OPERATIONS WITH RADICALS

Unit Overview

This unit is about radical expressions which are important in algebra and geometry. Understanding radicals is essential in solving quadratics using the quadratic formula and the distance formula.

Definition of a Square Root

Square Root

If x is a number greater than or equal to zero, $\sqrt{x}$ represents the positive square root of x and $-\sqrt{x}$ represents the negative square root of x.

$$\sqrt{x}\cdot\sqrt{x}=x,\ (-\sqrt{x})(-\sqrt{x})=x$$

To understand the terminology of radicals, study the illustration below.

radical sign

$$\pm\sqrt{a}$$

positive or negative

radicand

To evaluate a square root means to determine what number times itself is the radicand.

Examples: Simplify

a.) $\sqrt{36}$ b.) $-\sqrt{121}$ c.) $\pm\sqrt{16}$ d.) $\sqrt{5}$

a.) $\sqrt{36}=6$ because $6\cdot 6=36$, $\sqrt{36}$ means the positive square root.

b.) $-\sqrt{121}=-11$ because $-(\sqrt{121})=-(11)=-11$.

c.) $\pm\sqrt{16}=\pm 4$ because both the positive and negative roots are indicated.

d.) $\sqrt{5}$ is irrational because 5 is not a perfect square. The two perfect squares that 5 falls between are 4 and 9, so $\sqrt{5}$ will be between 2 and 3, the square roots of 4 and 9.

Simplifying Radicals

Sometimes it will be necessary to simplify radicals to produce like radicands. A square root is in simplest form when:

- There are no perfect square factors of the radicand.
- The radicand is not a fraction.
- No radical is in the denominator.

To simplify a radical

a.) Look for a perfect square factor.

Perfect squares are numbers that are squares of integers. Perfect squares can be calculated as follows: $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, $5^2 = 25$, and so on.

The first 15 perfect squares are listed below:

{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225...}

b.) Factor the radicand using the perfect square.

c.) Leave any factors that are not perfect squares under the radical.

Example #1: Simplify $\sqrt{18}$

a.) Eighteen contains a perfect square factor of 9.

b.) Factor the radicand using the perfect square of 9.

$$\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2}$$

c.) Write $\sqrt{9}$ as 3 and simplify.

$$\sqrt{18} = 3 \times \sqrt{2}$$

$$\sqrt{18} = 3\sqrt{2}$$

Example #2: Simplify $\sqrt{275}$

$$\sqrt{275} = \sqrt{25 \cdot 11}$$

$$= \sqrt{25} \cdot \sqrt{11}$$

$$= 5 \cdot \sqrt{11}$$

$$= 5\sqrt{11}$$

*If the radicand contains a variable, the rules are a little different.

Because the radical symbol designates the positive square root, the value of $\sqrt{a^2}$ must be positive. To ensure this, absolute value signs are used to indicate this when the exponent of a variable in the radicand is even and the simplified exponent outside is odd. For our purposes in this course, we will assume that the variables produce a positive result.

Example #3: Simplify $\sqrt{a^8 b^{12} c}$

*Simplify this radical by asking how many times "2" will go into each of the exponents. (This is because the index of a square root, $\sqrt{}$, is 2.) The result is what the new exponent will be outside the radical. If there is anything left over, this remains under the radical sign.

* 2 goes into 8, 4 times $\sqrt{a^8 b^{12} c}$

* 2 goes into 12, 6 times $\sqrt{a^8 b^{12} c}$

* 2 does not go into 1 evenly. $\sqrt{a^8 b^{12} c^1}$

$$\sqrt{a^8 b^{12} c} = \sqrt{\left(a^4\right)^2 \cdot \left(b^6\right)^2 \cdot c}$$

$$= a^4 b^6 \sqrt{c}$$

*No absolute value signs are necessary in this answer since the simplified exponents of the variables outside the radicand are even.

Example #4: $\sqrt{x^5y^{10}}$

*Simplify this radical by asking how many times "2" will go into each of the exponents. (This is because the index of a square root, $\sqrt{\ }$, is 2.) The result is what the new exponent will be outside the radical. If there is anything left over, this remains under the radical sign.

* 2 goes into 5, 2 times, with one left over.

* 2 goes into 10, 5 times, with none left over, but the result is an odd exponent, so you should use absolute value signs.

$$\sqrt{x^5y^{10}} \qquad = \sqrt{x^4\cdot x\cdot y^{10}} = \sqrt{\left(x^2\right)^2\cdot x\cdot\left(y^5\right)^2}$$

$$x^2y^5\sqrt{x}$$

$$x^2\,|\,y^5\,|\sqrt{x}$$

Example #5: Simplify $\sqrt{x^3y^7}$

-divide both, 3 and 7, by 2 to determine the new exponent outside the radical.

$$\sqrt{x^3y^7} \qquad = \sqrt{x^2\cdot x\cdot y^6\cdot y}$$

$$xy^3\sqrt{xy}$$

Try this: Simplify $\sqrt{a^5b^2c^4}$

Answer: $a^2bc^2\sqrt{a}$

Stop! **Go to Questions #1-13 about this section, then return to continue on to the next section.**

Operations with Radicals

In order to add or subtract radicals, the radicands **must** be the same. This rule is similar to combining like terms. For example, if given $4x^2 + 3y - 2x^2 + 5y$, you would be able to combine $4x^2 - 2x^2$ and $3y + 5y$ to produce the simplified expression $2x^2 + 8y$. Follow the examples below:

Example #1: Simplify $3\sqrt{2} + 6\sqrt{2}$

-Since both radicands are the same, you want to perform the operation on the numbers in front of the radicals.

$$3\sqrt{2} + 6\sqrt{2} = (3 + 6)\sqrt{2}$$
$$= 9\sqrt{2}$$

Example #2: Simplify $2\sqrt{7} + 6\sqrt{7} - 5\sqrt{7}$

-Since the radicands are the same, you want to perform the operation on the numbers in front of the radicals.

$$2\sqrt{7} + 6\sqrt{7} - 5\sqrt{7} = (2 + 6 - 5)\sqrt{7}$$
$$= 3\sqrt{7}$$

Example #3: Simplify $4\sqrt{3} + 7\sqrt{2} - \sqrt{3} + 5\sqrt{2}$

-Combine like radicands.

$$4\sqrt{3} + 7\sqrt{2} - \sqrt{3} + 5\sqrt{2} = 4\sqrt{3} - \sqrt{3} + 7\sqrt{2} + 5\sqrt{2}$$

$*-\sqrt{3} = -1\sqrt{3}$

$$= (4 - 1)\sqrt{3} + (7 + 5)\sqrt{2}$$
$$= 3\sqrt{3} + 12\sqrt{2}$$

Example #4: Simplify $3-\sqrt{24}+8-\sqrt{96}$

$3-\sqrt{24}+8-\sqrt{96}$	See if 24 and 96 can be factored using perfect squares.
$3-(\sqrt{4}\cdot\sqrt{6})+8-(\sqrt{16}\cdot\sqrt{6})$	Simplify each of the perfect squares.
$3-2\sqrt{6}+8-4\sqrt{6}$	Combine like terms.
$3+8-2\sqrt{6}-4\sqrt{6}$	Simplify.
$11-6\sqrt{6}$	

Example #5: Simplify $4+\sqrt{27}+15-\sqrt{48}$

$4+\sqrt{27}+15-\sqrt{48}$	Factor 27 and 48 using perfect squares.
$4+\sqrt{9}\cdot\sqrt{3}+15-\sqrt{16}\cdot\sqrt{3}$	
$4+3\cdot\sqrt{3}+15-4\cdot\sqrt{3}$	Simplify each perfect square.
$4+3\sqrt{3}+15-4\sqrt{3}$	Simplify.
$4+15+3\sqrt{3}-4\sqrt{3}$	Combine like terms.
$19+(3-4)\sqrt{3}$	
$19+(-1)\sqrt{3}$	Simplify.
$19-\sqrt{3}$	

Example #6: Simplify $7\sqrt{45}-11\sqrt{5}$

$7\sqrt{45}-11\sqrt{5}$	Factor 45 using a perfect square.
$7\sqrt{9}\cdot\sqrt{5}-11\sqrt{5}$	Simplify the perfect square.
$7\cdot 3\cdot\sqrt{5}-11\sqrt{5}$	
$21\cdot\sqrt{5}-11\sqrt{5}$	Multiply factors outside the radicand.
$21\sqrt{5}-11\sqrt{5}$	Simplify.
$(21-11)\sqrt{5}$	Combine like terms.
$10\sqrt{5}$	

Example #7: Simplify $4\sqrt{90}-3\sqrt{40}+5\sqrt{160}$

$4\sqrt{90}-3\sqrt{40}+5\sqrt{160}$	Factor 90, 40, and 160 using perfect squares.
$4\sqrt{9}\cdot\sqrt{10}-3\sqrt{4}\cdot\sqrt{10}+5\sqrt{16}\cdot\sqrt{10}$	Simplify the perfect squares.
$4\cdot 3\cdot\sqrt{10}-3\cdot 2\cdot\sqrt{10}+5\cdot 4\cdot\sqrt{10}$	
$12\cdot\sqrt{10}-6\cdot\sqrt{10}+20\cdot\sqrt{10}$	Multiply factors outside the radicand.
$12\sqrt{10}-6\sqrt{10}+20\sqrt{10}$	Simplify.
$(12-6+20)\sqrt{10}$	Combine like terms.
$26\sqrt{10}$	

Try this: Simplify $\sqrt{96}-4+\sqrt{24}+2$

Answer:

$$\sqrt{16}\cdot\sqrt{6}-4+\sqrt{4}\cdot\sqrt{6}+2$$
$$4\sqrt{6}\cdot -4+2\sqrt{6}+2$$
$$-4+2+(4+2)\sqrt{6}$$
$$-2+6\sqrt{6}$$

Stop! **Go to Questions #14-20 about this section, then return to continue on to the next section.**

Multiplying Radicals

To multiply radicals, the procedure is very similar to multiplying polynomials.

Example 1: Simplify $(3\sqrt{2})^2$

The second power means the product of two identical factors. Rearrange the factors and multiply.

$$
\begin{aligned}
(3\sqrt{2})^2 &= (3\sqrt{2})(3\sqrt{2}) \\
&= 3 \cdot 3 \cdot \sqrt{2} \cdot \sqrt{2} \\
&= 9\sqrt{4} \\
&= 9 \cdot 2 \\
&= 18
\end{aligned}
$$

Example 2: Simplify $\sqrt{2} \times \sqrt{6}$

The multiplication property of radicals allows multiplication of the separate radicals. Once the radicals are multiplied, simplify if necessary.

$$
\begin{aligned}
\sqrt{2} \cdot \sqrt{6} &= \sqrt{12} \\
&= \sqrt{4} \cdot \sqrt{3} && \text{Factor the perfect square out of 12.} \\
&= 2 \cdot \sqrt{3} \\
&= 2\sqrt{3}
\end{aligned}
$$

Example 3: Simplify $2\sqrt{75} \times \sqrt{5}$

Sometimes, simplifying the radical first may simplify the final steps.

$$
\begin{aligned}
2\sqrt{75}\times\sqrt{5} &= 2\cdot\sqrt{25}\cdot\sqrt{3}\times\sqrt{5} && \text{Factor the perfect square out of 75.}\\
&= 2\cdot 5\cdot\sqrt{3}\times\sqrt{5} && \text{Simplify the perfect square and multiply.}\\
& && \text{The factors in front of the radicands.}\\
&= 10\cdot\sqrt{3}\times\sqrt{5} && \text{Apply the multiplication property of radicals.}\\
&= 10\cdot\sqrt{15} && \\
&= 10\sqrt{15} && \text{Simplify.}
\end{aligned}
$$

Example 4: Simplify $\sqrt{3}(\sqrt{6}+\sqrt{12})$

Use the distributive property on this example and again simplify, if necessary.

$$
\begin{aligned}
\sqrt{3}(\sqrt{6}+\sqrt{12}) &= \sqrt{3}\cdot\sqrt{6}+\sqrt{3}\cdot\sqrt{12} && \text{Apply the distributive property.}\\
&= \sqrt{18}+\sqrt{36} && \\
&= \sqrt{9}\cdot\sqrt{2}+\sqrt{36} && \\
&= 3\cdot\sqrt{2}+6 && \\
&= 3\sqrt{2}+6 &&
\end{aligned}
$$

Example 5: Simplify $(2-\sqrt{3})(4+\sqrt{3})$

In this example, use FOIL just like multiplying two binomials, and then simplify.

$$
\begin{aligned}
& \qquad\ \ \text{F} \qquad\quad \text{O} \qquad\quad \text{I} \qquad\qquad \text{L}\\
(2-\sqrt{3})(4+\sqrt{3}) &= 2(4)+2\cdot\sqrt{3}-4\cdot\sqrt{3}-\sqrt{3}\cdot\sqrt{3}\\
&= 8+2\sqrt{3}-4\sqrt{3}-\sqrt{9}\\
&= 8+2\sqrt{3}-4\sqrt{3}-3\\
&= 8-2\sqrt{3}-3\\
&= 5-2\sqrt{3}
\end{aligned}
$$

Try this: Simplify $3\sqrt{6}\times\sqrt{24}$

Answer:

$$3\sqrt{6}\cdot\sqrt{4}\cdot\sqrt{6}$$
$$3\sqrt{6}\cdot 2\cdot\sqrt{6}$$
$$3\cdot 2\cdot\sqrt{6}\cdot\sqrt{6}$$
$$(3\cdot 2)\cdot(\sqrt{6}\cdot\sqrt{6})$$
$$6\cdot 6$$
$$36$$

Try this: Simplify $(5+\sqrt{7})(4-\sqrt{7})$

Answer:

$$\begin{array}{lcccccc}
F & & O & & I & & L \\
5(4) & - & 5\cdot\sqrt{7} & + & 4\cdot\sqrt{7} & - & \sqrt{7}\cdot\sqrt{7} \\
20 & - & 5\cdot\sqrt{7} & + & 4\cdot\sqrt{7} & - & \sqrt{49} \\
20 & - & 5\cdot\sqrt{7} & + & 4\cdot\sqrt{7} & - & 7 \\
20 & - & & \sqrt{7} & & - & 7
\end{array}$$

$$13-\sqrt{7}$$

Dividing Radicals

Follow along as we examine the various techniques used in dividing radicals.

Example #1: Simplify $\sqrt{\frac{9}{16}}$

Rewrite $\sqrt{\frac{9}{16}}$ as $\frac{\sqrt{9}}{\sqrt{16}}$ and simplify each perfect square root.

$$\frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$$

Example #2: Simplify $\sqrt{\frac{5}{36}}$

Rewrite $\sqrt{\frac{5}{36}}$ as $\frac{\sqrt{5}}{\sqrt{36}}$ and simplify the perfect square root in the denominator.

$$\frac{\sqrt{5}}{\sqrt{36}} = \frac{\sqrt{5}}{6}$$

Example #3: Simplify $\sqrt{\frac{7}{3}}$

$\sqrt{\frac{7}{3}} = \frac{\sqrt{7}}{\sqrt{3}}$ * In this case, there is a radical that remains in the denominator. In simplifying radicals this means that this particular problem is not completely simplified. To completely simplify this quotient, you will have to multiply the entire fraction by $\frac{\sqrt{3}}{\sqrt{3}}$ (which is the same as multiplying by one which doesn't change the value of the fraction).

$$\frac{\sqrt{7}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{21}}{\sqrt{9}} = \frac{\sqrt{21}}{3}$$

*This type of simplification is called rationalizing the denominator and is used when the denominator contains a radical that is not a perfect square root.

Example #4: Simplify $\frac{\sqrt{2}}{\sqrt{5}}$

Rationalize the denominator.

$$\frac{\sqrt{2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{10}}{\sqrt{25}} = \frac{\sqrt{10}}{5}$$

Example 5: Simplify $\frac{\sqrt{9}}{\sqrt{7}} \cdot \frac{\sqrt{2}}{\sqrt{9}}$

$$\frac{\sqrt{9}}{\sqrt{7}} \cdot \frac{\sqrt{2}}{\sqrt{9}} = \frac{3}{\sqrt{7}} \cdot \frac{\sqrt{2}}{3}$$

$$= \frac{\cancel{3}^{1}}{\sqrt{7}} \cdot \frac{\sqrt{2}}{\cancel{3}^{1}}$$

$$= \frac{\sqrt{2}}{\sqrt{7}}$$

$$= \frac{\sqrt{2}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$$ Rationalize the denominator.

$$= \frac{\sqrt{14}}{\sqrt{49}}$$ Simplify the perfect square.

$$= \frac{\sqrt{14}}{7}$$

* This problem cannot be simplified any further because the 14 and 7 do not cancel. (Since the 14 is under the radical and the 7 is not, they are not the same kinds of values and CANNOT be cancelled.) Another way to write this answer is $\frac{1}{7}\sqrt{14}$.

Example 6: Simplify $\sqrt{\frac{x^2 y^3}{z^2}}$

$$\sqrt{\frac{x^2 y^3}{z^2}} = \frac{\sqrt{x^2 y^3}}{\sqrt{z^2}} = \frac{\sqrt{x^2 y^2 y}}{\sqrt{z^2}} = \frac{xy\sqrt{y}}{z}$$

Remember from ***unit 5***, that a square root is also written as an exponent of ½. To review this concept, please see **unit 5, Rational Exponents**.

Another way to view this problem is as: $\left(\dfrac{x^2 y^3}{z^2}\right)^{\frac{1}{2}}$

To simplify, distribute the ½ to each variable:

$$= \frac{x^{2\times\frac{1}{2}} y^{3\times\frac{1}{2}}}{z^{2\times\frac{1}{2}}} = \frac{x^1 y^{\frac{3}{2}}}{z^1}$$

We can simplify x^1 and z^1 by just writing *x* and *z*. $y^{3/2}$ is the second (or square) root of $y \cdot y \cdot y$. This is simplified to $y\sqrt{y}$. This gives a final answer of $\dfrac{xy}{z}\sqrt{y}$. Notice this is the same as the answer given above.

Example 7: Simplify: $\sqrt{\dfrac{ab^2}{c}}$

$$\frac{\sqrt{ab^2}}{\sqrt{c}} \qquad \text{multiply by } \frac{\sqrt{c}}{\sqrt{c}}$$

$$\frac{\sqrt{ab^2}}{\sqrt{c}} \cdot \frac{\sqrt{c}}{\sqrt{c}}$$

$$\frac{\sqrt{ab^2 c}}{\sqrt{c^2}} = \frac{b\sqrt{ac}}{c}$$

*This problem cannot be simplified any further because the *c*'s do not cancel. (Since one is under the radical and the other is not, they are not the same value and CANNOT be cancelled.)

Example #8: Simplify using exponent rules: $\sqrt{(2^4 3^8 x^2)}$

First rewrite using exponents: $(2^4 3^8 x^2)^{\frac{1}{2}}$

Take each exponent to the ½ power individually: $2^{4\times\frac{1}{2}}\,3^{8\times\frac{1}{2}}\,x^{2\times\frac{1}{2}}$

Continue to simplify: $2^2 3^4 x^1 = 4 \cdot 81 \cdot x = 324x$

Try this: Simplify $\sqrt{\dfrac{96}{16}}$

Answer: First rewrite as: $\dfrac{\sqrt{96}}{\sqrt{16}}$

$$\frac{\sqrt{96}}{\sqrt{16}} = \frac{\sqrt{4}\sqrt{24}}{\sqrt{16}} = \frac{\sqrt{4}\sqrt{4}\sqrt{6}}{4}$$

Using the simplified amounts, you get: $\dfrac{4\sqrt{6}}{4} = \sqrt{6}$

Try this: Simplify $\sqrt{\dfrac{x^2 y^5 z}{z}}$

Answer: First rewrite as: $\dfrac{\sqrt{x^2 y^5 z}}{\sqrt{z}}$

$$\frac{\sqrt{x^2 y^5 z}}{\sqrt{z}} = \frac{xy^2\sqrt{yz}}{\sqrt{z}}$$

Multiply by $\dfrac{\sqrt{z}}{\sqrt{z}}$

$$\frac{xy^2\sqrt{yz}}{\sqrt{z}} \cdot \frac{\sqrt{z}}{\sqrt{z}} = \frac{xy^2\sqrt{y}\sqrt{z}\sqrt{z}}{\sqrt{z}\sqrt{z}} = \frac{xy^2 z\sqrt{y}}{z}$$

Try this: Simplify $\sqrt[3]{27x^3y^6}$

Answer:

$$27^{\frac{1}{3}}x^{3\times\frac{1}{3}}y^{6\times\frac{1}{3}}=3xy^2$$

Stop! **Go to Questions #21-42 to complete this unit.**