NONLINEAR FUNCTIONS

In this unit, you will examine several types of nonlinear functions. You will learn how to graph quadratic functions known as parabolas. A quadratic function models the height of a projectile; for example, the curved path that a golf ball would take through the air. You will also learn how to solve simple quadratic equations using square roots. You will then examine other types of functions and graphs such as direct and inverse functions, exponential functions of growth and decay, cube roots, and cubic graphs.

Graphing Quadratic Functions

Solve Equations Using Squares and Square Roots

Direct Variation

Inverse Variation

Exponential Growth and Decay

Perfect Cubes and Their Cube Roots

Graph Paper

Graphing Quadratic Functions

The graph of a quadratic function is a curve known as a **parabola** and is shown below. The lowest point on this parabola is the **minimum value** of the function, the point (0, 0) and is called the **vertex** of the parabola. In a parabola there is a vertical line called the axis of symmetry drawn through the vertex, that reflects the parabola across the line of x = h, or in other words, splits the parabola into two equal parts. In the case below, the axis of symmetry would be the y-axis or $x = 0$.

$$y = x^2$$
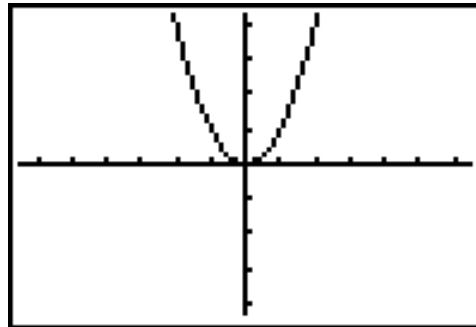
If "a" is negative ($y = -x^2$), the parabola opens down and therefore has a maximum value at the vertex (0, 0).
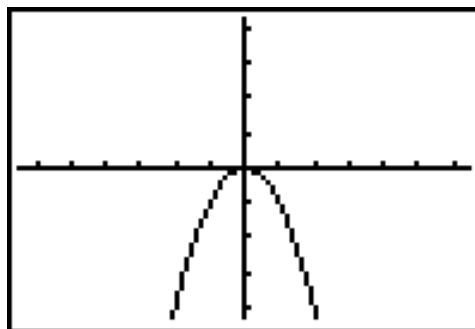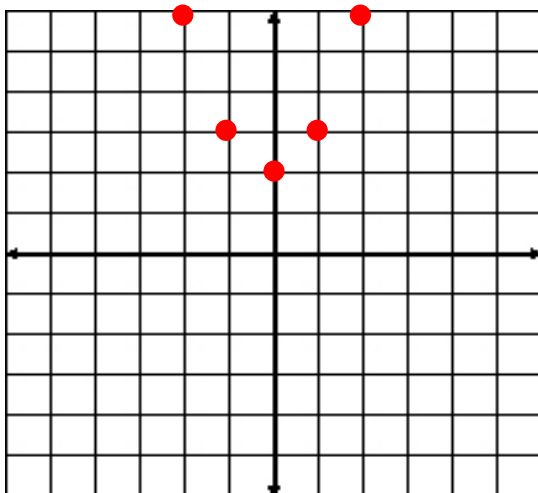
To graph a quadratic equation, you will need to make a table of values. Study the example below.

Example 1: Graph $y = x^2 + 2$

a) Make a table of values using positive and negative *x*-values.

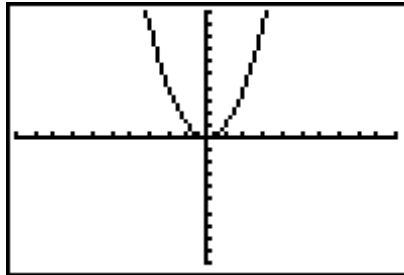| x	$x^2 + 2$	y
-2	$(-2)^2 + 2$	6
-1	$(-1)^2 + 2$	3
0	$(0)^2 + 2$	2
1	$(1)^2 + 2$	3
2	$(2)^2 + 2$	6

b) Graph the points from above on a coordinate plane.

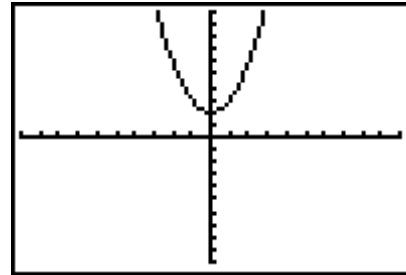c) Connect the points using a curve (remember that parabolas are curved).
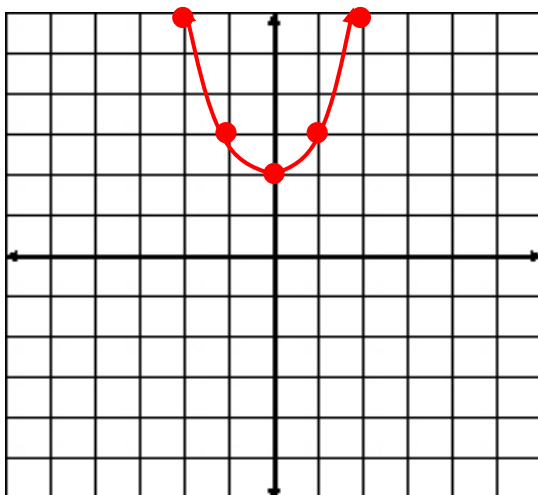
*Notice that the vertex of this parabola is located at (0, 2) as opposed to (0, 0) as shown in the first part of the unit. Let's take a look at both functions and decide why this happened.

a.) $y = x^2$ b.) $y = x^2 + 2$

If you compare both equations, you will notice that in the equation in example a, there is no constant term, whereas in the equation in example b, there is a constant of +2. The graph of equation in example b has moved up the y-axis 2 units; so you can conclude that a constant moves the graph vertically along the y-axis.

Let's use $y = x^2$ as our parent function with a vertex at (0, 0) and compare it with $y = x^2 - 3$. What do you think will happen in this case?
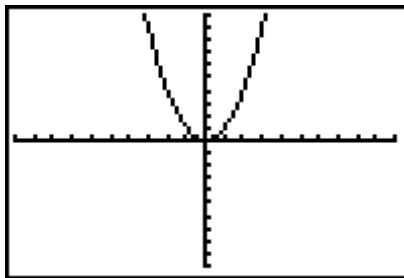
$y = x^2$ $y = x^2 - 3$ **Table of Values**

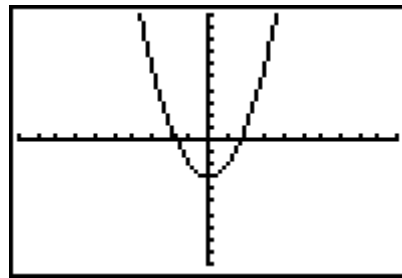| x	y
−2	1
−1	−2
0	−3
1	−2
2	1

*Due to the pixels in the graphing calculator, it may look like the second equation passes through (2, 0) and (–2, 0). This is not the case and at this point we are only concerned about what the constant does to the graph.

By examining the two equations you can see that by subtracting 3 the graph has moved down 3 units.

Example 2: Graph $y = (x+1)^2$

Make a table of values and graph the points.

x	y
−2	1
−1	0
0	1
1	4
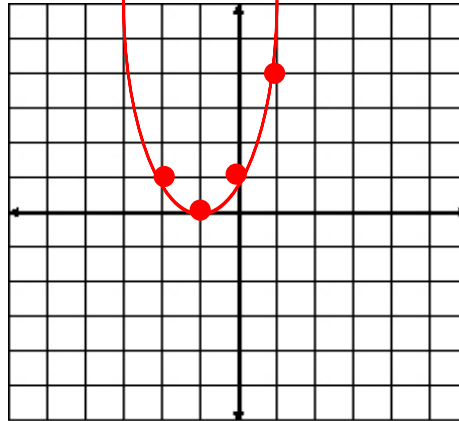
Notice that in this example we do not have the same number of points on each side of the vertex. If you want, you can substitute more values for *x* to determine how wide the parabola gets on the left.

Notice that when the constant term is in a quantity with the variable, the vertex moved horizontally along the *x*-axis. In the case above, it moved one unit to the **left**. What do you suppose will happen to the vertex of $y = (x-2)^2$? If you answered that the vertex would move 2 units to the right you are correct. This concept now brings us to the vertex form of a quadratic ($y = -3(x-8)^2 + 10$) which will be explained in another unit.

Solve Equations Using Squares and Square Roots

The equation $x^2 = 16$ can be read as the question, "What number multiplied by itself is 16?"

There are two possible solutions to this, 4 times itself is 16 and (–4) times itself is 16.

$$x^2 = 16$$

$$x = 4 \text{ or } x = -4$$

The positive square root of the number is called the principal square root. In general if $x^2 = y$ and $y > 0$, then $x = \sqrt{y}$ or $x = -\sqrt{y}$. This can be written as $x = \pm\sqrt{y}$ and read as "x equals plus or minus the square root of y".

Example 1: Solve: $x^2 = 36$

Take the **square root** of both sides to solve.

$$\sqrt{x^2} = \sqrt{36}$$

$$x = \pm\ 6$$

$$x = +\ 6 \text{ or } x = -6$$

Example 2: Solve: $y^2 = 18$

Take the **square root** of both sides to solve.

$$\sqrt{y^2} = \sqrt{18}$$

$$y = \pm\sqrt{18}$$

$$y \approx \pm\ 4.23 \qquad\qquad \text{-Round to the nearest hundredth.}$$

$$y \approx +4.23 \text{ or } y = -4.23$$

$\approx$ is read "approximately equal to"

Example 3: Solve: $\sqrt{x} = 10$

Square both sides to solve.

$$(\sqrt{x})^2 = 10^2$$

To find $(\sqrt{x})^2 \rightarrow \ (\sqrt{x})^2 = \sqrt{x} \times \sqrt{x} = \sqrt{x^2} = x$

To find $10^2 \rightarrow \ 10^2 = 10 \times 10 = 100$

Therefore, $x = 100$

Example 4: Solve: $\sqrt{a} = 7$

$\sqrt{a} = 7$	Given Problem.
$(\sqrt{a})^2 = 7^2$	Square both sides of the equation.
$a = 49$	Simplify.

Example 5: Solve: $\sqrt{y} = 2.5$

$\sqrt{y} = 2.5$	Given Problem.
$(\sqrt{y})^2 = (2.5)^2$	Square both sides of the equation.
$y = 6.25$	Simplify.

Example 6: Solve: $2x^2 = 100$

$2x^2 = 100$ Given Problem.

$\dfrac{2x^2}{2} = \dfrac{100}{2}$ Divide both sides of the equation by 2.

$x^2 = 50$ Simplify. $\left(\dfrac{\cancel{2}x^2}{\cancel{2}} = \dfrac{\overset{50}{\cancel{100}}}{\cancel{2}} \right)$

$\sqrt{x^2} = \sqrt{50}$ Take the square root of both sides.

$x \pm \sqrt{50}$ Simplify.

$x \approx \pm \, 7.1$

$x \approx +7.1$ or $x \approx -7.1$

Example 7: Solve: $x^2 + 12 = 48$

$x^2 + 12 = 48$ Given Problem.

$x^2 + 12 - 12 = 48 - 12$ Subtract 12 from both sides of the equation.

$x^2 = 36$ Simplify.

$\sqrt{x^2} = \sqrt{36}$ Take the square root of both sides.

$x \pm 6$ Simplify.

$x = +\,6$ or $x = -6$

Example 8: Solve: $\dfrac{x^2}{4} = 16$

$\dfrac{x^2}{4} = 16$ Given Problem.

$4(\dfrac{x^2}{4}) = (16)4$ Multiply both sides of the equation by 4. $\left(4(\dfrac{x^2}{4}) = (16)4 \right)$

$x^2 = 64$ Simplify.

$\sqrt{x^2} = \sqrt{64}$ Take the square root of both sides.

$x \pm 8$ Simplify.

$x = + 8 \text{ or } x = -8$

Direct Variation

The variable y varies directly as x if there is a nonzero constant k such that $y = kx$. The equation $y = kx$ is called a direct variation equation and the number k is called the constant of variation.

*There are many situations in which one quantity varies directly as another:

      -An employee's wages vary directly as the number of hours worked.

      -The amount of sales tax varies directly as the total price of the merchandise.

To find the constant of variation (k) and the direct-variation equation, use the following steps.

   Step 1: Replace the x and y with the given values.

   Step 2: Find k.

   Step 3: Replace k in the direct variation equation.

      Example 1: Find the constant of variation (k) and the direct-variation equation, if y varies directly as x and $y = -72$ when $x = -18$.

$$y = kx$$
$$-72 = k(-18) \qquad \textit{Step \#1}: \text{Replace } x \text{ and } y \text{ with the given values.}$$
$$\frac{-72}{-18} = \frac{k(-18)}{-18} \qquad \textit{Step \#2}: \text{Find } k.$$
$$4 = k$$

$$y = 4x \qquad \textit{Step \#3}: \text{Replace } k \text{ in the direct variation equation.}$$

Example 2: Each day Michael roller blades for exercise. When traveling at a constant rate, he travels 4 miles in about 20 minutes. At this rate, how long will it take Michael to travel 7 miles?

Write a direct variation equation, $d = rt$, that models Michael's distance as it varies with time.

$$rt = d$$
$$r(20) = 4$$

Find the constant of variation (r).

$$r = \frac{4}{20} = \frac{1\ \text{mi}}{5\ \text{min}} \quad \text{or} \quad \frac{1}{5} \text{ mile per minute}$$

Write the direct variation equation.

$$d = \frac{1}{5}t$$

Use the direct variation equation to solve the problem.

$$d = \frac{1}{5}t$$
$$7 = \frac{1}{5}t \qquad \text{--Substitution}$$
$$35 = t \qquad \text{--Multiply by 5.}$$

Thus, at this rate, it will take Michael 35 minutes to travel 7 miles.

The function above is graphed below. Study the graph carefully. As ***t* increases, *d* increases**. For example, when *t* is 5, *d* is 1. When *t* increases to 10, *d* increases to 2. When *t* increases to 35, *d* increases to 7. Note: the graph is **linear**.

$$d = \frac{1}{5}t$$

(35,5)
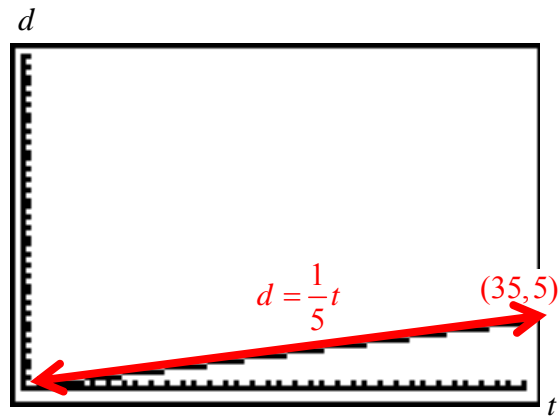
*This graph was created on a graphing calculator causing pixelation of the straight line. Thus, a **straight red line** was added to show the true appearance of the linear graph.

Inverse Variation

An inverse variation is a function that is defined by an equation in the following form: $xy = k$, where k is a nonzero real-number constant.

*There are many situations in which one quantity varies indirectly as another:

-As the rate increases, the time decreases when traveling a set distance.

-The volume of a gas in a container decreases as the pressure increases and the temperature remains constant.

The volume of a gas in a container with a constant temperature varies indirectly with the pressure.

Consider the following expressions:

$$xy = k$$

$$y = \frac{k}{x} \qquad \text{Divide both sides by } x.$$

$$y = k \cdot \frac{1}{x} \qquad \text{Another way to write } \frac{k}{x}.$$

y is directly proportional to the multiplicative inverse of x, $\frac{1}{x}$.

y varies inversely to x

Examine the following function: $xy = 30 \quad (k = 30)$

The given function is graphed below. Study the graph carefully. As **x increases, y decreases**. For example, when x is 1, y is 30. When x increases to 2, y decreases to 15. When x increases to 10, y decreases to 3. Note: the graph is **nonlinear**.
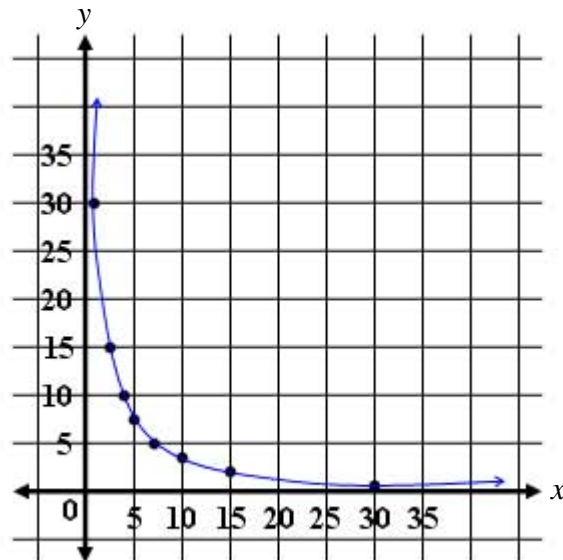
Exponential Growth and Decay

An **exponential function** is a nonlinear function in which the variable is in the exponent. The equation for the exponential function is in two variables that has the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$.

> **Exponential Function:**
>
> $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$.

Exponential Growth

Example 1: Graph the exponential function $y = 3(2)^x$.

*Step 1*: Make a table of values. Choose several *x*-values and find the corresponding *y*-values.

x	y
−3	$\frac{3}{8}$
−2	$\frac{3}{4}$
−1	$\frac{3}{2}$
0	3
1	6
2	12
3	24

$y = 3(2)^{-3}$ $\quad$ $y = 3(\frac{1}{2^3})$ $\quad$ $y = 3(\frac{1}{8})$ $\quad$ $y = \frac{3}{8}$

$y = 3(2)^{-2}$ $\quad$ $y = 3(\frac{1}{2^2})$ $\quad$ $y = 3(\frac{1}{4})$ $\quad$ $y = \frac{3}{4}$

$y = 3(2)^{-1}$ $\quad$ $y = 3(\frac{1}{2^1})$ $\quad$ $y = 3(\frac{1}{2})$ $\quad$ $y = \frac{3}{2}$

$y = 3(2)^0$ $\quad$ $y = 3(1)$ $\quad$ $y = 3$

$y = 3(2)^1$ $\quad$ $y = 3(2)$ $\quad$ $y = 6$

$y = 3(2)^2$ $\quad$ $y = 3(4)$ $\quad$ $y = 12$

$y = 3(2)^3$ $\quad$ $y = 3(8)$ $\quad$ $y = 24$

Step 2: Use the table to make a list of ordered pairs.

$$\left(-3, \frac{3}{8}\right), \ \left(-2, \frac{3}{4}\right), \ \left(-1, \frac{3}{2}\right), \ (0, 3), \ (1, 6), \ (2, 12), \ (3, 24)$$

Step 3: Graph the ordered pairs. Draw a smooth curve through the points as shown below.
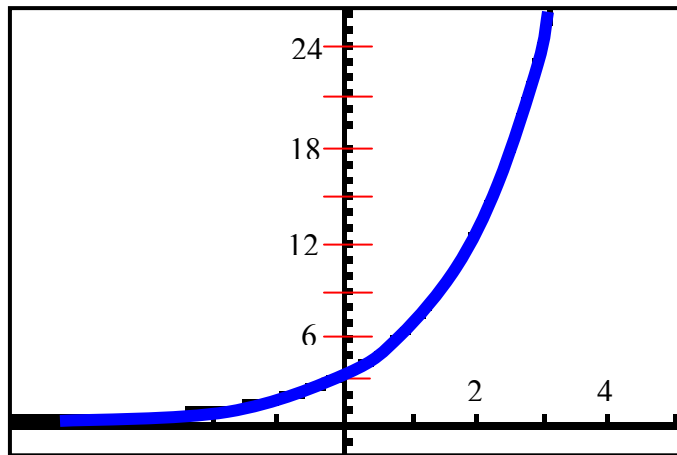
Exponential growth occurs in an exponential function when the quantity increases. The larger the quantity gets, the faster it grows. As x increases, y increases where $a > 0$ and $b > 1$.
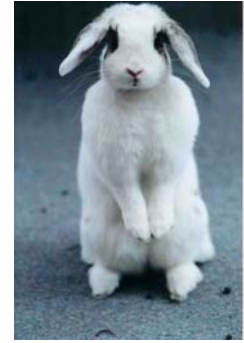
Exponential Growth:

$y = ab^x$, where $a > 0$ and $b > 1$

When exponential growth occurs, the growth appears to start out slow, but grows quickly (exponentially) as the x-values increase. Graphs of exponential functions are nonlinear.

Note: In Example 1, y increases as x increases where $a > 0$ and $b > 1$. Therefore, the function $y = 3(2)^x$ is an example of exponential growth.

Example 2: Thirty rabbits are introduced to a secluded area with no predators. Assume that the rabbit population doubles every six months. How many rabbits will be in the area after four (4) years?

    a.) Since the population doubles twice a year, in 4 years it will have doubled 8 times.

    b.) There are 30 original rabbits.

    c.) The rabbits are doubling every 6 months.

$$30 \times 2^8 = 7680$$

In this example of exponential growth, the population doubles (is multiplied by 2) every six months, so $b = 2$. There are 8 time periods (4 years = 8 half years) so $x = 8$. The original number of rabbits, 30, is a.

$$y = ab^x$$
$$y = (30)(2)^8 \qquad a = 30, \; b = 2, \; x = 8$$
$$y = (30)(256)$$
$$y = 7680$$

In four years there will be 7680 rabbits.

Exponential Decay

Exponential decay occurs in an exponential function when the quantity decreases. As x increases, y decreases as where $a > 0$ and $0 < b < 1$.

Exponential Decay:

$y = ab^x$, where $a > 0$ and $0 < b < 1$

Example 3: Graph the exponential function $y = \left(\dfrac{1}{4}\right)^x$.

*Note: $0 < \dfrac{1}{4} < 1$ This means $b\left(\dfrac{1}{4}\right)$ is between 0 and 1.

Step 1: Make a table of values. Choose several x-values and find the corresponding y-values.

x	y
−3	64
−2	16
−1	4
0	1
1	$\dfrac{1}{4}$
2	$\dfrac{1}{16}$
3	$\dfrac{1}{64}$

$y = \left(\dfrac{1}{4}\right)^{-3}$ $y = 4^3$ $y = 64$

$y = \left(\dfrac{1}{4}\right)^{-2}$ $y = 4^2$ $y = 16$

$y = \left(\dfrac{1}{4}\right)^{-1}$ $y = 4^1$ $y = 4$

$y = \left(\dfrac{1}{4}\right)^{0}$ $y = 1$

$y = \left(\dfrac{1}{4}\right)^{1}$ $y = \dfrac{1}{4}$
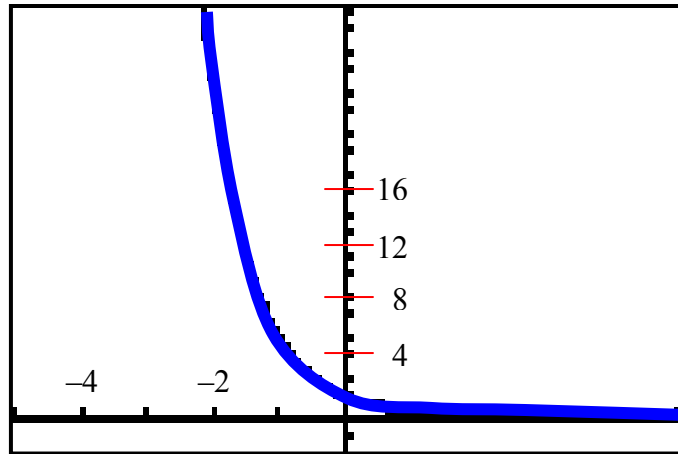
$y = \left(\dfrac{1}{4}\right)^{2}$ $y = \dfrac{1}{16}$

$y = \left(\dfrac{1}{4}\right)^{3}$ $y = \dfrac{1}{64}$

Step 2: Use the table to make a list of ordered pairs.

$(-3, 64), (-2, 16), (-1, 4), (0, 1), \left(1, \dfrac{1}{4}\right), \left(2, \dfrac{1}{16}\right), \left(3, \dfrac{1}{64}\right)$

Step 3: Graph the ordered pairs. Draw a smooth curve through the points as shown below.

Note: In this example, y decreases as x increases where $a > 0$ and $0 < b < 1$.

Therefore, exponential decay occurs in the exponential function $y = \left(\dfrac{1}{4}\right)^x$.

Example 4: A town with a population of 64,500 is losing 3% of its population each year. At this rate, how many people will be left in the town after 15 years?

   a.) The growth factor will be $1 - .03 = .97$

   b.) Population $= 64,500(.97)^{15}$

$$\approx 40,845$$

In this example of exponential decay, the population decreases (3%) every year, so $b = 0.97$. There are 15 time periods so $x = 15$. The original population, 64,500, is a.

$y = ab^x$

$y = (64,500)(0.97)^{15}$ $a = 64,500,\ b = 0.97,\ x = 15$

$y = (64,500)(0.633251)$ Round b to the nearest millionth.
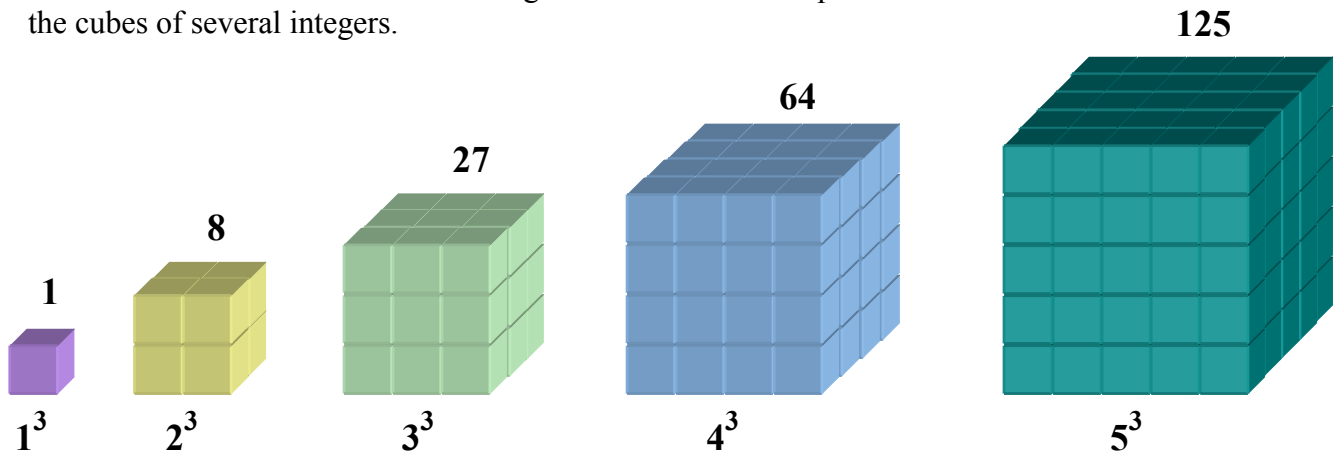
$y \approx 40,845$ Round to the nearest whole number.

After 15 years the population will be about 40,845 people in the town if the rate of decay continues at 3%.

Perfect Cubes and Their Cube Roots

Perfect Cubes x^3

Perfect cubes are numbers that are integers raised to the third power. Let's take a look at the cubes of several integers.

1 **8** **27** **64** **125**

1^3 2^3 3^3 4^3 5^3

Positive Integers	Perfect Cube
1^3 (1 cubed)	1
2^3 (2 cubed)	8
3^3 (3 cubed)	27
4^3 (4 cubed)	64
5^3 (5 cubed)	125

Negative Integers	Perfect Cube
$(-1)^3$ $(-1 \times -1 \times -1)$	-1
$(-2)^3$ $(-2 \times -2 \times -2)$	-8
$(-3)^3$ $(-3 \times -3 \times -3)$	-27
$(-4)^3$ $(-4 \times -4 \times -4)$	-64
$(-5)^3$ $(-5 \times -5 \times -5)$	-125

Cube Roots $\sqrt[3]{x}$

A **cube root** of a number is a number that, when cubed, is equal to the given number. It is denoted with an exponent of "1/3".

> ### Cube Root:
>
> If $r^3 = x$, then $r = x^{\frac{1}{3}}$ or $r = \sqrt[3]{x}$
>
> where r is a real number that can **positive** or **negative.**

The cube root of 27 is 3 written as $27^{\frac{1}{3}}$ or $\sqrt[3]{27} = 3$.

The cube root of –27 is –3 written as $(-27)^{\frac{1}{3}}$ or $\sqrt[3]{-27} = -3$.

One way to find the cube root of a perfect cube is to use "trial and error" guessing to find the solution.

Let's use "trial and error" to solve the following equation for x.

$$x^3 = 343$$

By testing several integers, we find that x equals 7.

$$7(7)(7) = 343 \quad \text{Thus, } x = 7.$$

An odd root of a negative number is a negative number.

Now let's solve:

$$x^3 = -343$$

Since the cubed number is negative, use trial and error trying negative roots. We find that x equals –7.

$$-7(-7)(-7) = -343 \quad \text{Thus, } x = -7.$$

This "trial and error" system for finding the cube roots of perfect cubes such as 343, 512, 729, 1000, or even the value of 1, is sufficient for this course.

The first twelve "perfect cubes" of the counting numbers is shown below.

$$\{1^3, 2^3, 3^3, 4^3, \ 5^3, \ 6^3, \ 7^3, \ 8^3, \ 9^3, \ 10^3, \ 11^3, \ 12^3...\}$$
$$\downarrow \downarrow \downarrow \ \downarrow \ \downarrow \ \downarrow \ \ \downarrow \ \ \downarrow \ \ \downarrow \ \ \downarrow \ \ \downarrow \ \ \downarrow$$
$$\{1, \ 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728...\}$$

*Note: You will examine the cube root of non-perfect cubes in a higher level math course.

Graph Paper