RATIONAL EXPRESSIONS AND EQUATIONS

Unit Overview

In this unit you will learn how to multiply, divide, add, and subtract rational expressions. You will also simplify rational expressions. The unit concludes with solving rational equations.

Simplifying Rational Expressions

A rational expression is in simplest form when its numerator and denominator are polynomials that have no common divisors.

To Simplify:

*factor the numerator and denominator

*cancel any common factors

Example #1: Simplify $\dfrac{x^2-10x+9}{x^2+2x-3}$.

$$\frac{(x-9)\cancel{(x-1)}}{(x+3)\cancel{(x-1)}}$$ Cancel the (x – 1) that is common.

$$=\frac{(x-9)}{(x+3)}$$ *You **can not** cancel the 9 and the 3 because they are connected to the x values by (–) and (+) signs.

Stop! **Go to Questions #1-3 about this section, then return to continue on to the next section.**

Multiplying and Dividing Rational Expressions

Multiplying and dividing rational expressions is similar to multiplying and dividing rational numbers.

To Multiply

*factor **all** numerators and denominators

*cancel any common factors

*multiply straight across

Example #2: Multiply $\frac{x^2-4x-5}{x^2-3x+2}\cdot\frac{x^2-4}{x^2-3x-10}$.

$$\frac{(x-5)(x+1)}{(x-2)(x-1)}\cdot\frac{(x+2)(x-2)}{(x-5)(x+2)}$$

$$\frac{(x-5)(x+1)}{(x-2)(x-1)}\cdot\frac{(x+2)(x-2)}{(x-5)(x+2)}$$

Cancel the $(x-5)$, $(x+2)$ and $(x-2)$ because these are all common factors.

$$=\frac{(x+1)}{(x-1)}$$

To divide rational expressions you multiply by the reciprocal of the divisor, just as you do when you divide rational numbers.

To Divide

*multiply by the reciprocal, change the problem to a multiplication problem and use the reciprocal (flip) of the rational expression you are dividing by

*factor all numerators and denominators

*cancel any common factors

*multiply straight across

Example #3: Divide $\dfrac{4x^3-9x}{2x-7} \div \dfrac{3x^3+2x^2}{4x^2-14x}$.

$$\frac{4x^3-9x}{2x-7} \cdot \frac{4x^2-14x}{3x^3+2x^2}$$

$$\frac{x(4x^2-9)}{2x-7} \cdot \frac{2x(2x-7)}{x^2(3x+2)}$$

$$\frac{\cancel{x}(2x+3)(2x-3)}{\cancel{2x-7}} \cdot \frac{2\cancel{x}\cancel{(2x-7)}}{\cancel{x^2}(3x+2)}$$

Cancel the x^2 with the *x* in the first expression and the *x* in the second expression and the 2*x* – 7's can cancel.

$$\frac{2(4x^2-9)}{3x+2} = \frac{8x^2-18}{3x+2}$$

Multiply straight across.

QuickTime Multiplying and Dividing Rational Functions (04:22)

Stop! **Go to Questions #4-10 about this section, then return to continue on to the next section.**

Adding and Subtracting Rational Expressions

To add and subtract rational expressions, you must have a common denominator. If you do, just add the numerators together and keep the common denominator.

Example #1: Find the sum: $\frac{9x}{x+7}+\frac{2x}{x+7}$.

$$\frac{9x}{x+7}+\frac{2x}{x+7}=\frac{11x}{x+7}$$

Example #2: Find the difference: $\frac{7x}{x-4}-\frac{4x+12}{x-4}$.

After adding or subtracting the numerators, if possible, factor and cancel the common factors.

$$\frac{7x}{x-4}-\frac{4x+12}{x-4}=\frac{7x-(4x+12)}{x-4}$$

$$=\frac{7x-4x-12}{x-4}$$

$$=\frac{3x-12}{x-4}$$

$$=\frac{3\cancel{(x-4)}}{\cancel{x-4}}=3$$

*If the denominators are not the same:

a.) factor all denominators

b.) find all common factors

c.) multiply the denominators by what is missing in the common denominator

Example #3: Find the difference: $\frac{x}{x+6}-\frac{72}{x^2-36}$.

$$\frac{x}{x+6}-\frac{72}{(x+6)(x-6)}$$

$$\frac{x}{x+6}\left(\frac{x-6}{x-6}\right)-\frac{72}{(x+6)(x-6)}$$

The least common denominator will be **(*x* + 6)(*x* – 6)** because those are the factors that are combined between the denominators.

$$\frac{x^2-6x}{(x+6)(x-6)}-\frac{72}{(x+6)(x-6)}$$

Multiply the first expression by $\frac{x-6}{x-6}$ so that you will have a common denominator.

$$\frac{x^2-6x-72}{(x+6)(x-6)}$$

Factor the numerator.

$$\frac{(x-12)\cancel{(x+6)}}{\cancel{(x+6)}(x-6)}$$

Cancel common factors.

$$=\frac{x-12}{x-6}$$

Let's try another example.

Example #4: Find the difference: $\frac{x}{3x-3}-\frac{x}{x^2-1}$.

$$\frac{x}{3(x-1)}-\frac{x}{(x-1)(x+1)}$$

LCD will be 3(*x* – 1)(*x* +1).

$$\frac{x}{3(x-1)}\left(\frac{x+1}{x+1}\right)-\frac{x}{(x-1)(x+1)}\left(\frac{3}{3}\right)$$

$$\frac{x^2+x}{3(x-1)(x+1)}-\frac{3x}{3(x-1)(x+1)}$$

$$\frac{x^2+x-3x}{3(x-1)(x+1)}$$

Factor the numerator to see if you can cancel any common factors; in this case you can not, so either of the answers are correct.

$$\frac{x^2-2x}{3(x-1)(x+1)} \quad \text{OR} \quad \frac{x(x-2)}{3(x-1)(x+1)}$$

Rational expression activity:

List the reasons to justify each of the following steps used to add:

$$\frac{x-1}{x^2-1}+\frac{3}{4x+4}$$

The reasons applied to each step are shown below in blue text. Drag each reason to the empty text box beside the step where the reason is applied in solving the rational expression.

$$=\frac{x-1}{(x+1)(x-1)}+\frac{3}{4(x+1)}$$

$$=\frac{4(x-1)}{4(x+1)(x-1)}+\frac{3(x-1)}{4(x+1)(x-1)}$$

$$=\frac{4x-4}{4(x+1)(x-1)}+\frac{3x-3}{4(x+1)(x-1)}$$

$$=\frac{7x-7}{4(x+1)(x-1)}$$

$$=\frac{7(x-1)}{4(x+1)(x-1)}$$

$$=\frac{7}{4(x+1)}$$

Add the numerators | Distributive Property | Divide out common factors | Reset

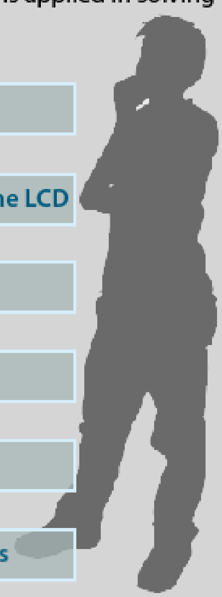
Factor the denominators | Factor the numerator | Rewrite each expression with the LCD

Answers for rational expression activity:

List the reasons to justify each of the following steps used to add:

$$\frac{x-1}{x^2-1}+\frac{3}{4x+4}$$

The reasons applied to each step are shown below in blue text. Drag each reason to the empty text box beside the step where the reason is applied in solving the rational expression.



$$=\frac{x-1}{(x+1)(x-1)}+\frac{3}{4(x+1)}$$ Factor the denominators

$$=\frac{4(x-1)}{4(x+1)(x-1)}+\frac{3(x-1)}{4(x+1)(x-1)}$$ Rewrite each expression with the LCD

$$=\frac{4x-4}{4(x+1)(x-1)}+\frac{3x-3}{4(x+1)(x-1)}$$ Distributive Property

$$=\frac{7x-7}{4(x+1)(x-1)}$$ Add the numerators

$$=\frac{7(x-1)}{4(x+1)(x-1)}$$ Factor the numerator

$$=\frac{7}{4(x+1)}$$ Divide out common factors

The examples illustrate that rational expressions form a system similar to the rational numbers, in that rational expressions are closed under addition, subtraction, multiplication, and division by a nonzero rational expression.

QuickTime Adding and Subtracting Rational Functions (08:54)

Stop! **Go to Questions #11-16 about this section, then return to continue on to the next section.**

Solving Rational Equations

To solve rational equations:

a.) find all excluded values by factoring the denominator and setting all factors equal to zero and solving

b.) find the LCD and multiply all terms by the LCD to eliminate fractions

c.) solve the equation and check your answer

Example #1: Solve $\frac{x}{x+3} = \frac{6}{x-1}$.

$x+3=0 \qquad x-1=0$

$x \neq -3 \qquad x \neq 1$

Both of these values will produce a zero in the denominator, so they must be excluded from the solution.

The LCD will be $(x + 3)(x - 1)$.

$$(x+3)(x-1)\left(\frac{x}{x+3}\right) = (x+3)(x-1)\left(\frac{6}{x-1}\right)$$

Multiply both sides of the equation by $(x + 3)(x - 1)$.

$$(\cancel{x+3})(x-1)\left(\frac{x}{\cancel{x+3}}\right) = (x+3)(\cancel{x-1})\left(\frac{6}{\cancel{x-1}}\right)$$

Cancel common factors.

Multiply and solve.

$$x^2 - x = 6x + 18$$

$$x^2 - 7x - 18 = 0$$

$$(x-9)(x+2) = 0$$

$x-9=0 \qquad x+2=0$

$x=9$ and $x=-2$

Check your answers by replacing x with 9 first then replacing it with –2.

$\frac{9}{9+3} = \frac{6}{9-1} \qquad \frac{-2}{-2+3} = \frac{6}{-2-1}$

$\frac{9}{12} = \frac{6}{8}$ **True** $\qquad \frac{-2}{1} = \frac{6}{-3}$ **True**

Example #2: Solve $\frac{x}{x+2}-\frac{8}{x^2-4}=\frac{2}{x-2}$.

Step #1: Factor all denominators to find the LCD and excluded values. In this case the LCD is $(x-2)(x+2)$ and the excluded values are 2 and –2 because if x is equal to either of these, the denominator will become zero.

$$\frac{x}{x+2}-\frac{8}{(x-2)(x+2)}=\frac{2}{x-2}$$

Step #2: Multiply each term by the LCD to eliminate all fractions.

$$(x-2)\cancel{(x+2)}\left(\frac{x}{\cancel{x+2}}\right)-\cancel{(x-2)}\cancel{(x+2)}\left(\frac{8}{\cancel{(x-2)}\cancel{(x+2)}}\right)=\cancel{(x-2)}(x+2)\left(\frac{2}{\cancel{x-2}}\right)$$

$$x(x-2)-8=2(x+2)$$

$$x^2-2x-8=2x+4$$

Step #3: Combine all like terms, factor and solve.

$$x^2-4x-12=0$$

$$(x-6)(x+2)=0$$

$$x-6=0 \qquad x+2=0$$

$$x=6 \qquad x=-2$$

Since –2 is an excluded value, the only solution is 6.

Using Rational Equations

Uniform Motion is motion at a constant speed. In uniform motion problems, we use the formula $d = rt$ (distance = rate × time). In problems where there rate or time is unknown, we use the formula $t = \frac{d}{r}$ or $r = \frac{d}{t}$. Both of these formulas use equations involving rational expressions.

Example #1: Allie drove 120 miles to visit her grandmother. On the return trip home, it was raining so she averaged 10 miles per hour less. The drive home took her 24 minutes longer. Find the average speed for both trips.

Use the formula $t = \dfrac{d}{r}$

Let r = the rate Allie travels to her grandmother's house

$\dfrac{120}{r}$ = time Allie travels to her grandmother's house

$r - 10$ = rate on the return trip

$\dfrac{120}{r-10}$ = time for the return trip

Since rate is usually measured in miles per hour change 24 minutes to hours, 24 minutes out of 60 minute is $\dfrac{24}{60} = \dfrac{2}{5}$ hours.

Because the difference between the two time is $\dfrac{2}{5}$, the set up for the equation will be: longer time – shorter time = $\dfrac{2}{5}$.

$$\frac{120}{r-10}-\frac{120}{r}=\frac{2}{5}$$

Multiply each term by the LCD $5(r)(r-10)$

$120(5r)-120(5)(r-10)=2(r)(r-10)$

Simply each term.

$600r-600r-6000=2r^2-20r$

Group all terms on one side of the equation.

$2r^2-20r+6000=0$

Divide each side by 2.

$r^2-10r+3000=0$

Factor and solve.

$(r-60)(r+50)=0$

$r-60=0$ or $r+50=0$

$r=60$ or $r=-50$

The answer r = –50 is a solution to the equation, but it cannot be the average speed of a car. Her average speed going to her grandmother's was 60 mph. Her average speed on the return trip was 50 mph.

Check:

$t=\frac{d}{r}=\frac{120}{60}=2$ hours Time it takes Allie to go to grandmother's

$t=\frac{d}{r}=\frac{120}{50}=2\frac{2}{5}$ hours Time it takes Allie to return home

$2\frac{2}{5}$ hours $-$ 2 hours $=\frac{2}{5}$ hours more

Work Problem

Work problems typically involve two people or machine working together to complete a task. The basic set up for a word problem is

Portion of job completed by 1st person + Portion of job completed by 2nd person = 1 (one completed job)

Tyler can shovel a rake the leaves in the yard in 30 minutes. If Josh helps, the job takes only 20 minutes. How long would it take Josh to do the job by himself?

Let x = number of minutes it takes Josh to rake the leaves working alone

$\frac{1}{30}$ = amount of job completed by Tyler in one minute

$\frac{1}{30}(20) = \frac{20}{30} = \frac{2}{3}$ amount of the job completed by Tyler in 20 minutes

$\frac{1}{x}$ = amount of the job completed by Josh in one minute

$\frac{1}{x}(20) = \frac{20}{x}$ amount of the job completed by Josh in 20 minutes

Write the equation using the basic setup above.

$$\frac{2}{3} + \frac{20}{x} = 1$$

Multiply each term by the LCD $3x$.

$$\frac{2}{3}(3x) + \frac{20}{x}(3x) = 1(3x)$$

Simplify each term.

$$2x + 60 = 3x$$

Subtract $2x$ from each side of the equation.

$$60 = x$$

Therefore it would take Josh 60 minutes working alone to rake the leaves in the yard.

Check:

If Josh takes 60 minutes to rake the leaves by himself he completes $\frac{1}{60}$ of the job in one minute. So in 20 minutes Josh completes $\frac{20}{60} = \frac{1}{3}$ of the job. Tyler completes $\frac{2}{3}$ of the job in 20 minutes. Working together they complete $\frac{2}{3} + \frac{1}{3}$ or 1 job.

Stop! **Go to Questions #17-30 to complete this unit.**