TRANSFORMATIONS

In this unit, you will start with graphing and analyzing two-dimensional shapes in the coordinate plane. You will then examine translations (slides), reflections (flips), and rotations (turns), along with ways to represent these transformations in the coordinate plane. You will then explore shapes that tessellate a plane.

Graphing in Quadrant I of the Coordinate Plane

Graphing in Quadrants II, III, and IV

Analyze Shapes Using Coordinate Geometry

Types of Transformations

Translations

Reflections and Symmetry

Rotations

Dilations

Tessellations

Graphing in Quadrant I of the Coordinate Plane

Ordered pair - An ordered pair is a pair of numbers that represent the location in a grid. In the coordinate plane, an ordered pair is the *x* and *y*-coordinate of a point represented as (*x*, *y*).

Origin - The origin is the beginning point in the coordinate plane. It is the point where the *x*-axis and the *y*-axis intersect. The coordinates of the origin are (0,0).

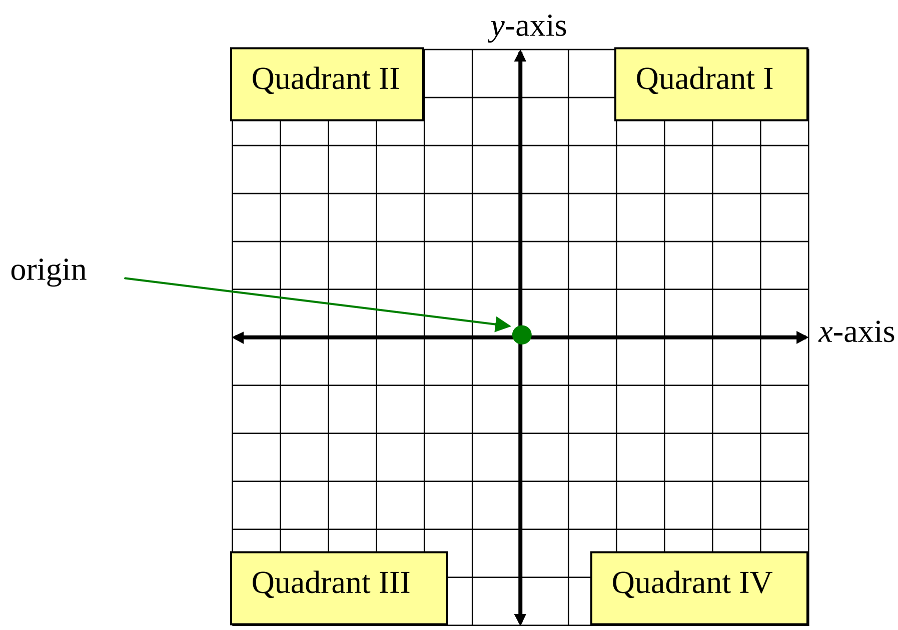
Quadrants - Quadrants are the four regions of the coordinate plane. The *x* and *y*-axis divide the coordinate plane into four quadrants.

Axes - Axes is the plural of axis. There are two axes in a coordinate plane. The *x*-axis is the horizontal axis and is a number line. The *y*-axis is the vertical axis and is a vertical number line.

Coordinates - Coordinates are the components of an ordered pair. In an ordered pair, the first number is called the *x*-coordinate and the second number is called the *y*-coordinate.

Coordinate plane - The coordinate plane is a numbered grid system that has a horizontal and a vertical number line in the center. These lines are perpendicular to each other and meet at the origin, the point considered the starting point of the system. The origin is numbered as (0, 0).

In a **coordinate plane**, points may be located by **plotting** them. The coordinate plane is divided into **four quadrants** by the ***x*–axis** and the ***y*-axis**. The starting point, the **origin**, is the center, or point where the *x* and *y*-axis intersect (cross).



A point is designated by both an ***x*-coordinate** and a ***y*-coordinate.** The origin's coordinates are (0, 0). The *x*-coordinate is the first number and the *y*-coordinate is the second number.

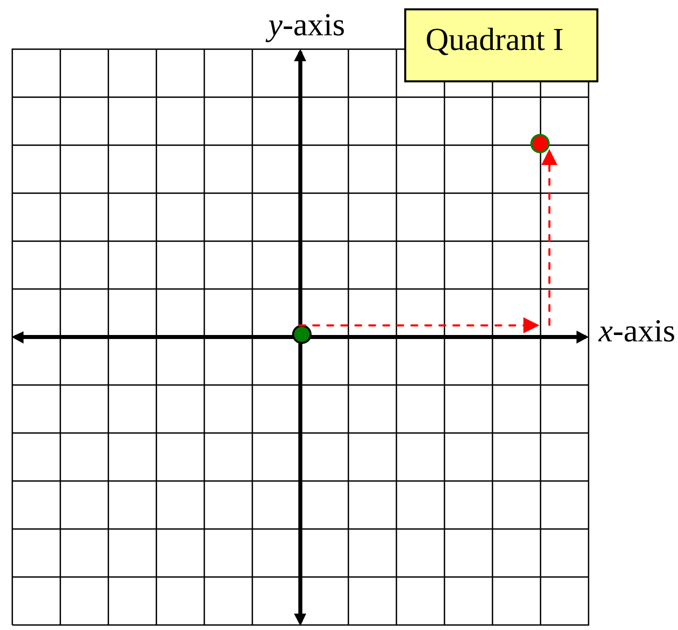
The ***x*-coordinate** is how far you count **right or left** of the origin. The ***y*-coordinate** is how far you then count **up or down**. A point's location is written as an **ordered pair (*x*, *y*)**.

In this grid, each space represents one unit.

Plot (5, 4)

When plotting points, start at the origin. Count right if the *x*-coordinate is positive, left if it is negative. Then count up if the *y*-coordinate is positive, count down if it is negative.

To plot (5,4) start at the origin, count 5 units to the right, and then count 4 units up.



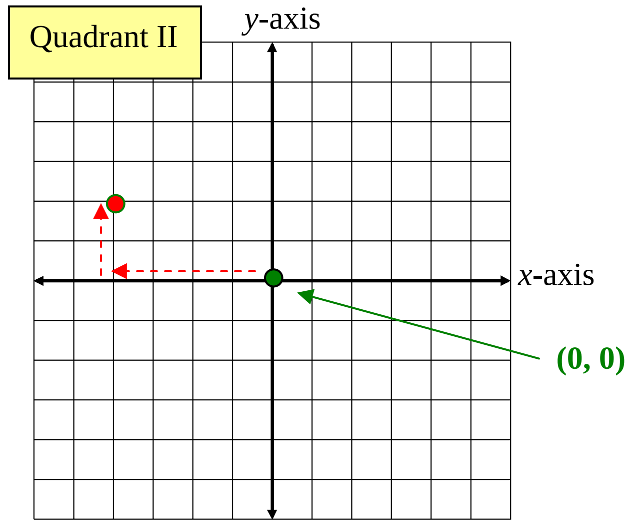
Graphing in Quadrants II, III, and IV

In these grids each space represents one unit. The starting point is the origin (0,0).

Plot (–4, 2)

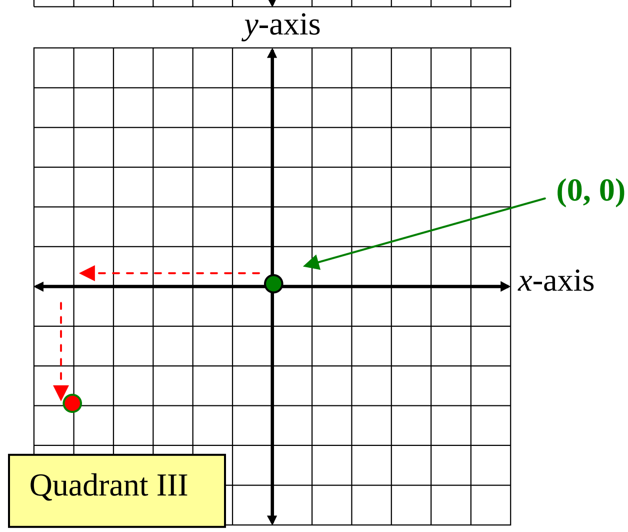
When plotting points, start at the origin. Count right if the *x*-coordinate is positive, left if it is negative. Then count up if the *y*-coordinate is positive, count down if it is negative.

To plot (–4, 2) starting at the origin, count 4 units to the left, and then count 2 units up.



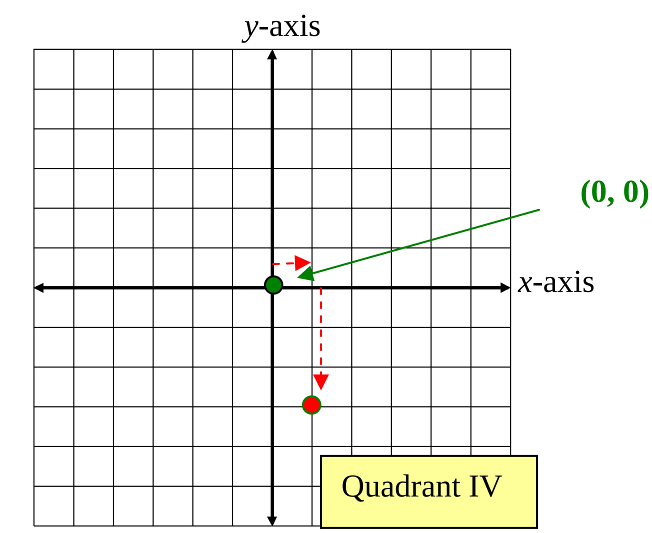
Plot (–5, –3)

To plot (–5, 3), start at the origin, count 5 units to the left, and then count 3 units down.



Plot (1, –3)

To plot (1, –3), start at the origin, count 1 unit to the right, and then count 3 units down.



Analyze Shapes Using Coordinate Geometry

Now that we have become more experienced with graphing on a coordinate plane and connecting graphs with algebraic phrases, let's consider some ideas more closely related to geometry.

First we will review a few shapes and their properties.

square- 4 sides congruent, 4 right angles

rectangle - 4 right angles, 2 pairs of parallel and congruent sides

parallelogram - opposite sides parallel and congruent

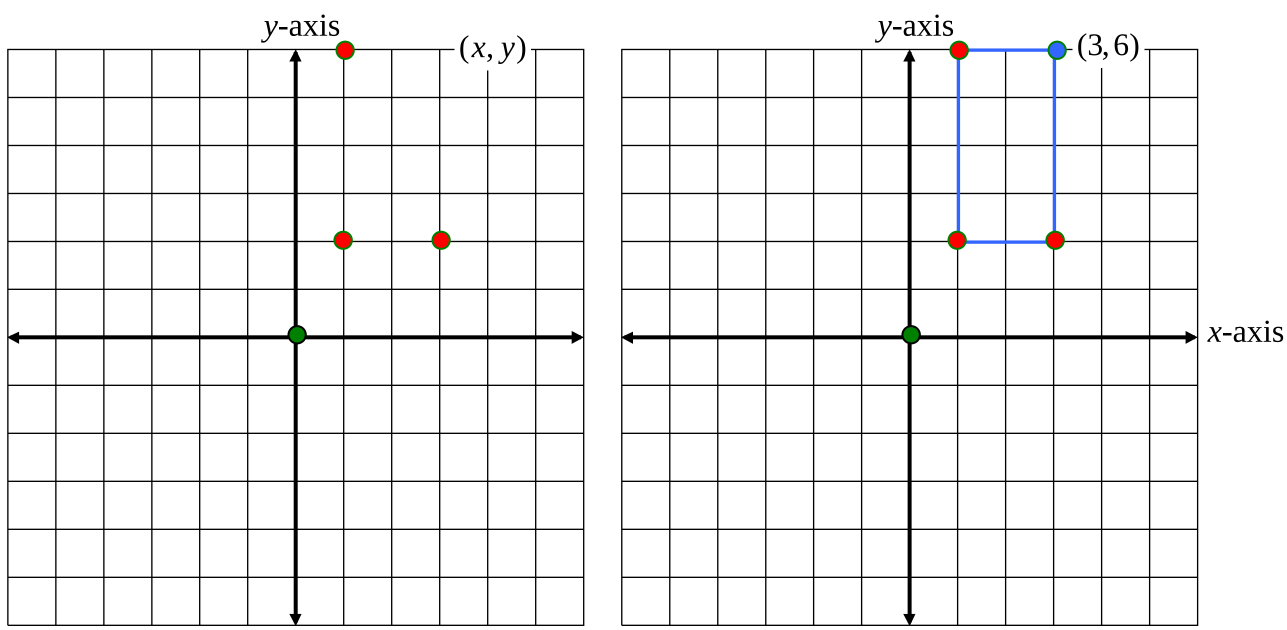
trapezoid - one pair of parallel sides

equilateral triangle - all three sides congruent, all angles 60˚

If we take certain points on a coordinate plane and connect them to a geometric idea, we should be able to predict a missing point.

Example 1: Consider a **rectangle**: $(1,2),(1,6),(3,2),$ and (x,y). Find the coordinates for a point that completes the shape into a rectangle.

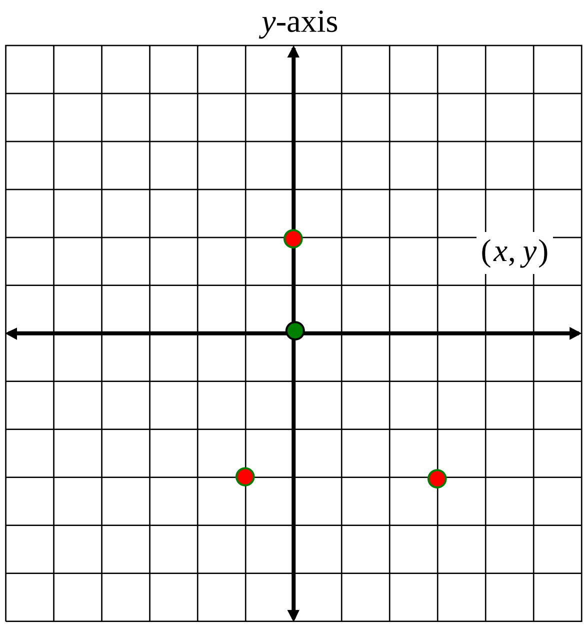
If we plot the points and take a look, we can find the missing ordered pair.

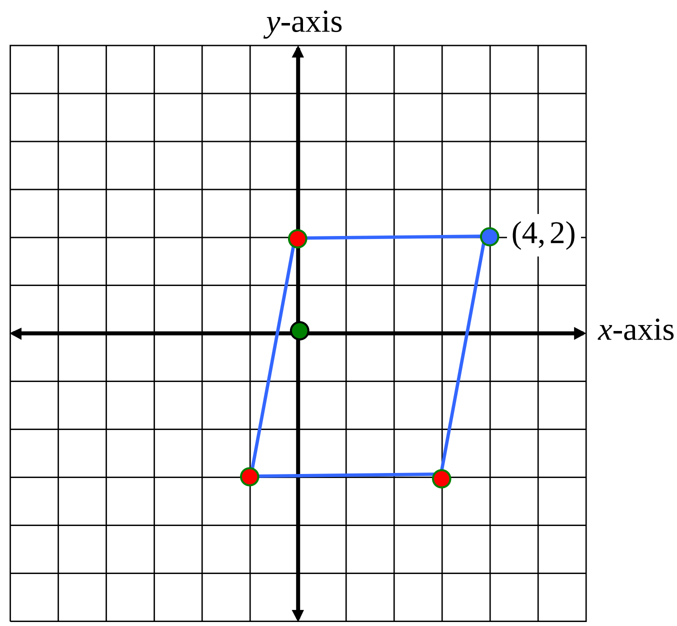


The missing point must be $(3,6)$ to complete the shape into a rectangle.

Example 2: Consider the parallelogram $(-1,-3), (0,2), (3,-3),$ and (x, y). Find the coordinates for a point that completes the shape into a parallelogram.

Plot the points and take a look to find the missing ordered pair.



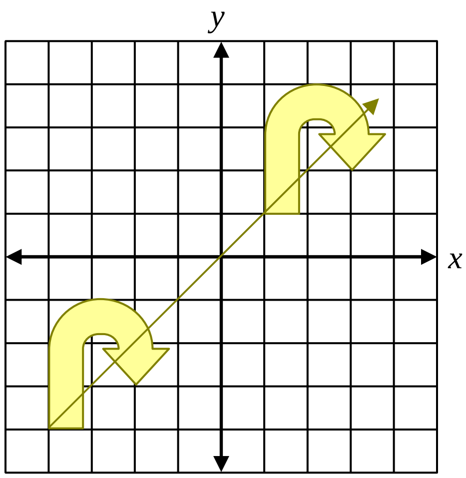


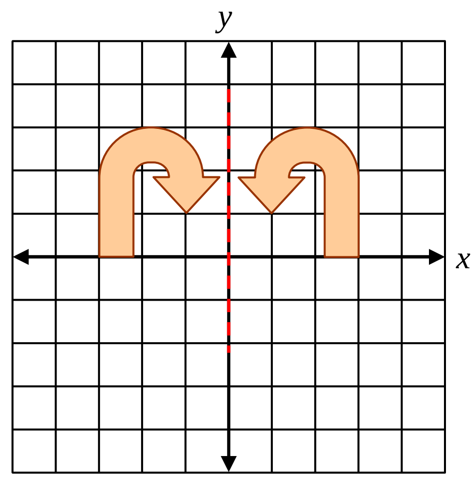
The missing point must be $(4, 2)$ to complete the shape into a parallelogram.

Types of Transformations

transformation - A transformation is a change made to the size or location of a figure. The new figure formed by the translation is called an image.

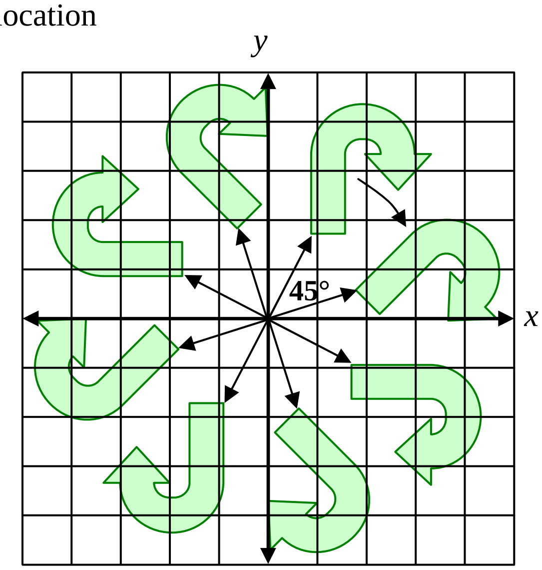
Types of transformations are *translations, dilations, reflections, and rotations.*

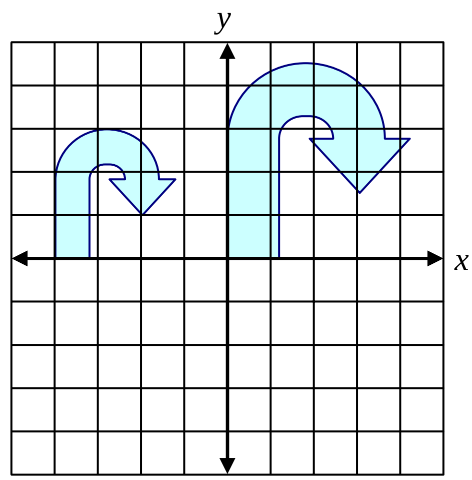




translation – A translation is the sliding a figure along a straight line without turning to another location

reflection – A reflection is the flipping a figure over a line of reference creating a mirror image.





rotation – A rotation is the turning a figure around a fixed point called the center of rotation

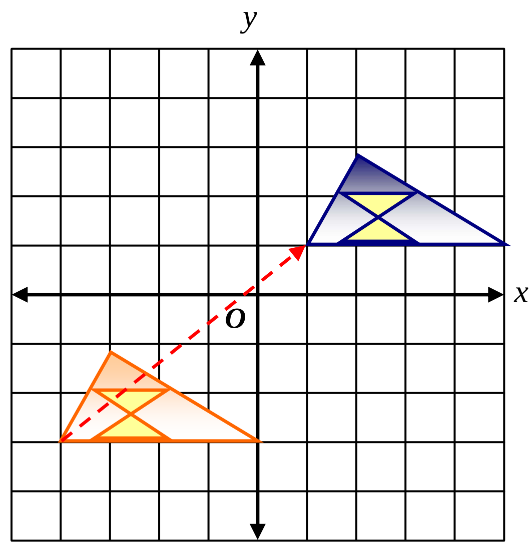
The **angle of rotation** is an angle formed by rays drawn from the center of rotation through corresponding points on an original figure and its image.

This curved arrow has been rotated 45° each time around the center of rotation, $(0,0)$.

dilation – A dilation is the enlarging or reducing a figure

Translations

A **translation** is a transformation in which each point of the figure moves the same distance in the same direction. A figure and its image are congruent.



A translation slides a figure along a line without turning.

Example 1: Describe the translation in words.

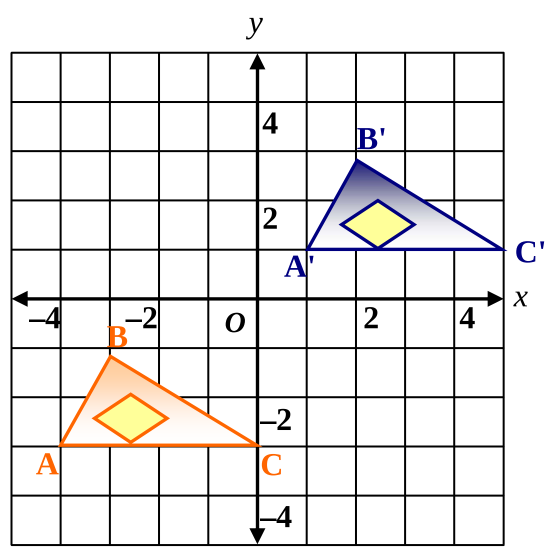


Figure ABC is translated to figure A'B'C'. (A'B'C' is read "A-primed, B-primed, C-primed".) The translation is 5 units to the right, 4 units up.

To check, start at point A and count 5 units to the right and 4 units up. You will end up at point A'. Do the same for the other two pairs of corresponding points.

Notation in the Coordinate Plane: You can describe a translation of each point (x, y) of a figure using coordinate notation.

Translation: $(x, y) \rightarrow (x + a, y + b)$

a = how many units a point moves horizontally

b = how many units a point moves vertically

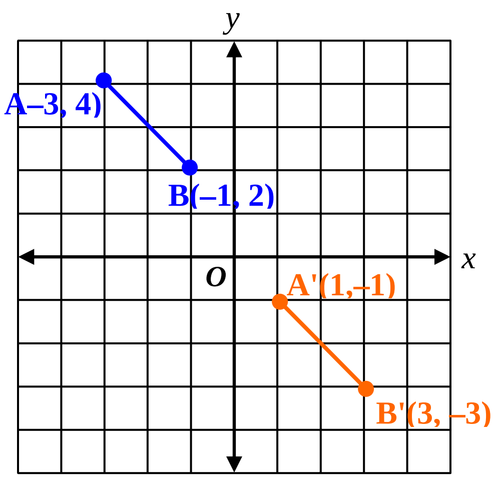
If $a > 0$, the point moves to the right.

If $a < 0$, the point moves to the left.

If $b > 0$, the point moves up.

If $b < 0$, the point moves down.

Example 2: Given segment AB with vertices A(–3,4) and B(–1,2). Find the coordinates of the image after the translation $(x, y) \rightarrow (x+4, y-5)$ and draw the image.



Translation:

$(x, y) \rightarrow (x+a, y+b)$

$(x, y) \rightarrow (x+4, y-5)$

$A(-3, 4) \rightarrow A'(-3+4, 4-5) = A'(1, -1)$

$B(-1, 2) \rightarrow B'(-1+4, 2-5) = B'(3, -3)$

*Note: To check your work, make sure that both point A and point B were translated such that they were moved to the **right four** units (x + 4) and **down five** units (y – 5).

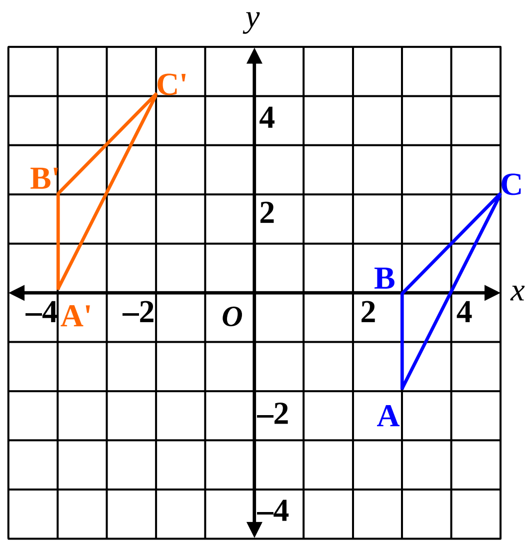
Example 3: Draw triangle ABC with vertices of A(3, –2), B(3, 0), and C(5, 2). Then find the coordinates of the vertices of the image after the translation. Draw the image.

Translation: $(x, y) \rightarrow (x - 7, y + 2)$

Step 1: Draw triangle ABC with vertices of A(3, –2), B(3, 0), and C(5, 2).

Step 2: Find the coordinates of the vertices of the image. Subtract 7 from each *x*-coordinate. Add 2 to each *y*-coordinate.

Original		Image
(x, y)	$\rightarrow$	$(x - 7, y + 2)$
$A(3, -2)$	$\rightarrow$	$A'(-4, 0)$
$B(3, 0)$	$\rightarrow$	$B'(-4, 2)$
$C(5, 2)$	$\rightarrow$	$C'(-2, 4)$



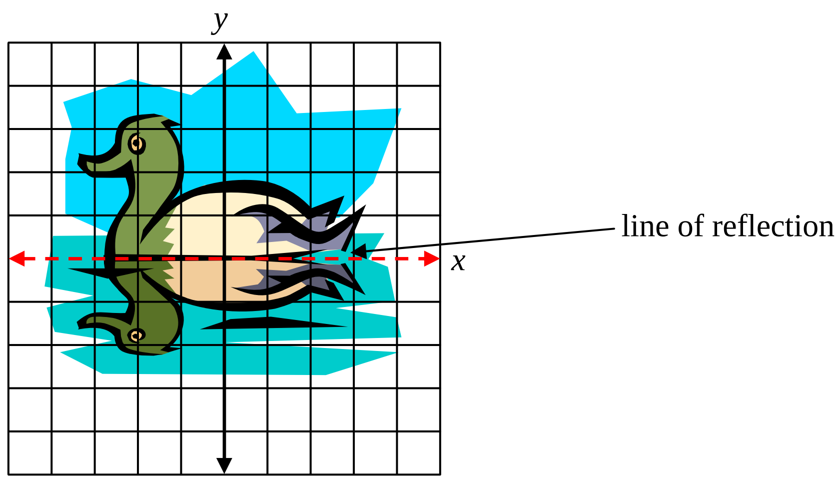
Step 3: Draw the image, A'B'C'.

Each point on triangle *ABC* moves 7 units to the left and 2 units up to translate to triangle *A'B'C'*.

*Note: To check your work, make sure that both point A, point B, and point C were translated such that they were moved to the **left seven** units ($x - 7$) and **up two** units ($y + 2$).

Reflections and Symmetry

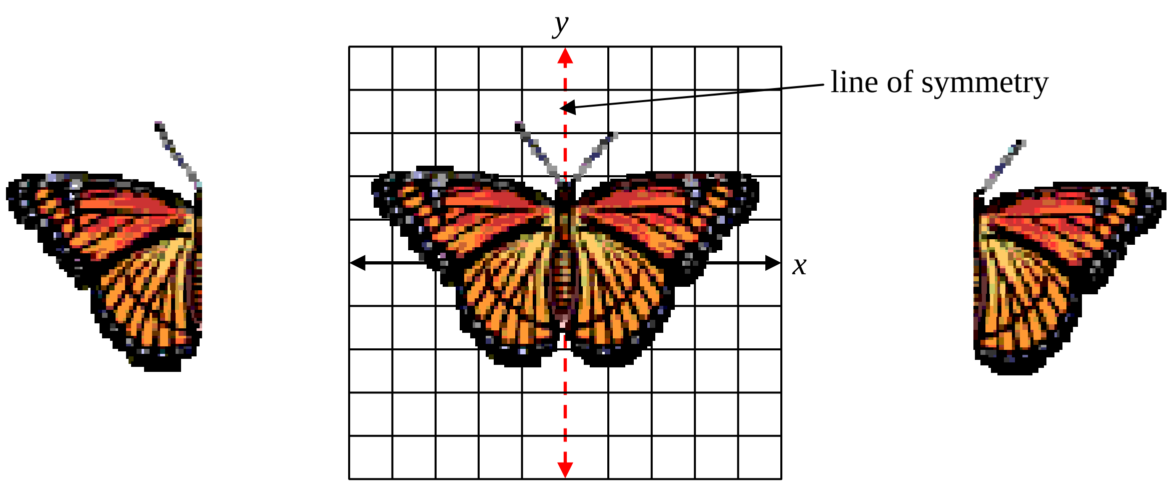
A **reflection** is a transformation of a figure in which the figure is reflected or flipped across the line. The line is called the **line of reflection**.

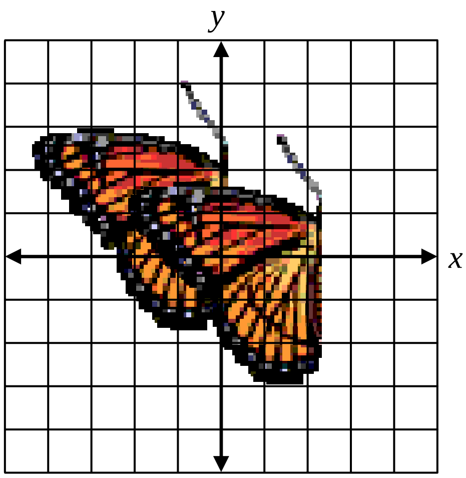


The duck is reflected in a pool of water creating a mirror image.

A figure has **line symmetry** if it can be folded over a line so that one half of the figure matches the other half. The line is called the **line of symmetry**.

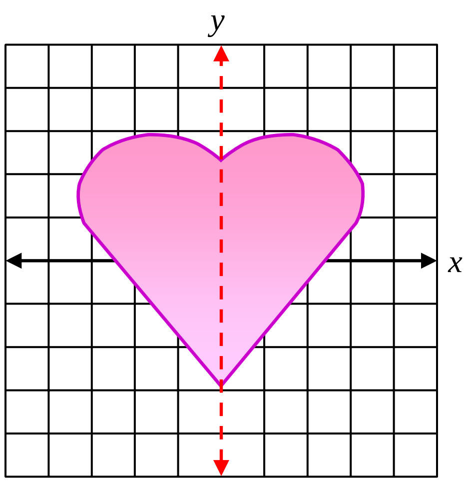
The **line of symmetry** divides the figure into two parts that are reflections of each other. The two sides are mirror images of each other.

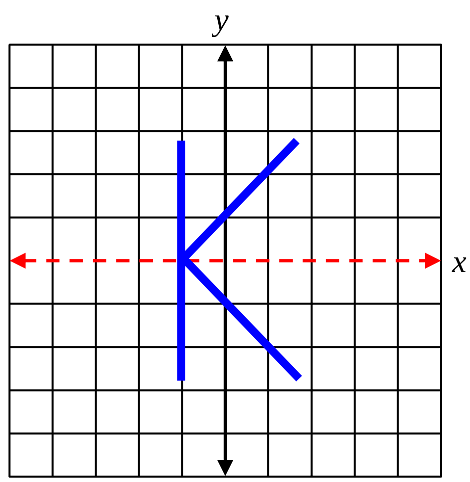




The left side of the butterfly is symmetrical to the right side. Symmetrical objects have parts that are congruent. The wings are congruent.

Samples of Line Symmetry



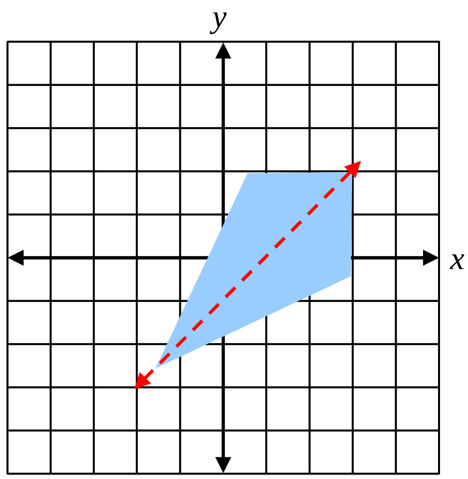


Vertical Line Symmetry

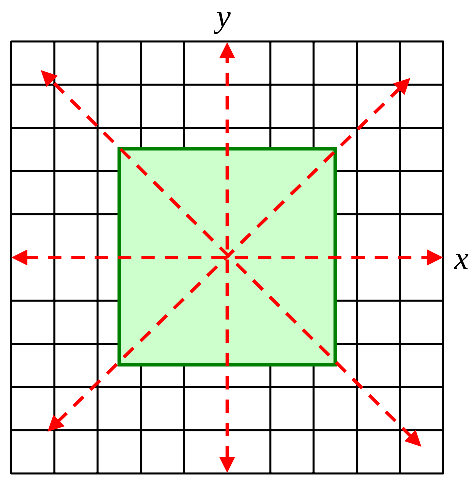
The left side of the heart is a reflection of the right side.

Horizontal Line Symmetry

The top part of the K is a reflection of the bottom part.



Diagonal Line Symmetry

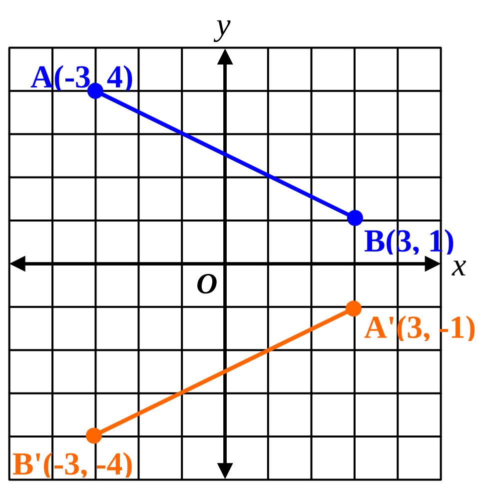


Four Lines of Symmetry

Notation in the Coordinate Plane: Reflections may be described using coordinate notation.

When the *y*-coordinate is multiplied by –1, a reflection over the *x*-axis occurs with the *x*-axis being the line of symmetry.

Reflection over the *x*-axis



Multiply the y-coordinate by -1.

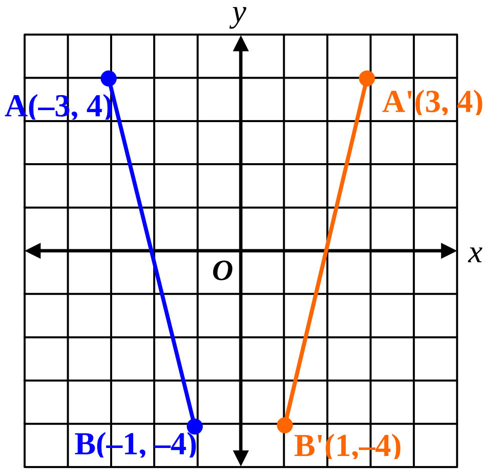
$(x, y) \rightarrow (x, -y)$

$(-3, 4) \rightarrow (-3, -4)$

$(3, 1) \rightarrow (3, -1)$

When the *x*-coordinate is multiplied by –1, a reflection over the *y*-axis occurs with the *y*-axis being the line of symmetry.

Reflection over the *y*-axis



Multiply the x-coordinate by -1.

(x, y)	$\rightarrow$	$(-x, y)$
$(-3, 4)$	$\rightarrow$	$(3, 4)$
$(-1, -4)$	$\rightarrow$	$(1, -4)$

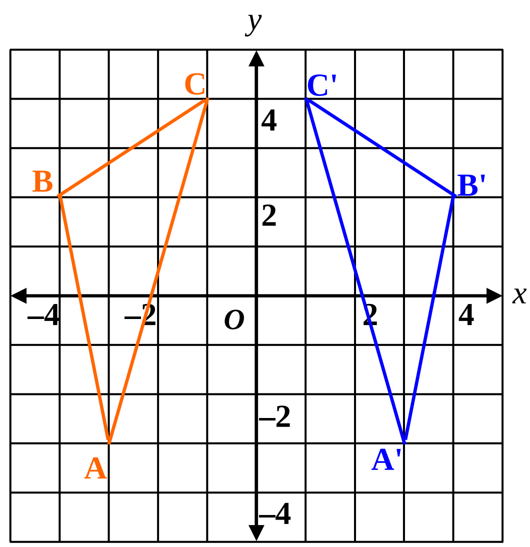
Example: Draw triangle ABC with vertices of $A(-3,-3)$, $B(-4,2)$, and $C(-1,4)$. Then find the coordinates of the vertices of the image after a reflection over the *y*-axis. Draw the image.

Step 1: Draw triangle ABC.

Step 2: Find the coordinates of the vertices of the image by reflecting triangle ABC over the *y*-axis. .

Original		Image
$A(-3, -3)$	$\rightarrow$	$A'(3, -3)$
$B(-4, 2)$	$\rightarrow$	$B'(4, 2)$
$C(-1, 4)$	$\rightarrow$	$C'(1, 4)$

Step 3: Draw the image $A'B'C'$.

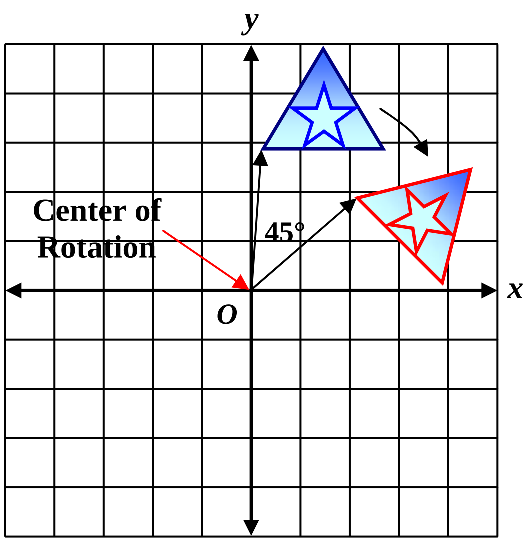


Triangle $A'B'C'$ is a reflection of triangle ABC. The *y*-axis is the line of symmetry.

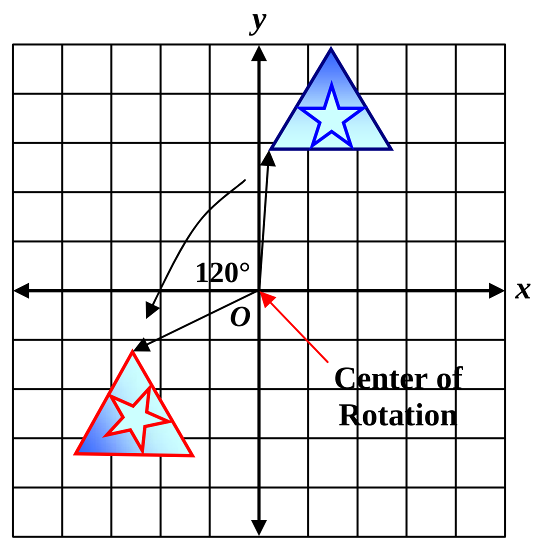
Rotations

rotation - A rotation is a transformation in which a figure is turned about a fixed point, called the center of rotation.

angle of rotation - The angle of rotation is an angle formed by rays drawn from the center of rotation through corresponding points on an original figure and its image. The direction of rotation can be *clockwise* or *counterclockwise*. The figure and its image are congruent.

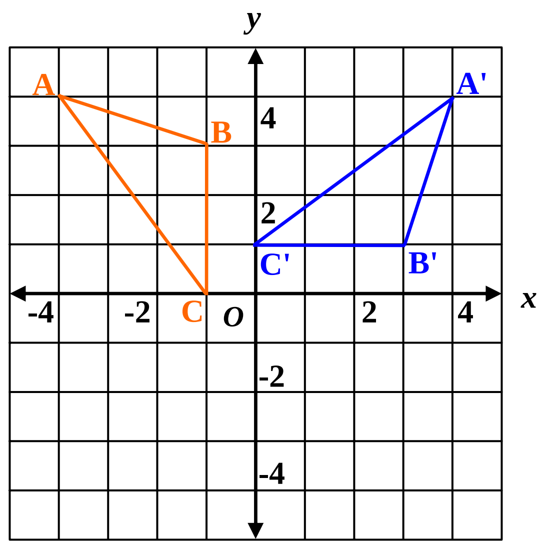


45° clockwise rotation



120° counterclockwise rotation

Example 1: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0). Then find the coordinates of the vertices of the image after a **90°** clockwise rotation around the origin. Draw the image, A'B'C'.



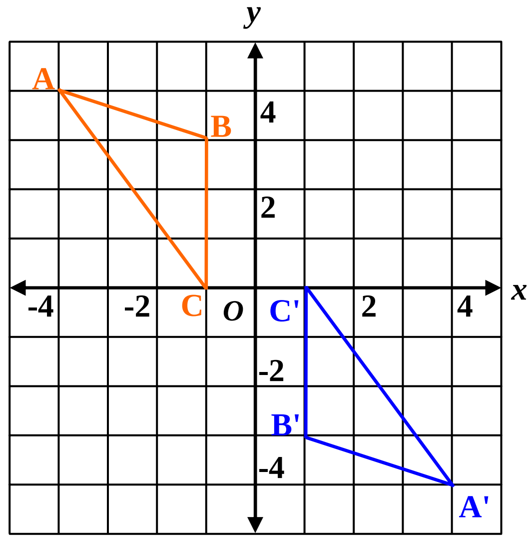
Step 1: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0).

Step 2: Find the coordinates of the vertices of the image. Note: You can switch the coordinates and multiply the new *y*-coordinate by –1.

Original		Image
(x, y)	$\rightarrow$	$(y, -x)$
$A(-4, 4)$	$\rightarrow$	$A'(4, 4)$
$B(-1, 3)$	$\rightarrow$	$B'(3, 1)$
$C(-1, 0)$	$\rightarrow$	$C'(0, 1)$

Step 3: Draw the image, A'B'C'.

Example 2: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0). Then find the coordinates of the vertices of the image after a **180°** rotation around the origin. The image is the same whether you rotate the figure clockwise or counterclockwise. Draw the image, A'B'C'.



Step 1: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0).

Step 2: Find the coordinates of the vertices of the image. Note: You multiply both *x*- and *y*-coordinates by –1.

Original		Image
(x, y)	$\rightarrow$	$(-x, -y)$
$A(-4, 4)$	$\rightarrow$	$A'(4, -4)$
$B(-1, 3)$	$\rightarrow$	$B'(1, -3)$
$C(-1, 0)$	$\rightarrow$	$C'(1, 0)$

Step 3: Draw the image, A'B'C'.

Dilations

dilation - A dilation is a transformation in which the size is changed, but not the shape. It can be an enlargement or a reduction of a figure. The dilation of a figure is similar to the original image.

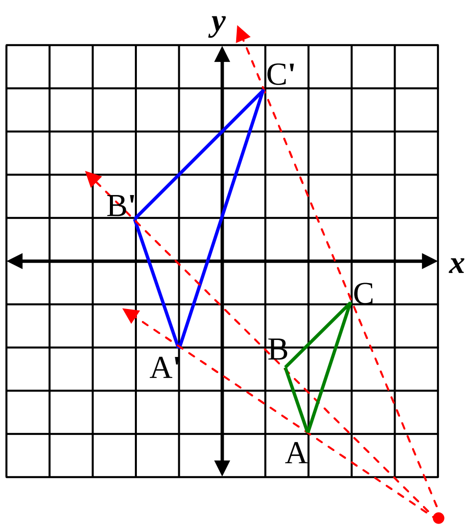
A dilation has a fixed point that is the **center of dilation**. The figure stretches or shrinks with respect to the **center of dilation**. To find the center of dilation, draw a line that connects each pair of corresponding vertices.

The **scale factor** describes how much a figure is enlarged or reduced. It is the ratio of the side length of the image to the corresponding side length of the original figure. A scale factor can be expressed as a decimal, fraction, or percent.

A **dilation** produces an image similar to the original figure.

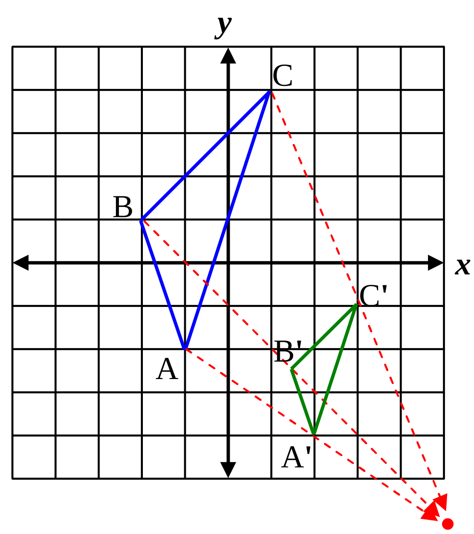
Multiplying by a scale factor > 1 → Enlarges a figure

Multiplying by a scale factor < 1 → Reduces a figure



Center of Dilation

Triangle A'B'C' is a dilation of triangle ABC and has a 100% increase in size. The scale factor is 2.

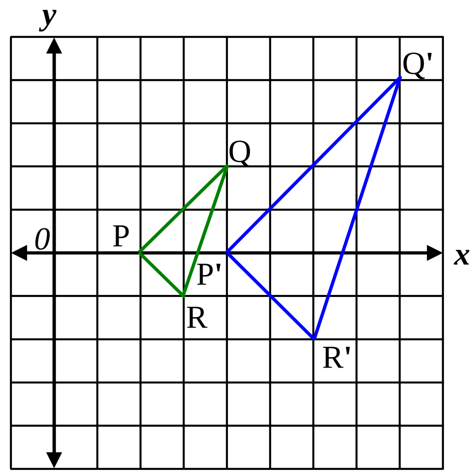


Center of Dilation

Triangle A'B'C' is a dilation of triangle ABC and has a 100% decrease in size. The scale factor is 0.5.

Example 1: Draw triangle PQR with vertices of P(2, 0), Q(4, 2), and R(3, –1). Then find the coordinates of the vertices of the image after the dilation having a scale factor of 2. Draw the image, P'Q'R'.

Step 1: Draw triangle PQR with vertices of P(2, 0), Q(4, 2), and R(3, –1).



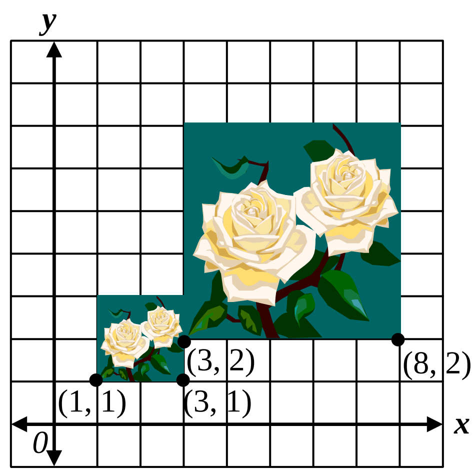
Step 2: Find the coordinates of the vertices of the image. To dilate triangle PQR, multiply the *x*- and *y*-coordinates of each vertex by 2.

Original	Image
(x, y)	$(2x, 2y)$
P(2, 0)	P'(4, 0)
Q(4, 2)	Q'(8, 4)
R(3, –1)	R'(6, –2)

Step 3: Draw the image, P'Q'R'.

Example 2: An artist uses a computer program to enlarge a design. Find the scale factor of the design.

Step 1: Find the width of the original design.



The width of the original design is the differences of the original design’s *x*-coordinates, 3 – 1 = 2 units.

The scale factor is 2.5.

Step 2: Find the width of the image.

The width of the image is the difference of the image’s *x*-coordinates, 8 – 3 = 5.

Step 3: Find the scale factor.

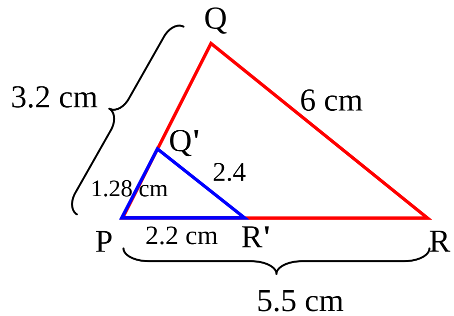
The scale factor is:

$$\frac{\text{image}}{\text{original design}} = \frac{5}{2} = 2.5$$

Example 3: Dilate triangle PQR by a scale factor of 0.4. Let vertex P be the center of dilation. Segment PR has a length of 5.5 cm, segment PQ has a length of 3.2 cm and segment QR has a length of 6 cm. P and P' are the same point.

Step 1: Multiply each side by 0.4.

Original	**Image**
(Length)	(0.4 × Length)
PR = 5.5 cm	PR' = 2.2 cm
PQ = 3.2 cm	PQ' = 1.28 cm
QR = 6 cm	Q'R' = 2.4 cm



Step 2: Draw the image, P'Q'R'.

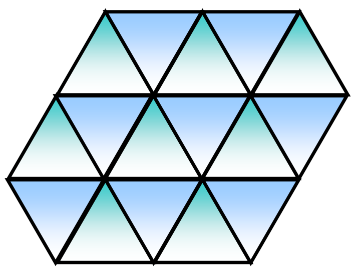
Triangle PQ'R' is a dilation of triangle PQR by a scale factor is 0.4 with point P as the center of dilation.

Tessellations

tessellation - A tessellation is a covering of a plane with a repeating pattern of one or more shapes. It has no gaps or overlaps. You can create a tessellation by translating a shape. These designs are used in art and architecture.

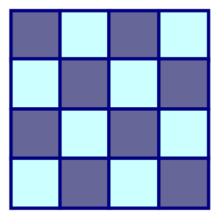
In a **regular tessellation**, a regular polygon (all sides congruent), is repeated to fill a plane. The angles at each vertex total 360°. Exactly three regular tessellations exist. One is a tessellation of equilateral triangles, a second one is a tessellation of squares, and a the third one is a tessellation of hexagons.

Regular Tessellations



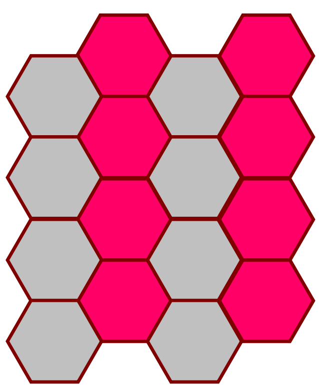
Equilateral triangle translated to create a regular tessellation. There are six 60° angles at each vertex.

$$6 \times 60° = 360°$$



Square translated to create a regular tessellation. There are four 90° angles at each vertex.

$$4 \times 90° = 360°$$

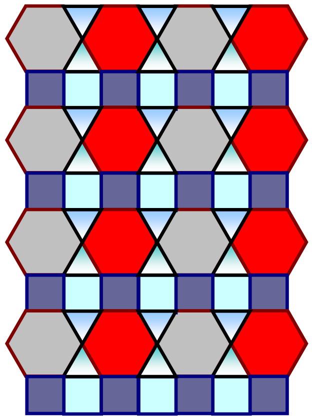


Regular hexagon translated to create a regular tessellation. There are three 120° angles at each vertex.

$$3 \times 120° = 360°$$

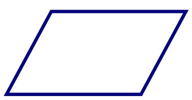
In a **semiregular tessellation**, two or more regular polygons are repeated to fill the plane.

Semiregular tessellation

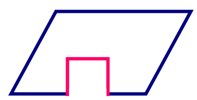


Translation of a Polygon

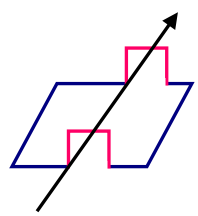
A polygon that tessellates the plane can be altered to create other tessellations. The polygon below will be altered and translated.



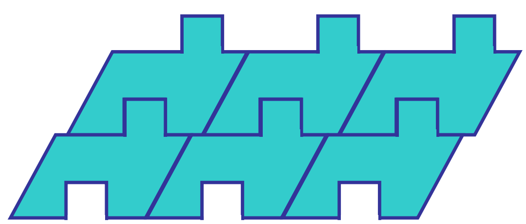
Step 1: Cut a piece from the polygon that tessellates.



Step 2: Translate the piece to any part on the opposite side.

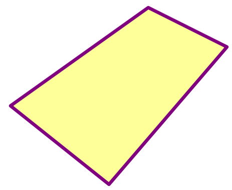


Step 3: Translate the new figure repeatedly to create a tessellation.

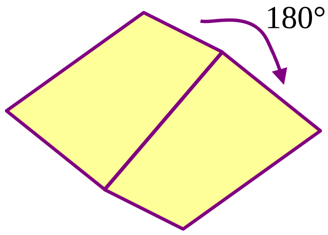


Rotation and Translation of a Polygon

A polygon that tessellates the plane can be altered to create other tessellations. The polygon below will be rotated and translated.



Step 1: Rotate the quadrilateral 180 degrees.



Step 2: Translate the new figure repeatedly to create a tessellation.