PYTHAGOREAN THEOREM AND RIGHT TRIANGLES

In this unit, you will explore the Pythagorean Theorem and extend its application to the distance formula in coordinate graphing. You will examine special right triangles, the 45-45-90 degree right triangle and the 30-60-90 degree right triangle. You will also examine graphing irrational numbers using the Pythagorean Theorem.

Pythagorean Theorem and Distance Formula

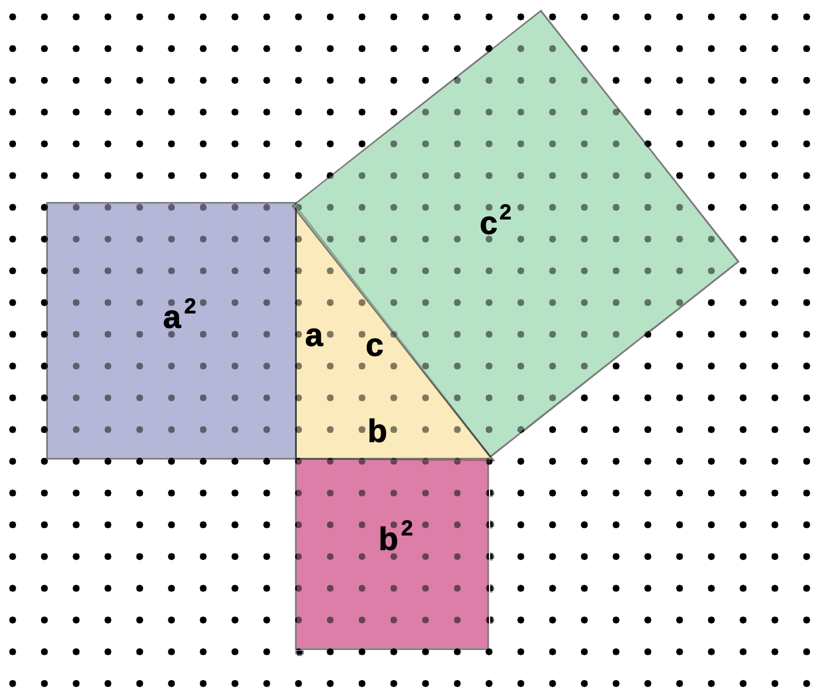
Special Right Triangles

Graphing Irrational Numbers

Pythagorean Theorem and Distance Formula

Theorem - A theorem is a mathematical statement that must be proven before it is accepted as being true.

Pythagorean Theorem - The Pythagorean Theorem is a relationship between the three sides of a right triangle. The sum of the squares of the two sides of the right triangle that make up the right angle are equal to the square of the third side, the hypotenuse which is the side opposite the right angle.



Special names are given to the sides of a right triangle. The two sides that make up the right triangle are called "**legs**" and the side opposite the right angle is called the "**hypotenuse**".

A special relationship exists between the sides of a right triangle. The sum of the squares of the two legs equals the square of the hypotenuse.

$$c^2 = a^2 + b^2$$

In the example above, the legs measure 6 and 8 units. What does the diagonal measure?

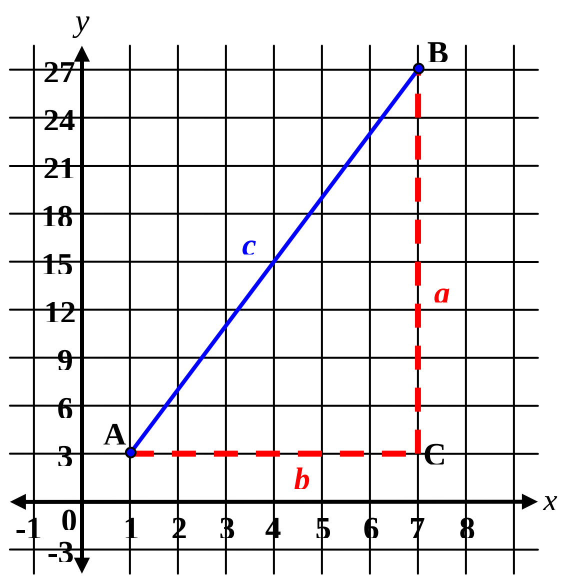
$c^2 = 6^2 + 8^2$
$c^2 = 36 + 64$
$c^2 = 100$
$c = \sqrt{100}$
$c = 10$

Pythagorean Theorem

In a right triangle, the sum of the squares of the measures of the legs equals the square of the measure of the hypotenuse.

Example 1: Use the Pythagorean Theorem to find the distance from A to B.

*Note: Each space on the *x*-axis equals one unit. Each space on the *y*-axis equals 3 units.



Point A (1, 3) Point B (7, 27) Point C (7, 3)

Leg *a*: Find the length of $\overline{BC}$.

Point C (7, 3) to Point B (7, 27)

To find the distance, look at the change in the y-coordinates.

a = $\overline{BC} = |\,3 - 27| = |-24| = 24$

Leg *b*: Find the length of $\overline{AC}$.

Point A (1, 3) to Point C (7, 3)

To find the distance, look at the change in the *x*-coordinates.

b = $\overline{AC} = |\,1 - 7| = |-6| = 6$

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
24^2 + 6^2 &= c^2 \\
576 + 36 &= c^2 \\
612 &= c^2 \\
\pm\sqrt{612} &= c \\
24.7 &\approx c
\end{aligned}
$$

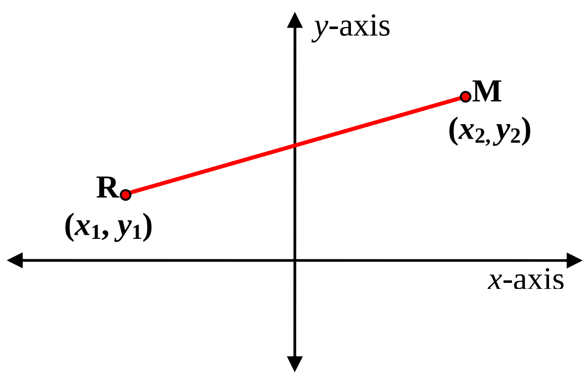
*612 is an irrational number

*612 is an irrational number and its root extends on forever and never develops into a repeating pattern. For this course, round irrational answers as directed. The symbol for approximately equal is $\approx$. In geometry, the negative root is often ignored because the problems are mostly about distance as is true in this problem.

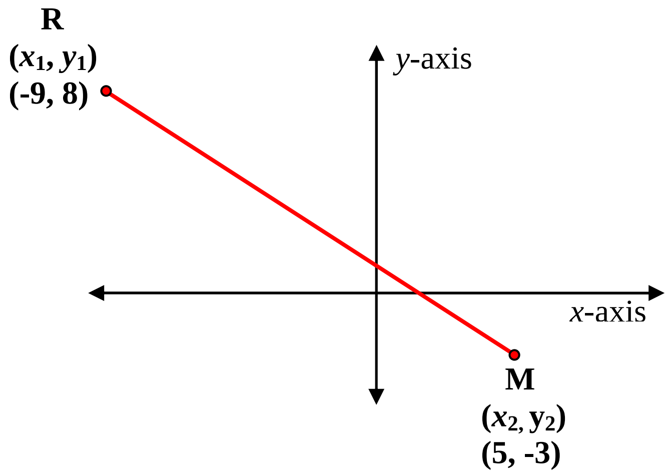
In coordinate geometry, the Pythagorean Theorem can be adapted to the Distance Formula.

Distance Formula

The distance *d* between any two points with coordinates (x_1, y_1) and (x_2, y_2) is given by the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **.**



Example 2: Find the length of $\overline{RM}$ for R(–9, 8) and M(5, –3).



Let R be Point 1 and represented by (x_1,y_1). $(x_1,y_1) = (–9, 8)$

Let M be Point 2 and represented by (x_2,y_2). $(x_2,y_2) = (5, –3)$

Thus,

$x_1 = –9$ $y_1 = 8$ $x_2 = 5$ $y_2 = –3$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{(5-(-9))^2 + (-3-8)^2}$$
$$d = \sqrt{(14)^2 + (-11)^2}$$
$$d = \sqrt{196+121}$$
$$d = \sqrt{317}$$
$$d \approx 17.8$$

Pythagorean Triples

Pythagorean triple – A Pythagorean triple is a set of three "whole" numbers that satisfies the Pythagorean Theorem where *c* is the greatest of the three whole numbers.

The whole numbers, 3, 4, and 5 are an example of a Pythagorean triple.

$$a^2 + b^2 = c^2$$
$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$
$$25 = 25$$

The wholes numbers 4, 5, and 6 are an example of three numbers that are NOT a Pythagorean triple.

$$a^2 + b^2 = c^2$$
$$4^2 + 5^2 \neq 6^2$$
$$16 + 25 \neq 36$$
$$41 \neq 36$$

Example 3: Are the lengths, 13, 84, and 85, lengths of the sides of a right triangle?

Since these three numbers are whole numbers, we will determine if they form a Pythagorean triple.

$$a^2 + b^2 = c^2$$
$$\text{Does } 13^2 + 84^2 \; = \; 85^2\,?$$
$$\text{Does } 169 + 7056 \; = \; 7225?$$
$$7225 = 7225 \quad \text{Yes!}$$

The whole numbers 13, 84, and 85 forma a Pythagorean triple.

Here is an interesting formula to investigate:

> Suppose that *m* and n are two positive integers with $m < n$,
> then $n^2 - m^2,\ 2mn,$ and $n^2 + m^2$ is a **Pythagorean triple**.

Let's test this formula.

Example 4: If $n = 5$ and $m = 3$, what Pythagorean triple will these numbers produce?

$n^2 - m^2$	$2mn$	$n^2 + m^2$
$5^2 - 3^2$	$2(5)(3)$	$5^2 + 3^2$
16	30	34

Check: Does $16^2 + 30^2 = 34^2$?

$$16^2 + 30^2 = 34^2$$
$$256 + 900 = 1156$$
$$1156 = 1156$$

Yes!

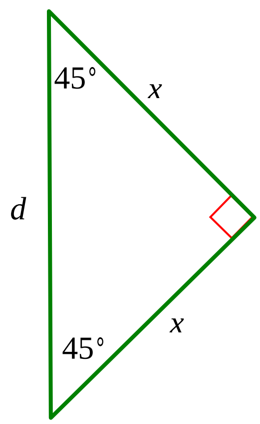
Special Right Triangles

45-45-90 Triangle

In a 45-45-90 degree triangle, the length of the hypotenuse can be determined by multiplying $\sqrt{2}$ times the leg.

Let's take a look at why this is so!

When given a right isosceles triangle, the angles opposite the congruent sides are congruent 45 degree angles. Determine the length of *d* for any isosceles right triangle with hypotenuse *d* and leg *x*.



We will use the Pythagorean Theorem to develop the formula for finding the hypotenuse (*d*) when given the length of either of the congruent sides.
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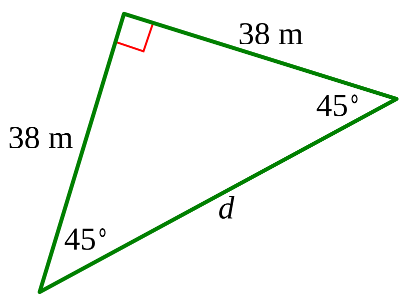
$a^2 + b^2 = c^2$	Pythagorean Theorem
$x^2 + x^2 = d^2$	Substitute $(a = x, b = x, c = d)$
$2x^2 = d^2$	Collect like terms.
$\sqrt{2x^2} = \sqrt{d^2}$	Take the square root of each side.
$\sqrt{2}\sqrt{x^2} = \sqrt{d^2}$	Split the radical.
$d = x\sqrt{2}$	

To summarize the relationship of the lengths of the three sides of a 45-45-90 degree triangle:

leg a = leg b	$\rightarrow$	x
hypotenuse	$\rightarrow$	$x\sqrt{2}$

Example 1: Find the length of the hypotenuse of a right isosceles triangle with a leg measuring 38 meters.



$d = x\sqrt{2}$	Relationship of the legs and hypotenuse of a 45-45-90 degree triangle
$d = 38\sqrt{2}$	Substitute $(x = 38)$
$d \approx 53.7$	Multiply if a decimal answer is specified.

Example 2: Find the length of the leg of a right isosceles triangle that has a hypotenuse measuring $25\sqrt{2}$ yards.

$d = x\sqrt{2}$	Relationship of the legs and hypotenuse of a 45-45-90 degree triangle
$25\sqrt{2} = x\sqrt{2}$	Substitute $(d = 25\sqrt{2})$
$x = 25$	Divide both sides by $\sqrt{2}$.

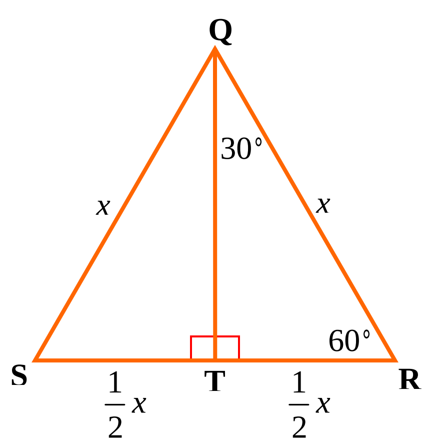
30-60-90Triangle

In a 30-60-90 degree triangle, the length of the hypotenuse is twice as long as the shorter leg and the longer leg equals the shorter leg multiplied by $\sqrt{3}$.

Let's take a look at why this is so!

First we can start with an equilateral triangle and draw its altitude.

The altitude of an equilateral triangle divides it into two 30-60-90 degree triangles.



*Notice that in either of the smaller triangles, the shorter leg is opposite a 30 degree angle and is half of the length of the of one side of the equilateral triangle.

$$\text{length of shorter leg } = \frac{1}{2}(\text{length of hypotenuse})$$

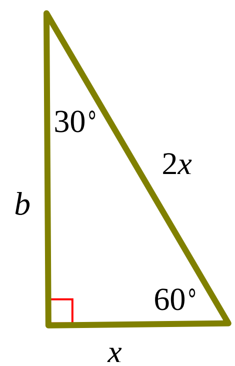
OR

$$2\left(\text{length of shorter leg}\right) = 2\left(\frac{1}{2}(\text{length of hypotenuse})\right)$$

EQUALS

$$2\left(\text{length of shorter leg}\right) = \text{length of hypotenuse}$$

Thus, if x equals the length of the shorter leg, then the length of the hypotenuse is $2x$.



We can then use the Pythagorean Theorem to develop a formula for finding the length of the side that is opposite the 60 degree angle.

Let x represent the length of the shorter leg and b represent the length of the longer leg.

Thus, the length of the hypotenuse equals $2x$.

$a^2+b^2=c^2$	Pythagorean Theorem
$x^2+b^2=(2x)^2$	Substitute $(a=x, c=2x)$
$x^2+b^2=4x^2$	Square $2x$ $(2x\cdot 2x=4x^2)$
$b^2=3x^2$	Subtract and collect like terms.
$\sqrt{b^2}=\sqrt{3x^2}$	Take the square root of each side.
$\sqrt{b^2}=\sqrt{3}\sqrt{x^2}$	Separate the radical.
$b=x\sqrt{3}$	Simplify

Now, we can use $x\sqrt{3}$ to represent the length of the longer leg.

To summarize the relationship of the three sides of a 30-60-90 degree triangle:

shorter leg	$\rightarrow$	x
longer leg	$\rightarrow$	$x\sqrt{3}$
hypotenuse	$\rightarrow$	$2x$

Example 3: What are the measures of the legs of a 30-60-90 degree triangle with a hypotenuse that measures 14 feet?

Step 1: Given: hypotenuse = 14, therefore 2*x* = 14

Step 2: shorter leg = *x*

$2x = 14$ length of hypotenuse

$x = 7$ Divide by 2

The shorter leg = 7 feet.

Step 3: longer leg = $x\sqrt{3}$

$x\sqrt{3}$ Length of longer leg

$7\sqrt{3}$ Substitute $(x = 7)$

The longer leg = $7\sqrt{3}$ feet.

Graphing Irrational Numbers

Irrational numbers are numbers that have decimals that go on forever, but never develop a repeating pattern. They are numbers that **cannot** be written as fractions where both the numerator and the denominator are integers.

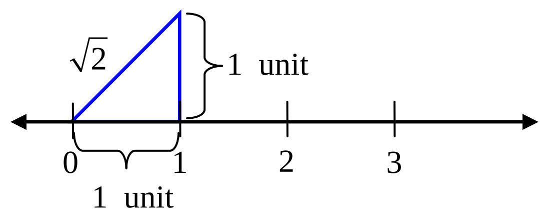
Irrational numbers do not repeat or terminate.

For example:

First 12 digits of $\pi\,(pi)$ = 3.14159265359 and so on...

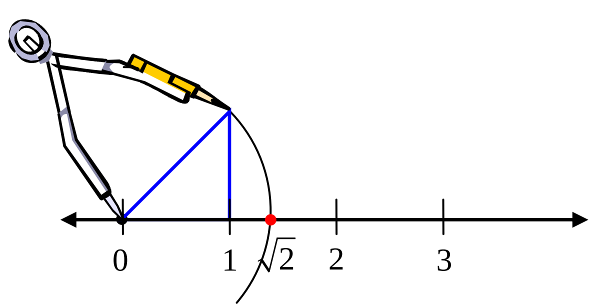
First 7 digits of $\sqrt{2} = 1.414213$ and so on...

Rational numbers can be graphed on a number line. To graph, $\sqrt{2}$, draw a right triangle with one leg on the number line and each leg with a length of 1 unit. By the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$.



Using a compass, you can transfer the length of the hypotenuse to the number line.

Place the metal point of the compass at zero (0) and the pencil point at the top of the right triangle. This length is the length of the square root of 2. Without changing the setting of the compass, draw an arc that passes through the number line. The point of intersection is the location of the $\sqrt{2}$ on the number line.



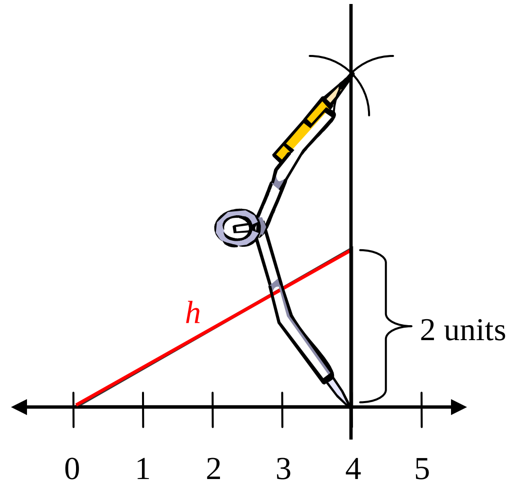
How would you locate an irrational number like $\sqrt{20}$ on the number line?

Determine two integers that when squared equal 20.

$2^2 = 4,\ 4^2 = 16,\ 4 + 16 = 20$

Therefore, graph 4 and 2 as the legs of the right triangle.

Step 1: Draw a number line. At 4, construct a perpendicular line segment 2 units in length.

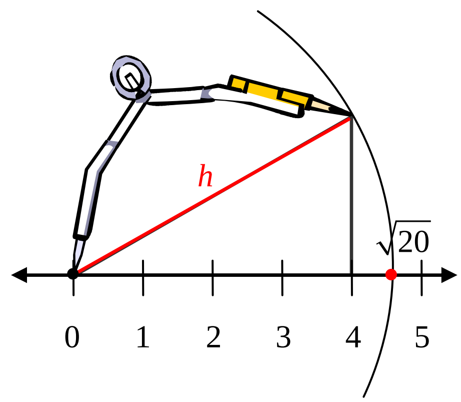


Step 2: Draw the line segment from zero to the end of the line segment that is 2 units in length and label it *h* for hypotenuse.

Step 3: Use the Pythagorean Theorem to show that the hypotenuse is $\sqrt{20}$ units long.

$$a^2 + b^2 = c^2$$
$$2^2 + 4^2 = c^2$$
$$20 = c^2$$
$$\sqrt{20} = c$$

Step 4: Open the compass to the length of *h*. With the tip of the compass at zero and the pencil point at the top of the right triangle, draw an arc that intersects the number line. The distance from 0 to the point on the number line is $\sqrt{20}$ units.



*Note: $\sqrt{20} \approx 4.5$.