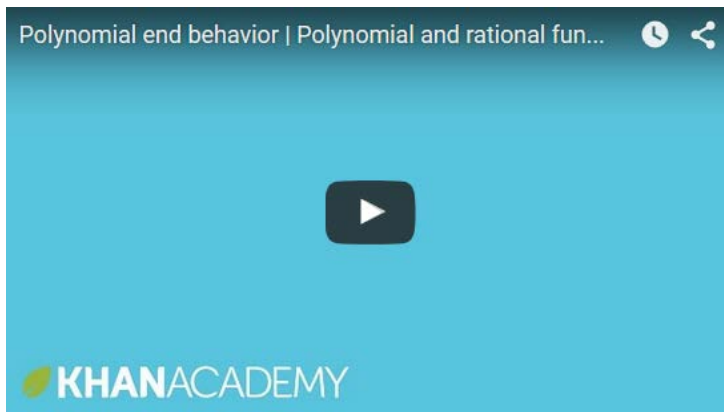
End Behavior

End behavior is how we describe what a graph does at the “ends” or how it continues on in both directions. We describe what a graph does as x continues to positive infinity and what the graph does as x continues to negative infinity.

Watch these videos to see examples of end behavior.

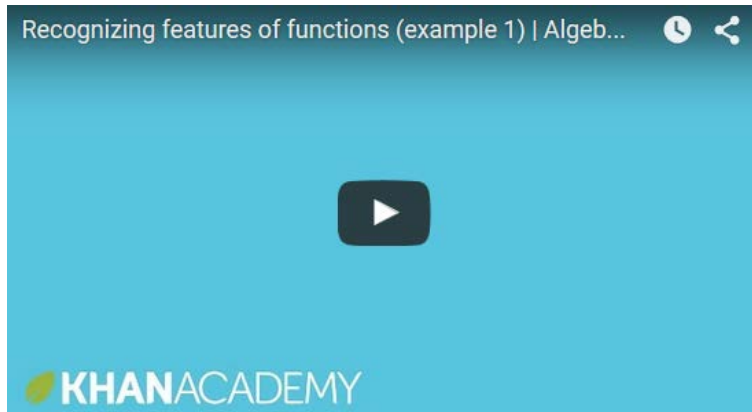
Video 1:

Click on the link to watch the video "[Polynomial end behavior | Polynomial and rational functions](#)" or click on the video.



Video 2:

Click on the link to watch the video "[Recognizing features of functions \(example 1\) | Algeb...](#)" or click on the video.



You will not need to understand how to graph the equations of higher degree for this section. You will need to be able to look at a graph and tell the end behavior by seeing the graph. You are expected to know how to graph linear and quadratic equations.

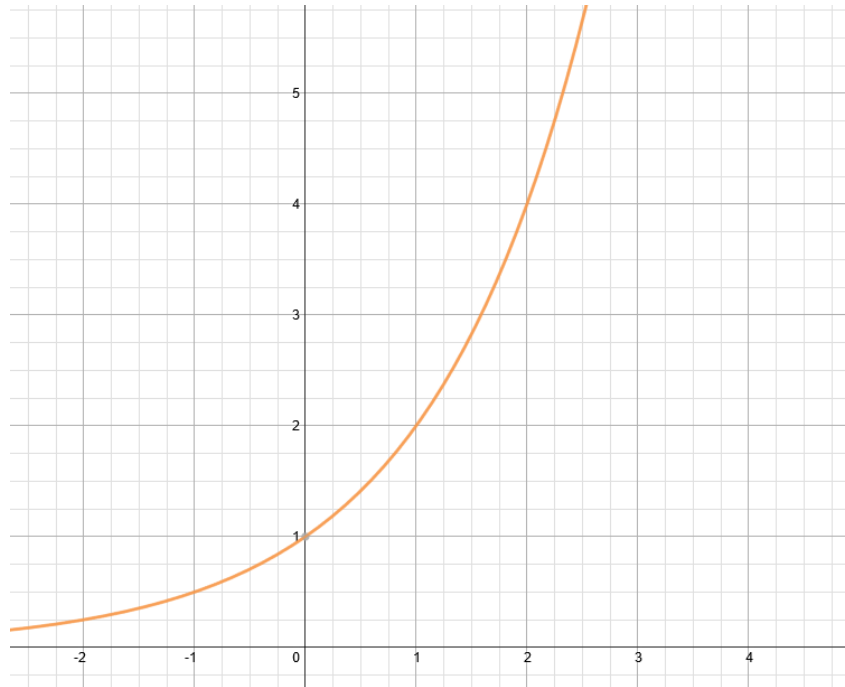
Example #1: Consider the logarithmic function $y = \log x$. The graph looks like this:



The end behavior as x approaches 0, y approaches negative infinity. As x approaches positive infinity, y approaches positive infinity. If asked, you could also identify where the function is positive as $x > 1$.



Try this! Look at the graph below and state the end behavior as x becomes more positive and as x becomes more negative. As review, also state the y -intercept by looking at the graph as well as the interval where the function is positive. This is an exponential function, $f(x) = 2^x$.



[“Click here” to check your answer](#)

1. End Behavior

As x goes to negative infinity (or becomes more negative), the y -value approaches zero.

As x goes to positive infinity (or becomes more positive), the y -value approaches positive infinity.

2. y -intercept

The y -intercept is $(0, 1)$ and is read from the graph. Remember the y -intercept is the point where the graph crosses the y -axis.

3. Interval where the function is positive

The function is positive (meaning y is positive) from negative infinity to positive infinity.

The function is always positive for all values of x .

Stop! Go to Questions #37-41 about this section, then return to continue on to the next section.

Word Problems

Example #1: Study the two functions shown below. Compare the x -intercepts and find the difference between the minimum values.

1. $f(x) = x^2 + 8x + 15$

2. $g(x)$ see table.

x	-7	-6	-5	-4	-3	-2	-1
$g(x)$	4	1.5	0	-0.5	0	1.5	4

To compare these functions, it will help if the functions are both in the same form. We could try writing $g(x)$ as an equation or $f(x)$ as a table or we could try graphing both of them. Let's try making a table for $f(x)$. For ease of comparison, we will use the same x values used in $g(x)$. Start by making a table using the same x -values as $g(x)$.

x	-7	-6	-5	-4	-3	-2	-1
$f(x)$							

Notice that the 2nd table is labeled for $f(x)$. Now plug these values into $f(x)$ and fill in the table.

$$f(-7) = (-7)^2 + 8(-7) + 15 = 49 + -56 + 15 = 8$$

$$f(-6) = (-6)^2 + 8(-6) + 15 = 36 + -48 + 15 = 3$$

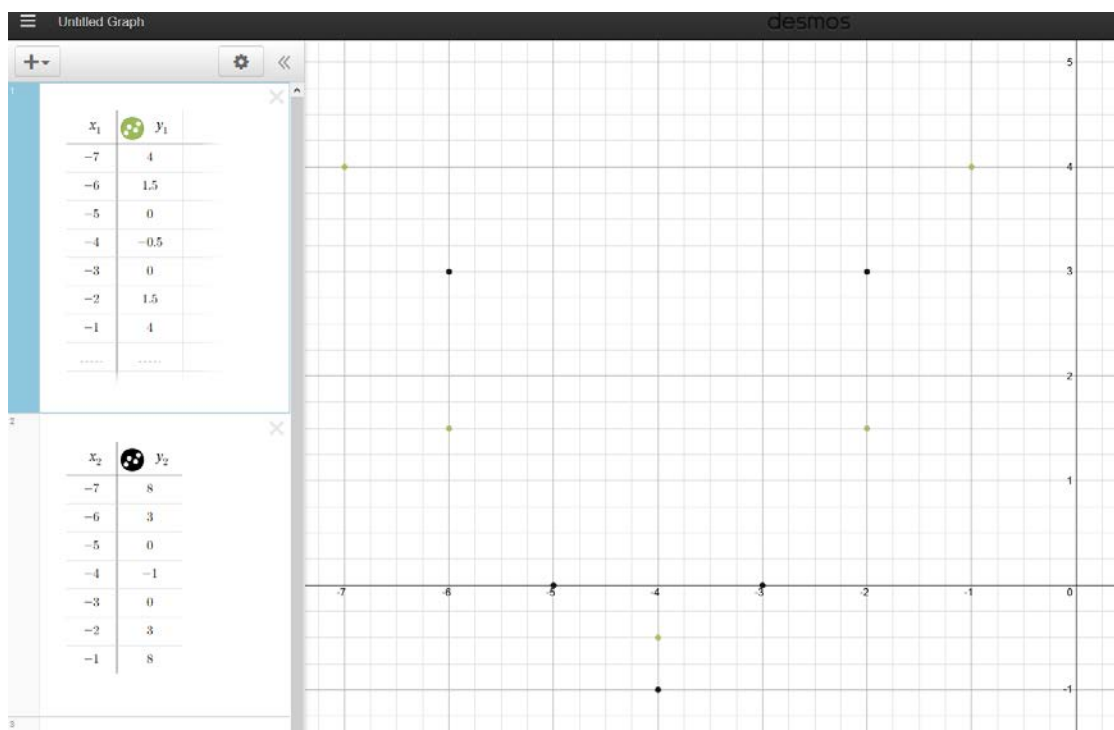
$$f(-1) = (-1)^2 + 8(-1) + 15 = 1 + -8 + 15 = 8$$

Not all values have been done for you. You will need to finish. When you fill in the table, it should look like this: Click [here](#) to view the table.

x	-7	-6	-5	-4	-3	-2	-1
$f(x)$	8	3	0	-1	0	3	8

Now, we can compare the tables. We were asked to compare the x -intercepts. Remember that x -intercepts are points directly on the x -axis (where $y = 0$). So, let's look for function values of 0. This happens for $g(x)$ at -5 and -3 . Therefore, the x -intercepts for $g(x)$ are $(-5, 0)$ and $(-3, 0)$. The x -intercepts for $f(x)$ are -5 and -3 (read from table where $f(x) = 0$). Therefore, the x -intercepts for $f(x)$ are $(-5, 0)$ and $(-3, 0)$. The x -intercepts are the same for both functions! We are also asked to find the difference between the minimum values. The minimum value is the lowest function value. For $g(x)$, the lowest value is -0.5 . For $f(x)$, the lowest function value is -1 . The difference is what we get when we subtract these values. Because we are looking at distance, we want a positive answer. You can subtract in any order, but keep the answer positive. So, $-1 - (-0.5) = -1 + 0.5 = -0.5$. The difference (or distance) is 0.5 .

Another way to compare these functions is to make a graph for each. You could do this easily by hand and plot the points. When you do, you will notice that they both take a parabola shape. You could also use a graphing utility. In the following graph, <https://www.desmos.com/calculator> was used. To do this yourself, click the drop down menu on the $+$ on the top left. Choose table. You will then be able to enter table values instead of an equation. Click $+$ again to add the second table. In the graph below, the green points represent $g(x)$ and the black points represent $f(x)$. However, since some points were the same, they are the second color black.



Using the graph, you can easily read the x -intercepts from the graph. If you find $g(x)$ intercepts first before entering $f(x)$, they will still be green and you will then see that you've written the same points for both functions. You can also read the minimum points from the graph by finding the points with the lowest y -value. To find the difference, you will still need to subtract.

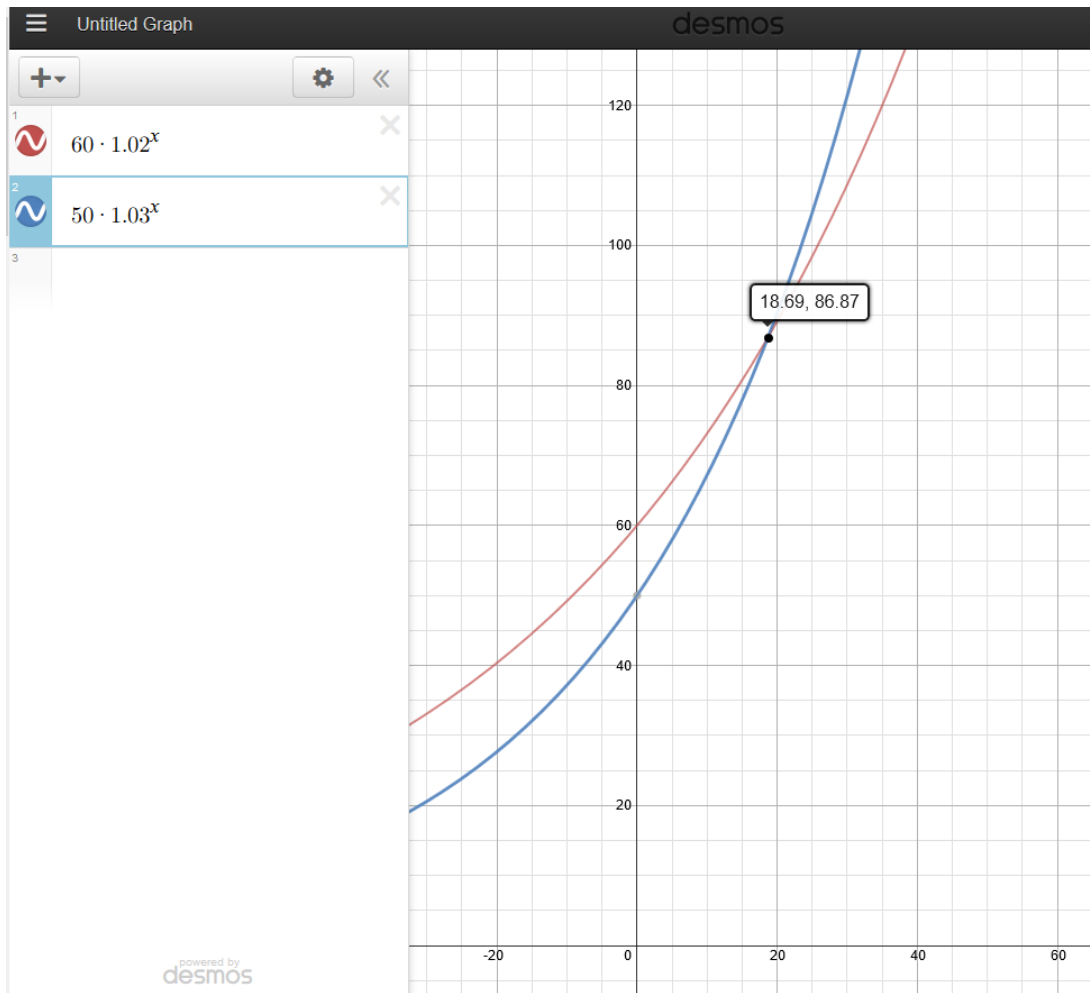
Example #2: Jacob has two job offers. Job A offers him \$60 per week with a 2% raise each year. Job B offers him \$50 per week with a 3% raise each year. Which job would earn Jacob the most money over several years?

To solve this, make a table for each scenario. Remember that each year's raise is based off a percentage of the current salary. So, do not just find 4% of \$60 for the second year. Find and add 2% to year 1 for year 2. Round all answers to the nearest penny.

	Job A	Job B
Year 0 (first year of working, no raise yet)	\$60.00	\$50.00
Year 1 (at end of first year, 1 st raise)	$60 + (.02 \times 60) = 61.20$	$50 + (.03 \times 50) = 51.50$
Year 2	$61.20 + (.02 \times 61.20) = 62.42$	$51.50 + (.03 \times 51.50) = 53.05$
Year 3	$62.42 + (.02 \times 62.42) = 63.67$	$53.05 + (.03 \times 53.05) = 54.64$
Year 4	$63.67 + (.02 \times 63.67) = 64.94$	$54.64 + (.03 \times 54.64) = 56.28$
Year 5	$64.94 + (.02 \times 64.94) = 66.24$	$56.28 + (.03 \times 56.28) = 57.97$
Year 6	$66.24 + (.02 \times 66.24) = 67.56$	$57.97 + (.03 \times 57.97) = 59.71$

Notice that at year 6, job A has increased \$7.56. Job B has increased \$9.71. Job B is increasing at a faster rate, and given enough years, will be more than Job A. However, Job A started out paying more and is still paying more. To decide the better job for pay, you will need to know how long you will stay at a job to see if it is worth it to go with a lower salary with a greater raise each year.

*Note: Using your knowledge of exponential growth, you may also recognize this as an exponential growth function. Job A pays $60(1.02)^t$ where t is in years. Job B pays $50(1.03)^t$. If you recognize this, you could graph these equations and see the points at which the jobs pay the same and when Job B begins paying more than Job A. Plug these equations into <https://www.desmos.com/calculator> and see what happens.

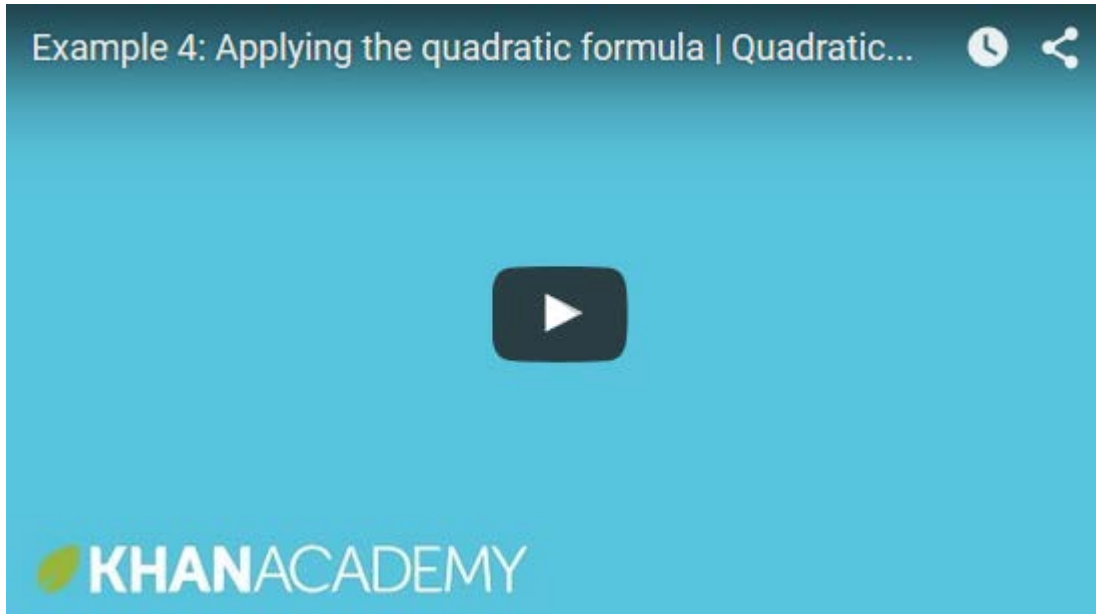


Looking at this graph, you see the initial starting points of $(0,50)$ and $(0,60)$. Notice the intersection at $(18.69,86.87)$. This represents the points at which the salaries will be the same. Since the salary is only calculated each year, at year 19, the salaries are the same and would be near \$87. Not knowing if Jacob would want to work at a job that long, he may be better off taking Job A and making more money for 19 years.

Example #3: A ball is thrown straight up into the air starting at 5 feet with an initial upward velocity of 45 feet per second. Write a function rule that relates the height (in feet) of the ball and the time (in seconds). Note that gravity has an effect on the height with the distance fallen represented by $16t^2$ where t is time in seconds.

To write this equation, look at what you know. The ball starts at 5 feet in the air. This could be where a person is holding it with their arm (above the ground of 0 feet) or maybe the ball is tossed from someone standing on something. We know the upward velocity is 45 feet per second or $45t$. Due to gravity always pushing down on an object, we must take into account that what goes up, must come down. For this, we will use $-16t^2$. Let h represent height and t represent seconds. Put all this together for $h(t) = -16t^2 + 45t + 5$.

Click on the link to watch the video "[Applying the quadratic formula | Quadratic equations](#)" or click on the video.



Example #4: Alliyah has a monthly car payment of \$375. She has estimated an average cost of \$0.42 per mile for gas and maintenance. She plans to budget for the car payment as well as an additional 4% for incidentals that may occur. Write a function that gives the amount Alliyah needs to budget as a function of the number of miles driven.

First, think about all of the costs involved in this scenario: the car payment, gas and maintenance, incidentals (emergencies, things that come up). To find a total, add.

Car payment = \$375

Gas and maintenance = 0.42/mile. If x = # miles driven, then $0.42 \times x$ = total for gas and maintenance.

Incidentals = 4% of car payment = $0.04 \times 375 = \$15.00$.

Write the function:

$f(x)$ = total cost where x = # miles driven

$$f(x) = 375 + 0.42x + 15$$

Simplify

$$f(x) = 390 + 0.42x$$

Watch these videos for additional examples:

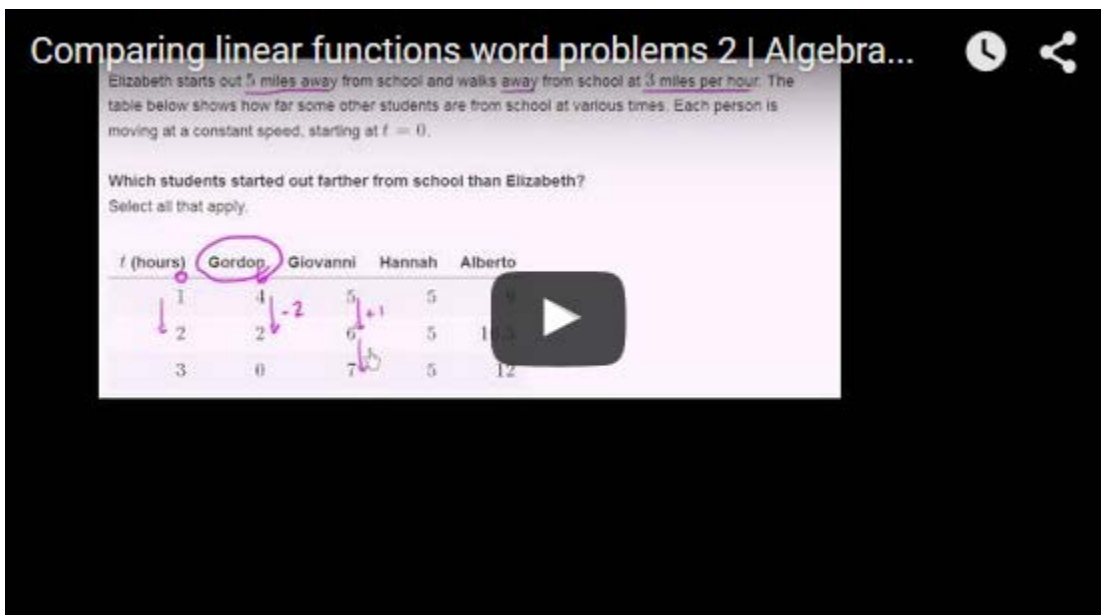
Video 1:

Click on the link to watch the video "[Modeling with combined functions | Functions and their graphs](#)" or click on the video.



Video 2:

Click on the link to watch the video "[Comparing linear functions word problems 2](#)" or click on the



Video 3:

Click on the link to watch the video "[Comparing linear functions word problems 3](#)" or click on the video.

Comparing linear functions word problems 3 | Algebra

Charles and Tammy live the same distance from work, and they started walking to work at the same time. They both walked at constant speeds, though not necessarily at the same speed. Charles's distance from work is shown in the following table:

t (seconds)	Charles's distance from work (meters)
0	900
50	830
100	760
150	690

Which of these sentences could possibly be true? Select all that apply.

- Tammy started 900 m from work, and walked towards work at 2 meters per second.
- Tammy started 900 m from work, and walked towards work at $\frac{7}{5}$ meters per second.
- Tammy started 900 m from work, and walked towards work at $\frac{5}{7}$ meters per second.
- Tammy started 760 m from work, and walked towards work at $\frac{7}{5}$ meters per second.
- Tammy started 830 m from work, and walked towards work at 1.4 meters per second.

Practice:

[Model situations by combining functions using arithmetic operations](#)

[Constructing linear functions word problems](#)

Stop! Go to Questions #42-47 to complete this unit.