INEQUALITIES AND ABSOLUTE VALUE

In this unit, you will review graphing and solving inequalities. You will then examine compound inequalities in one variable joined by “and” or “or”. You will also solve absolute value equations and inequalities.

Solving One-Step Inequalities

An **inequality** is a mathematical sentence that contains one of the following symbols:

$<$	"less than"
$>$	"greater than"
$\leq$	"less than or equal to"
$\geq$	"greater than or equal to"

Just like equations, inequalities are solved by using opposite operations. The one exception is that **when multiplying or dividing by a negative number, flip the inequality sign**. We will address this issue in the next unit when we study integers.

Let's take a look at a few examples of solving inequalities. ***Note: it will be easier to understand the solution set if you write your answer with the variable on the left.** (You will see what this means in example 2.)

Example 1: Solve $n - 7 < 22$

-The opposite of – 7 is + 7.

$$\begin{array}{r} n - 7 < 22 \\ +7 \quad +7 \\ \hline n < 29 \end{array}$$

This solution means that all numbers (n) less than 29 are a solution to the inequality.

Example 2: Solve $48 \geq 6y$

-The opposite of multiplying by 6 is dividing by 6.

$$\frac{48}{6} \geq \frac{6y}{6}$$

$$8 \geq y$$

Rewrite this solution with the variable *y* on the left. When doing this, make sure the inequality sign is pointing to the same term as in the original solution (in this case the *y*).

$y \leq 8$ *Notice the inequality is still pointing to the variable.

This solution means that all numbers *y*, that are less than or equal to 8, are a solution.

Graphing Inequality Solutions

The solution set of an inequality can be graphed on a number line in the following manner:

-If < or >, use an open circle (○) on the number line because the solution set **does not** include the solution (it includes only values less than or more than the solution).

-If $\leq$ or $\geq$, use a closed circle (●) on the number line because the solution set **does** include the solution.

Once you have determined if you are going to use an open circle or a closed circle, you will shade the part of the number line that includes the solution set.

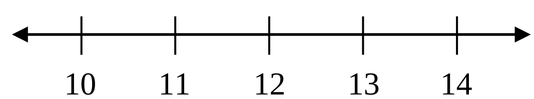
Let's take a look at a couple of examples:

Example 1: Solve and graph.

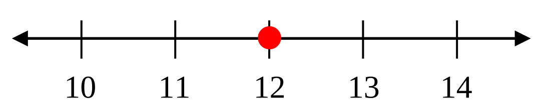
$$\begin{array}{r} m+3 \leq 15 \\ -3 \quad -3 \\ \hline m \leq 12 \end{array}$$

This solution tells us that all values "less than or equal" to 12 will result in a true statement. Follow the steps below to graph this inequality solution.

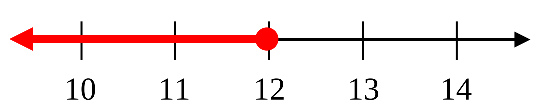
a) Draw a number line and label it with a few numbers.



b) Determine which circle to use. In this case, $\leq$ means to use a closed circle. Place a closed circle on the 12, since this was the solution to the inequality.



c) Determine which direction to shade on the number line. In this case, we want numbers less than 12; so, shade to the left of 12.



Example 2: Solve and graph.

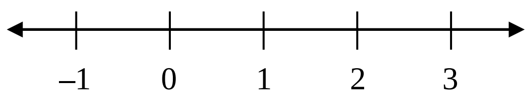
$8 < 4n$

$\frac{8}{4} < \frac{4n}{4}$

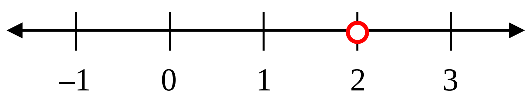
$2 < n$ *rewrite with the variable on the left

$n > 2$

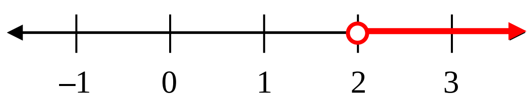
a) Draw a number line



b) Use an open circle around the solution because the inequality is >.



c) Shade the number line to the right, as the solution states that all values greater than 2 will be in the solution.



Solving Two-Step Inequalities

Just like equations, inequalities are solved by using opposite operations. **The one exception is that when multiplying or dividing by a negative number, flip the inequality sign.**

Let's take a look at a few examples of solving two-step inequalities.

*Note: it will be easier to understand the solution set if you write your answer with the variable on the left.

Example 1: Solve $4n + 9 < 53$

-the opposite of +9 is –9

$$\begin{array}{r} 4n + 9 < 53 \\ -9 \quad -9 \\ \hline 4n < 44 \end{array}$$

$$n < 11$$

This solution means that all numbers (n) less than 11 are a solution to the inequality.

Example 2: Solve $67 \geq 5y - 8$

-the opposite of –8 is +8

$$\begin{array}{r} 67 \geq 5y - 8 \\ +8 \qquad +8 \\ \hline 75 \geq 5y \end{array}$$

$$15 \geq y$$

Rewrite this solution with the variable y on the left. When doing this, make sure the inequality sign is pointing to the same term as in the original solution (in this case the y).

$y \leq 15$ *Notice the inequality is still pointing to the variable.

This solution means that all numbers (y) that are less than or equal to 15 are a solution.

Example 3: Solve $-6t + 5 > -13$

-the opposite of +5 is –5

$$\begin{array}{r} -6t + 5 < -13 \\ \underline{\quad -5 \quad -5} \\ \underline{-6t} < \underline{-18} \\ -6 \qquad -6 \\ t > 3 \end{array}$$

*Notice the **inequality** is pointing in the **opposite direction** since the last step was to **divide by (–6).**

This solution means that all numbers (*t*) that are greater than 3 are a solution to the inequality.

Let’s take a closer look at the meaning of the answer above.

For example, 10 is greater than 3, and therefore should prove TRUE. Let’s check.

Thus, $-6(10) + 5 < -13$
$-60 + 5 < -13$
$-55 < -13$ (a TRUE statement)

Ten (a number greater than 3) is a member of the solution set.

Another example is that 0 is less than 3, and therefore, should prove FALSE. Let’s check.

Thus, $-6(0) + 5 < -13$
$0 + 5 < -13$
$5 < -13$ (a FALSE statement)

Zero (a number that is NOT greater than 3) is NOT a member of the solution set.

Example 4: Write an inequality for the following math statement, and then solve: "Twice a number increased by 11 is at least 41".

$$\underbrace{\text{Twice a number}}_{2x}\ \underbrace{\text{increased by 11}}_{+\ 11}\ \underbrace{\text{is at least 41}}_{\geq\ 41}$$

$2x + 11 \geq 41$
$2x + 11 - 11 \geq 41 - 11$
$2x \geq 30$
$x \geq 15$

Thus, twice any number greater than or equal to 15, plus 11, is at least 41.

Example 5: Ella's bookcase has four shelves. The top three shelves hold the same number of books. The bottom shelf has 13 books on it. If the bookcase will hold, at most, 64 books, what is the highest number of books that could be on the top shelf?

Think algebraically.

Let b = the number of books on one shelf.

$3b$	-number of books on three shelves
$3b + 13$	-number of books on three shelves plus bottom shelf
64	-most amount of books that can fit on the bookcase
$3b + 13 \leq 64$	-total number of books as an expression set "less than or equal" to total possible number of books (at most)

Now solve.

$3b + 13 \leq 64$
$3b + 13 - 13 \leq 64 - 13$
$3b \leq 51$
$b \leq 17$

Each of the top three shelves could hold up to (at most) 17 books. Thus, the highest number of books on the top shelf could be 17 books.

Solve Inequalities in One Variable

inequality: a mathematical statement involving one of the following symbols:

$$<,\ >,\ \leq,\ \geq,\ \neq$$

*Solving inequalities is just like solving equations, use opposite operations to isolate the variable.

When multiplying or dividing by a negative number, the inequality sign must be reversed.

Example 1: $2y + 9 < 5y + 15$

$-5y \qquad -5y$

$-3y + 9 < 15$

$-9 \qquad -9$

$$\frac{-3y}{-3} < \frac{6}{-3}$$

$y > -2$ **Notice that the inequality sign is flipped because of the division by –3.**

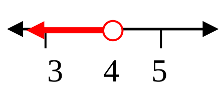
You can represent the solution of an inequality in one variable on a number line.

For < and > an **open circle** is used to denote that the solution number **is not** included in the solution.

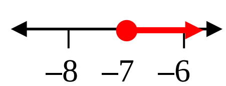
For $\leq$ and $\geq$ a **closed circle** is used to denote that the solution number **is** included in the solution.

Example 2: Graph the given inequalities.

$x < 4$



$y \geq -7$



compound inequalities: a pair of inequalities joined by "**and**" or "**or**".

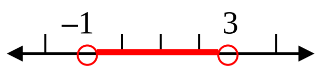
To solve a compound inequality joined with "**and**", find the values of the variable that satisfy **BOTH** inequalities.

*"**and**" means the **intersection** of the solutions

Example 3: $2x + 3 > 1$ and $5x - 9 < 6$

$2x > -2$ and $5x < 15$

$x > -1$ and $x < 3$



The solution is written $\{x/ -1 < x < 3\}$ (set notation) "all numbers *x*, such that –1 is less than *x* is less than 3".

To solve a compound inequality joined with "**or**", find the values of the variable that satisfy at **least one** inequality.

"**or**" means the **union** of the solutions

Example 4: $3b + 7 \leq 1$ or $2b - 3 \geq 1$

$3b \leq -6$ or $2b \geq 4$

$b \leq -2$ or $b \geq 2$

The solution is written $\{b/b \leq -2 \text{ or } b \geq 2\}$ (set notation) "all numbers *b* such that *b* is less than or equal to –2 OR *b* is greater than or equal to 2".

Solving Absolute Value Equations and Inequalities

absolute value - the distance a number is from zero (always positive).

*Two bars around the number denote absolute value.

$|-5| = 5$ $\qquad$ $|6| = 6$

To solve absolute value equations:

1. Rewrite the equation without the absolute value notation.
2. Rewrite a second time using the opposite of what the original equation was equal to, and connect with the word "**or**".
3. Solve both equations and check both answers in the original equation.

Example 1: Solve $|2x-1| = 3$

$2x - 1 = 3$ or $2x - 1 = -3$ $\qquad$ the –3 is the opposite of what the original was equal to.

$2x = 4$ or $2x = -2$

$x = 2$ or $x = -1$

Check $|2(2)-1| = 3$ or $|2(-1)-1| = 3$

$|4-1| = 3$ or $|-2-1| = 3$

$|3| = 3$ or $|-3| = 3$

$3 = 3$ or $3 = 3$

Therefore, the solution to the example is 2 or –1.

Example 2: Solve $|2x+1| = x + 5$

$2x + 1 = x + 5$ or $2x + 1 = -x - 5$ $\qquad$ *again use the opposite

$x = 4$ or $3x = -6$

$x = -2$

Check: $|2(4)+1| = 4 + 5$ or $|2(-2)+1| = -2 + 5$

$|8+1| = 9$ or $|-4+1| = 3$

$|9| = 9$ or $|-3| = 3$

$9 = 9$ or $3 = 3$

Therefore, the solution to the example is 4 or –2.

absolute value inequalities - an absolute value that contains an inequality.

To solve absolute value inequalities:

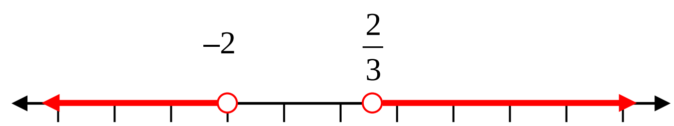
1.) Rewrite the inequality without the absolute value notation.
2.) Rewrite a second time, change the inequality sign, and use opposites.
3.) Solve both inequalities and check both answers in the original inequality.
4.) If the inequality is a < or ≤ , connect with the word "and".
5.) If the inequality is a > or ≥ , connect with the word "or".

Example 3: Solve $|3x+2| > 4$

$3x + 2 > 4$ **or** $3x + 2 < -4$ *flip the sign and use the opposite

$3x > 2$ **or** $3x < -6$

$x > \frac{2}{3}$ **or** $x < -2$



Check: $|3x+2| > 4$

To check this problem you will have to choose a number that is less than –2, and then choose a number that is greater than $\frac{2}{3}$.

Check (-3)		Check (1)
$\|3(-3)+2\| > 4$	or	$\|3(1)+2\| > 4$
$\|-9+2\| > 4$	or	$\|3+2\| > 4$
$\|-7\| > 4$	or	$\|5\| > 4$
$7 > 4$ (true)		$5 > 4$ (true)

Therefore, the solution to this absolute value inequality is $\{x / x < -2 \text{ or } x > \frac{2}{3}\}$.

Example 4: Solve $\frac{1}{2}|5x-12|+4 \le 13$

$$-4 \quad -4$$

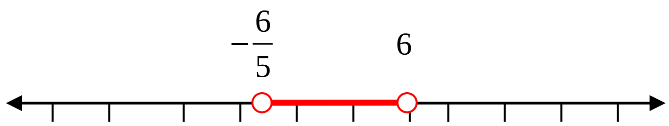
$$(2)\frac{1}{2}|5x-12| \le (2)9$$

$$|5x-12| \le 18$$

$5x-12 \le 18$ **and** $5x-12 \ge -18$ *flip the sign and use opposite

$5x \le 30$ $\qquad$ $5x \ge -6$

$x \le 6$ **and** $x \ge \frac{-6}{5}$



Check: $\frac{1}{2}|5x-12|+4 \le 13$

To check this problem, you will have to choose a number that is greater than $-\frac{6}{5}$ and also less than 6.

Check (0)

$$\frac{1}{2}|5x-12|+4 \le 13$$

$$\frac{1}{2}|5(0)-12|+4 \le 13$$

$$\frac{1}{2}|-12|+4 \le 13$$

$$\frac{1}{2}(12)+4 \le 13$$

$$6+4 \le 13$$

$10 \le 13$ (true)

Therefore, the solution to this absolute value inequality is $\{x/\ x > -\frac{6}{5}$ and $x < 6\}$.

Linear Inequalities

A linear inequality is like a linear equation, except the equal sign is replaced with an inequality sign, and the solution to the inequality is a region of the coordinate plane.

Example 1:

A linear equation would be in the form of $y = \frac{-2}{3}x + 4$

A linear inequality would look like $y \geq \frac{4}{5}x - 2$

Notice that the equal sign from the top equation was replaced by an inequality sign in the bottom equation.

A linear inequality can have one of the following signs:

$<$ “less than”

$>$ “greater than”

$\leq$ “less than or equal to”

$\geq$ “greater than or equal to”

To graph a linear inequality, you will follow similar steps to solving linear equations:

1) Solve the inequality for y and make sure it is in the slope-intercept form ($y = mx + b$) except you will have an inequality sign.

2) Plot the y-intercept $(0, b)$.

3) Use the slope ratio $\left(\frac{rise}{run}\right)$ to plot more points.

4) Draw a dashed line through the points if the inequality is $<$ or $>$ (◄- - - -►), or a solid line if the inequality is $\leq$ or $\geq$ (◄————►).

5) Shade a region of the graph that satisfies the inequality.

Let’s practice a couple of examples.

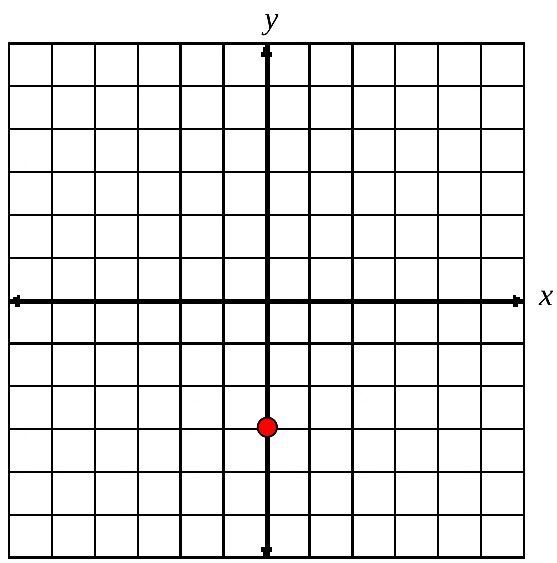
Example 1: Graph $x + y < -3$

a.) Solve for y.

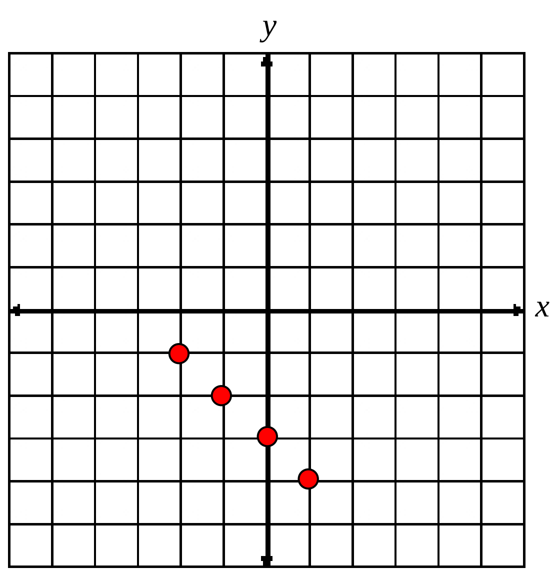
$x + y < -3$

$y < -x - 3$

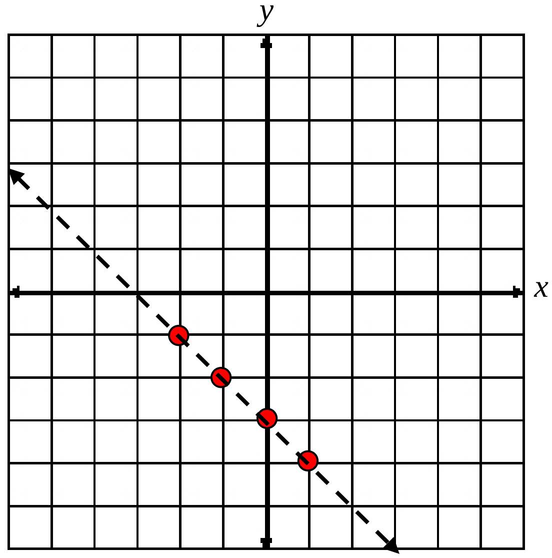
b.) Plot the y-intercept (0, –3).



c.) Use the slope, $\frac{-1}{1}$, to plot more points.



d.) Draw a dashed line through the points because the inequality is less than (<).



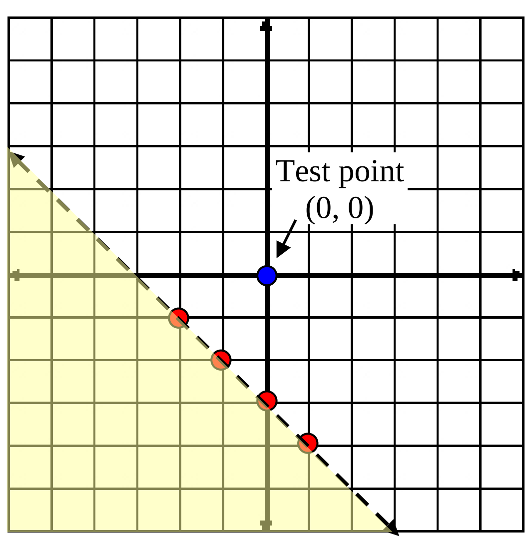
e.) Determine the region to shade. If the line you have just graphed is the boundary of your region, where would the *y*-values be less than this boundary, up to the right or down to the left? In this case, down to the left.

*If you are not sure which region to shade, you could always pick a test point to help you determine the shading. The easiest point to choose is (0, 0). If you replace *x* and *y* with (0, 0) in the original inequality and the result is a **true** statement, you will shade the region that contains the point (0, 0). If you test the point (0, 0) and the result is a **false** statement, you will shade the region that does not contain the point (0, 0).

Test: $x + y < -3$

$0 + 0 < -3$

$0 < -3$ False: this means that you should shade the region that **DOES NOT** contain (0, 0).



Example 2: Graph $-2x + 3y \geq -6$

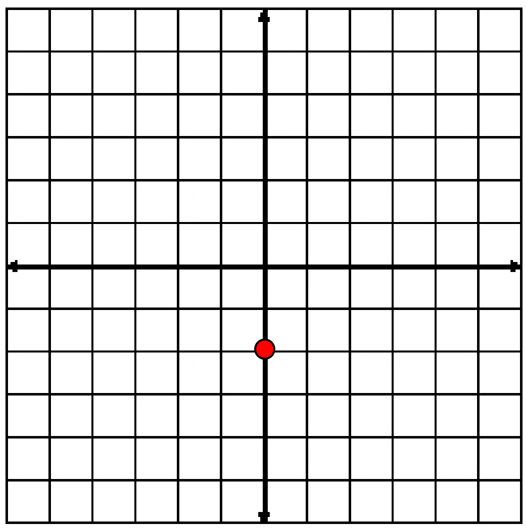
a.) Solve for y.

$-2x + 3y \geq -6$

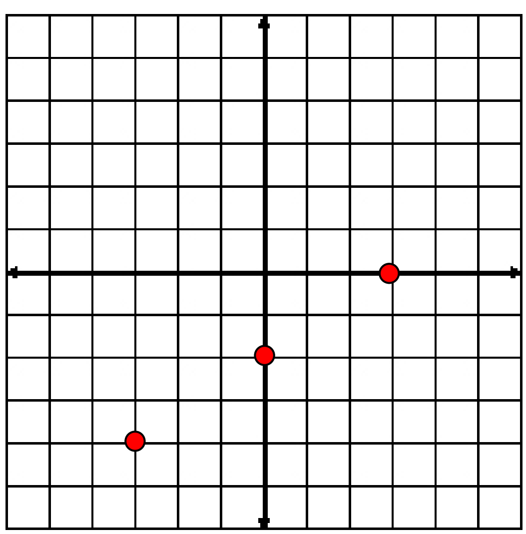
$3y \geq 2x - 6$

$$y \geq \frac{2}{3}x - 2$$

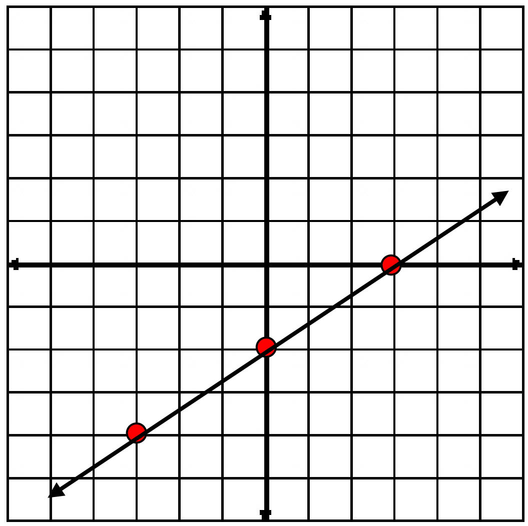
b.) Plot the y-intercept $(0, -2)$.



c.) Use the slope $\frac{2}{3}$, to plot more points.



d.) Draw a solid line through the points because the inequality is “greater than or equal to” ($\geq$).



e.) Determine which region you need to shade, either:

1) choose a test point, (0, 0) or

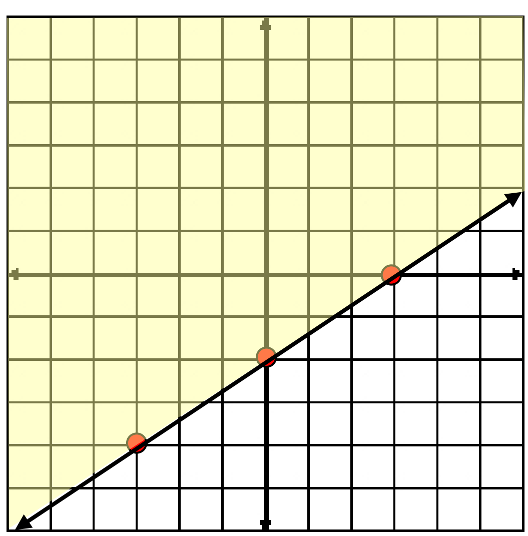
2) determine where the *y*-values are greater than the boundary line, up to the left, or down to the right.

*In this case you will shade up to the left of the boundary line because this is where the *y*-values are greater than the boundary. You could also check by replacing *x* and *y* with (0, 0).

Test: $-2x + 3y \geq -6$

$-2(0) + 3(0) \geq -6$

$0 \geq -6$ True: So shade the region that contains the point (0, 0).



Graphing Systems of Linear Inequalities

Now that you have learned how to graph a linear inequality, we will expand on this and graph a system of linear inequalities. A system of linear inequalities is like a system of equations except that the solution to a system of inequalities is a region on the coordinate plane that represents the intersection of both inequalities.

To solve a system of inequalities:

1) Graph each inequality on a coordinate plane (one at a time).

2) Shade the solution to each inequality.

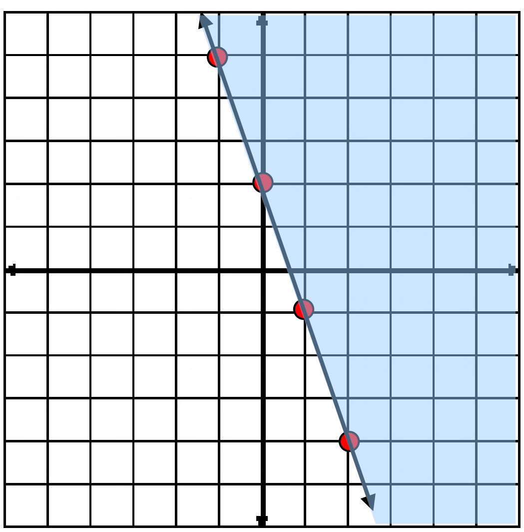
3) Determine where the shading of both inequalities intersects.

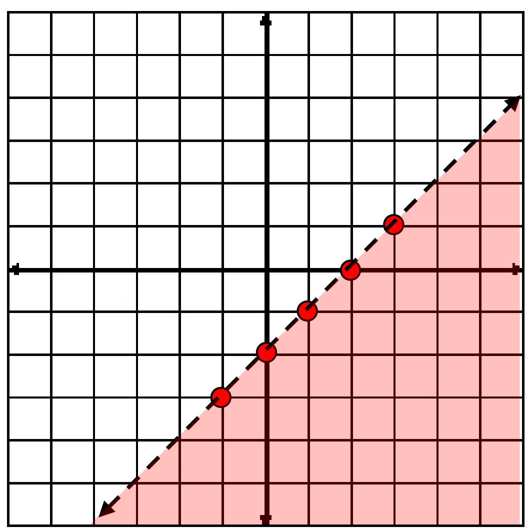
Example: Graph the solution to the system of inequalities shown below.

$$y \geq -3x + 2$$
$$y < x - 2$$

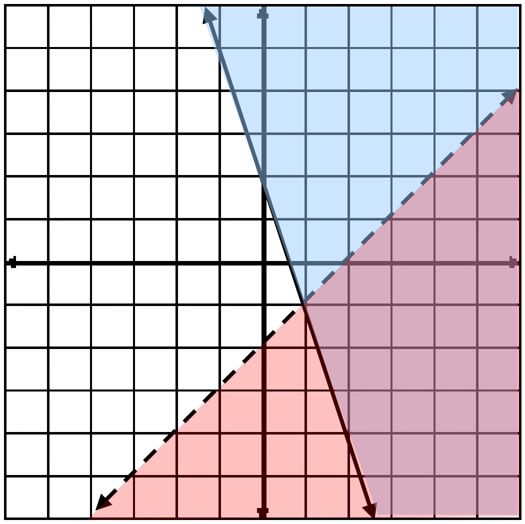
A. Graph the boundary line of $y \geq -3x + 2$ and shade the solution which is to the right of line.

B. Graph the boundary line of $y < x - 2$ and shade the solution which is below the line.





The two inequalities are shown on separate planes so you can see the solutions easier. When you solve a system of inequalities you will do so on one plane as shown below.



*Notice that the intersection of the two inequalities is the darker shading (purple area) in the lower right of the coordinate plane. This region represents the solution to the system. This means that any point in this region represents a solution to the system.

Graph Paper