PROBABILITY: FUNDAMENTAL COUNTING PRINCIPLE, PERMUTATIONS, COMBINATIONS

Unit Overview

In this unit you will begin with an introduction to probability by studying experimental and theoretical probability. You will then study the fundamental counting principle and apply it to probabilities. The unit concludes by exploring permutations, which are used when the outcomes of the event(s) depend on order, and combinations, which are used when order is not important.

Introduction to Probability

Probability is the likelihood of an event occurring.

Terminology (a coin is used for each of the examples)	
Definition	**Example**
Trial: a systematic opportunity for an event to occur	tossing a coin in the air
Experiment: one or more trials	tossing a coin 6 times
Sample space: the set of all possible outcomes of an event	H or T
Event: an individual outcome or any specified combination of outcomes.	landing H or landing T

Probability is expressed as a number from 0 to 1. It is written as a fraction, decimal, or percent.

- an impossible event has a probability of 0
- an event that must occur has a probability of 1

- the sum of the probabilities of all outcomes in a sample space is 1

The probability of an event can be assigned in two ways:

1.) **experimentally**: approximated by performing trials and recording the ratio of the number of occurrences of the event to the number of trials. (as the number of trials in an experiment increases, the approximation of the experimental probability increases).

2.) **theoretically**: based on the assumption that all outcomes in the sample space occur randomly.

Experimental Probability

$$P(\text{event}) = \frac{\text{number of times an event occurs}}{\text{total number of trials}}$$

Example #1: You tossed a coin 10 times and recorded a tail 4 times and a head 6 times.

A head showed up 4 times out of 10.

$$\text{P(tail)} = \frac{4}{10} = \frac{2}{5}$$

The experimental probability of tossing a tail was 2/5.

A head showed up 6 times out of 10.

$$P(\text{head}) = \frac{6}{10} = \frac{3}{5}$$

The experimental probability of tossing a head was 3/5.

Example #2: A basketball player made a free throw shot in 36 out his last 50 attempts. What is the experimental probability that he will make a free throw shot the next time he makes an attempt?

$$P(\text{making free throw}) = \frac{36}{50} = \frac{18}{25}$$

The probability that he makes the free throw is 18 out of 25 = 0.72 or 72%.

Example #3: A car manufacturer inspected 360 cars at random. The manufacturer found 352 of the cars had no defects. Predict how many cars will have no defects out of 1280.

$$P(\text{no defects}) = \frac{\text{number of time event occurs}}{\text{total number of trials}}$$

$= \frac{352}{360}$ Substitute

$= 0.978$ Simplify, rounded to nearest thousandth

$= 97.8\%$ Write as a percent

The probability that a car has no defects is 97.8%

To predict how many cars will have no defect out of 1280:

Number with no defect = P(no defects) × number of cars

= 0.978 × 1280 Substitute. Use 0.978 for 97.8%

= 1251.84 Simplify

Predictions are not exact, so round your results. Approximately 1.252 cars are like to have no defect.

Theoretical Probability

If all outcomes in a sample space are equally likely, then the theoretical probability of event *B*, denoted *P*(*B*), is defined by:

$$P(B) = \frac{\text{number of outcomes in event B (favorable outcomes)}}{\text{number of outcomes in the sample space (possible outcomes)}}$$

Example #4: Find the probability of randomly selecting an orange marble out of a jar containing 3 blue, 3 red, and 2 orange marbles.

$$P(\text{1 orange}) = \frac{\text{favorable}}{\text{possible}} = \frac{\text{2 orange}}{\text{8 possible}}$$

$$= \frac{2}{8} = \frac{1}{4} = 0.25 \text{ or } 25\%$$

Example #5: Find the probability of rolling an even number on a die. Die is the singular for dice. In this example, a 6-sided die is used.

The possible outcomes of rolling a cube are: 1, 2, 3, 4, 5, or 6.

$$P(\text{even number}) = \frac{\text{favorable}}{\text{possible}} = \frac{\text{3 even}}{\text{6 possible}}$$

$$= \frac{3}{6} = \frac{1}{2} = 0.5 \text{ or } 50\%$$

Let’s compare the experimental probability of flipping a fair coin to the theoretical probability.

The theoretical probability of flipping a coin. There are two choices, heads or tails.

$$P(\text{heads}) = \frac{1}{2}$$

$$P\ (\text{tails}) = \frac{1}{2}$$

Below are the results of flipping a coin 10, 100, and 1000 times.

Number of times a coin is flipped	10	100	1000
Number of time the coin is tails	3	43	502
Experimental probability of tails	$\frac{3}{10}$	$\frac{43}{100}$	$\frac{502}{1000}$

You may want to conduct your own coin tossing experiment to see what kind of results you get. Your results will more than likely be different, making your experimental probability different. The more trials you conduct in a experiment, the closer your experimental probability will be to the theoretical probability. This is called the *Law of Large Numbers.* For a small number of events, they may not match.

Example #6: A model says a spinning coin falls heads up with a probability 0.5 or ½. Would a result of 5 tails in a row cause you to question the model?

Answer: No, it cannot be determined that the coin is not fair due to the small number of trials.

QuickTime Introduction--Theoretical vs. Experimental (02:28)

QuickTime Theoretical Probability--Genders (02:26)

QuickTime Probability--Batting Average (02:35)

QuickTime Finding the Total Number of Outcomes (05:41)

Stop! **Go to Questions #1-12 about this section, then return to continue on to the next section.**

Fundamental Counting Principle

> **Fundamental Counting Principle**
>
> If there are *m* ways that one event can occur and *n* ways that another event can occur, then there are ***m*** x ***n*** ways that both events can occur.

Example #1 Emily is choosing a password for access to the Internet. She decides not to use the digit 0 or the letters A, E, I, O, or U. Each letter or number may be used more than once. How many passwords of 3 letters followed by 2 digits are possible?

Use the fundamental counting principle. There are 21 possible letters and 9 possible digits.

1st letter		2nd letter		3rd letter		1st digit		2nd digit
21	×	21	×	21	×	9	×	9

The number of possible passwords for Emily is $21^3 \cdot 9^2$ or 750,141.

Example #2 In the school cafeteria, you may choose from 4 entrees, 5 sides, 3 desserts and 4 drinks for lunch. How many different lunches can be made, if you choose one from each category?

Entrees		Sides		Desserts		Drinks
4	×	5	×	3	×	4

The number of possible lunches is 240.

QuickTime Using the Fundamental Counting Principle (06:13)

Stop! **Go to Questions #13-15 about this section, then return to continue on to the next section.**

Permutations

Before we look at permutations, we need to understand factorial numbers. Let's take a look.

$n!$ is read "n factorial"

6! means $6 \times 5 \times 4 \times 3 \times 2 \times 1$ which is a product of all the natural numbers, starting with 6 and going down to 1.

$6! = 720$

Example #1: Evaluate 10!

$10!$ means $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$10! = 3{,}628.800$

If you want to use your calculator to find 10! Press 10, [MATH], move the cursor over to PRB, and go down to 4:! Then press [ENTER].

Now we are ready for permutations.

A **permutation** is an arrangement of objects in a specific order. When objects are arranged in a row, the permutation is called a **linear permutation**.

Permutations of *n* Objects

The number of permutations of n objects is given by $n!$ (! is called factorial and means to multiply all consecutive natural numbers starting with n).

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

Example #1: On a baseball team, nine players are designated as the starting line up. Before a game, the coach announces the order in which the nine players will bat. How many different orders are possible?

$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 362,880 \text{ possible orders.}$$

*Note: When the coach is choosing, on his first choice he has nine players to choose from. Once he makes that choice, he has eight players left to choose from, then seven, then six, and so on.

If you want to use your calculator to find 9! Press 9, [MATH], move the cursor over to PRB, and go down to 4:! Then press [ENTER].

Permutations of *n* Objects Taken *r* at a Time

The number of permutations of *n* objects taken *r* at a time, denoted by $P(n, r)$ or ${}_nP_r$ is given by:

$$P(n,r) = {}_nP_r = \frac{n!}{(n-r)!}, \text{ where } r \le n$$

Example #2: Find the number of ways to listen to 6 different CD's from a selection of 18 CD's.

Method 1 Use the Fundamental Counting Principle.

$$18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 = 13{,}366{,}080$$

*Note: There are 18 choices of CD's to listen to first. Once that choice is made, there are 17 CD's left to choose from, then 16 CD's, then 15 CD's, and so on until 6 CD'S are chosen.

Method 2 Use the permutation formula.

Note: Since the order in which the CD's will be played **is** important, this is a "permutation" problem.

There are *n* = 18 CD's to arrange taking 6 CD's at a time.

$${}_{18}P_6 = \frac{18!}{(18-6)!}$$

$$= \frac{18!}{12!} = \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{12!}$$

$$= 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13$$

$$= 13{,}366{,}080$$

Instead of writing down all the factors for 18!, just write down the factors starting at 18 until you get to a factor which matches the 12! in the denominator. Cancel the 12!'s. This leaves the numerator with the factors $18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13$. Find the product.

Example #3: There are 20 members in the drama club. How many ways can a president, vice-president, secretary and treasurer be elected?

Method 1 Use the Fundamental Counting Principle.

There are 20 choices for president. Once the president is chosen, there are 19 choices left for the office of vice-President. Once the Vice-President is chosen, there are 18 choices for secretary and then 17 choices for treasurer.

$$20 \cdot 19 \cdot 18 \cdot 17 = 116,280$$

Method 2 Use the permutation formula.

Note: Order is important in this example because it makes a difference which student is chosen for each office. Since the order in which the officers are chosen **is** important, this is a "permutation" problem.

There are n = 20 members to arrange taking 4 at a time.

$$_{20}P_4 = \frac{20!}{(20-4)!}$$

$$= \frac{20!}{16!} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16!}{16!}$$

$$= 20 \cdot 19 \cdot 18 \cdot 17$$

$$= 116,280$$

Defining Permutations (08:42)

Permutations and Combinations: Part 1 (05:11)

Calculate the Number of Possible Outcomes

As we try to calculate probability, we find that each situation may be slightly different. For instance, if we considered randomly selecting letters from a word and the word we chose had repeated letters, we would not get a clear picture of the probability.

Example #1: What is the probability of selecting the letter "r" from the letters in the word random?

r-a-n-d-o-m

1-r 6-letters total probability $= \frac{1}{6}$

Example #2: What is the probability of selecting the letter "s" from the word success?

3-s 7-letters total probability $= \frac{3}{7}$

There is a higher probability when there are more chances of success.

When considering the arrangement of letters, use permutations.

Example #3: How many ways can the letters of the word "random" be arranged?

This example requires a permutation. It's formula is $P(n,n) = n!$ where we are selecting all of the letters in the arrangement.

$$P(n,n) = n!$$
$$P(6,6) = 6!$$
$$P(6,6) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$
$$P(6,6) = 720$$

There are 720 ways the letters in the word "random" may be arranged.

Example #4: If we look at arranging letters in the word "success", we need to realize that when an s or c is selected, it does not matter which is which. So there are less ways to select the arrangement.

This is called a permutation with repetition and is given by the following formula $P = \frac{n!}{a!b!}$ where "a" and "b" are repeating letters.

How many ways are there to arrange the letters in the word success?

We are using all 7 letters but the “s” has 3 repeats and the “c” has 2 repeats.

$$P = \frac{7!}{3!2!}$$

$$P = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1}$$

$$P = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4}^{2} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{2} \cdot 1}$$

$$P = \frac{7 \cdot 6 \cdot 5 \cdot 2}{1}$$

$$P = 420$$

The letters in the word “success” may be arranged 420 different ways.

QuickTime Solving Problems with the Fundamental Counting Principle and Permutations (04:48)

Stop! **Go to Questions #16-24 about this section, then return to continue on to the next section.**

Combinations

An arrangement of objects in which order is **not** important is called a combination.

Combinations of *n* Objects Taken *r* at a Time

The number of combinations of *n* objects taken *r* at a time is given by:

$$C(n,r) = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}, \text{ where } 0 \le r \le n$$

$C(n,r)$, ${}_nC_r$, and $\binom{n}{r}$ have the same meaning.

All are read "n choose r".

Example #1: How many ways are there to give 4 honorable mention awards to a group of 10 students?

Note: Since the order in which the honorable mention awards are presented is **not** important, then this is a "combination" problem.

$$ {}_{10}C_4 = \frac{10!}{4!(10-4)!}$$

$$= \frac{10!}{4!(6!)} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot (6!)}{4!(6!)}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot (\cancel{6!})}{4!(\cancel{6!})} \qquad \text{-cancel the } 6!\text{'s}$$

$$= \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}$$

$$= 210$$

There are 210 ways to give 4 honorable mention awards to 10 students.

Example #2: There are 20 students in the drama club. A four-person committee will be chosen to arrange a bake sale. In how many different ways can the committee be formed?

Note: Since the order the students are chosen to be on the committee is **not** important, this is a "combination" problem.

$$_{20}C_4 = \frac{20!}{4!(20-4)!}$$

$$= \frac{20!}{4!(16!)} = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot (16!)}{4!(16!)}$$

$$= \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot (16!)}{4!(16!)}$$ -cancel the $16!$'s

$$= \frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 4{,}845$$

There are 4,845 ways to choose a committee of 4 from 20 students.

QuickTime Introduction (01:06)

QuickTime Calculating Combinations (08:56)

QuickTime Identifying Strategies for Solving Problems with Combinations (06:32)

Stop! **Go to Questions #25-33 to complete this unit.**