

# **REAL NUMBER SYSTEM**

In this unit, you will examine the real number system closely. You will work with rational and irrational numbers that together make up the real number system. You will identify properties for operations. You will evaluate expressions using the “order of operations” and examine the use of variables. You will also solve simple equations.

Real Numbers

Rational and Irrational Numbers

Compare Real Numbers

Properties of Real Numbers

Order of Operations with Real Numbers

Introduction to Variables and Expressions

Solving One-Step Equations

Solving Equations with Rational Numbers

## Real Numbers

Real numbers are the numbers that can be represented on the number line. They include the whole numbers, their opposites, and all the other numbers in between them.

The real numbers are a union of the rational and irrational numbers.

**Rational Numbers** - Rational numbers are numbers that can be written as a quotient of two integers.

The natural numbers, whole numbers, and integers are subsets of the rational numbers.

Note: A member of any of these sets can be expressed as a quotient of two integers.

**Natural Numbers:** {1, 2, 3, 4, 5...}

**Whole Numbers:** {0, 1, 2, 3, 4, 5...}

**Integers:** {...-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5...}

Also included in the rational numbers are **fractions** (where the numerator and denominators are integers) and **repeating and terminating decimals**. Fractions may not have 0 as the denominator.

**Irrational Numbers** – Irrational numbers are numbers that cannot be expressed as the ratio of two integers. Examples of irrational numbers are (a) square roots of non-perfect squares, (b) pi ( $\pi$ ), and (c) decimals that do not develop into a repeating pattern.

(a)  $\sqrt{94}$

(b)  $\pi \approx 3.1415926535897932384626433832795...$

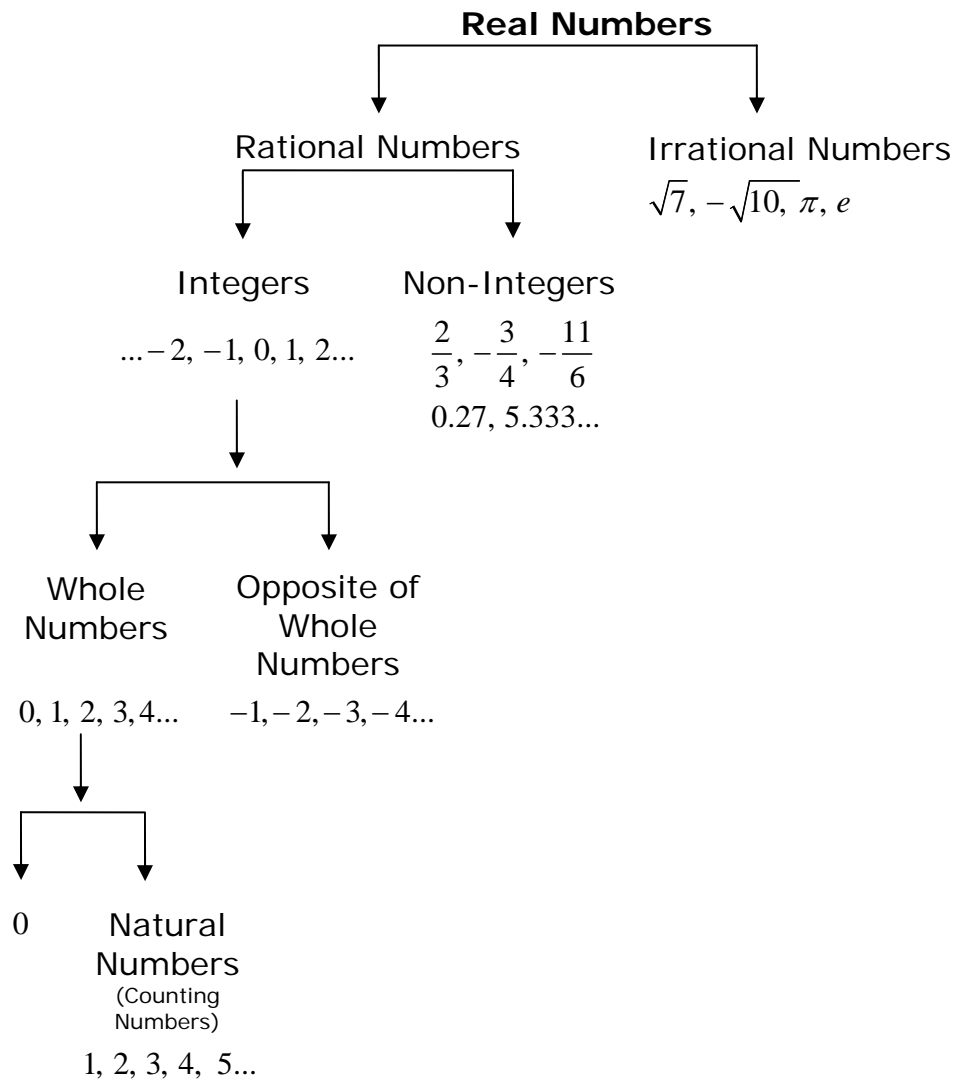
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**real numbers = rational numbers + irrational numbers**

Study the two sketches below to better understand the subsets of the real numbers.

The first sketch shows how each subset can be divided into further subsets. For example, 0 is a whole number, but is also an integer and a rational number because it is a member of all those subsets. The only subset of the rational numbers that 0 is not a part of is the natural numbers.

The second sketch shows the connection between the subsets using a Venn diagram. Again, looking at zero, it is in all of the ovals of the rational numbers with the exception of the natural numbers.

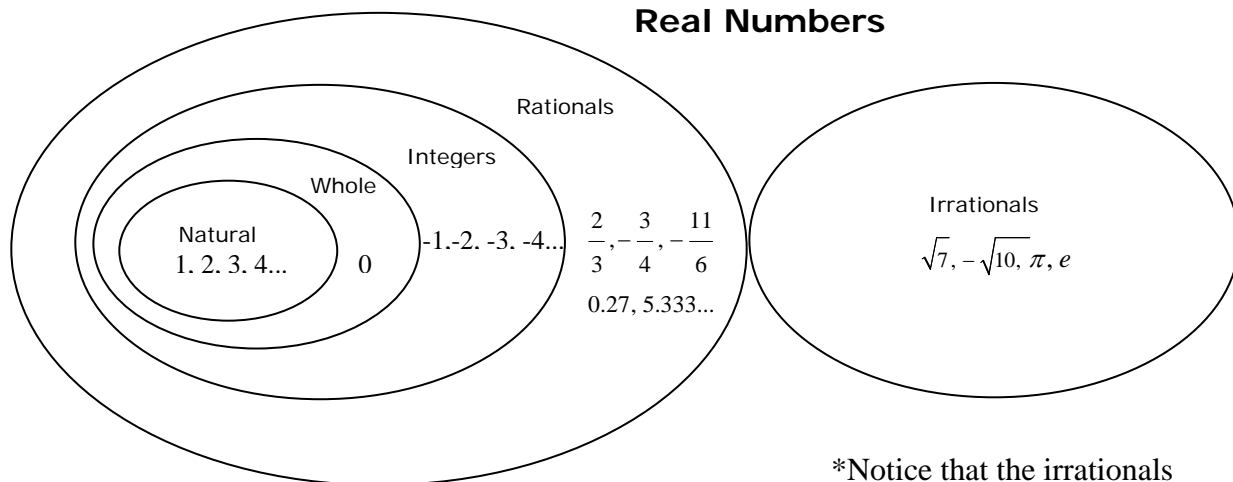


**Irrational Numbers**

$\sqrt{7}, -\sqrt{10}, \pi, e$

\* The numbers listed in the subsets are just a few examples of the infinite possibilities.

The next figure shows the connection between the subsets of real numbers using a Venn diagram.



The **rational numbers** are the types of numbers shown within the “rational circle”. The “rational circle” includes all of the numbers within the circles inside the it. Some examples of rational numbers are  $\left(\frac{-11}{6}\right)$ ,  $\left(-4 = \frac{-4}{1}\right)$ ,  $\left(0 = \frac{0}{1}\right)$ , and  $\left(4 = \frac{4}{1}\right)$ .

The **integers** are the types of numbers shown within the “integer circle”. The “integer circle” includes all of the numbers within the circles inside it. Some examples of integers are  $-4$ ,  $0$ ,  $4$ .

The **whole numbers** are the types of numbers shown within the “whole number circle”. The “whole number circle” includes all of the numbers within the circle inside it. Some examples of whole numbers are  $0$  and  $4$ .

The **natural numbers** are the types of numbers shown within the “natural number circle”. Some examples of whole numbers are  $3$  and  $4$ .

The **irrational numbers** are in a “league of their own”. Some examples of rational numbers are  $\sqrt{26}$ ,  $-\sqrt{43}$ ,  $\pi$ , and  $e$ .

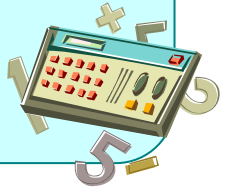
The **real numbers** are the types of numbers shown within all of the circles. The real numbers are the union of the rational and the irrational numbers. Some examples of real numbers are  $\left(\frac{-11}{6}\right)$ ,  $\left(-4 = \frac{-4}{1}\right)$ ,  $\left(0 = \frac{0}{1}\right)$ , and  $\left(4 = \frac{4}{1}\right)$ ; and also  $\sqrt{26}$ ,  $-\sqrt{43}$ ,  $\pi$ , and  $e$ .

## Rational and Irrational Numbers

**Calculators** can help us understand the difference between **rational** and **irrational** numbers.

**Rational numbers** are numbers that can be written as fractions, and when expressed as decimals are either a repeating or terminal decimal. **Irrational numbers** are numbers that have decimals that go on forever, but never develop a repeating pattern.

To express a fraction as a decimal, divide the numerator (top number) by the denominator (bottom number).



Definition	Type of Decimal	Calculator Buttons
<p><b>Rational numbers</b> are numbers that <b>can</b> be expressed as a fraction where both the numerator and the denominator are integers (with the exception of 0 as a denominator*).</p>	<p><b>Repeating</b>  <math>\frac{5}{11} = 0.454545\dots</math></p> <p><b>Terminating</b>  <math>\frac{7}{8} = 0.875</math></p>	<p>5 ÷ 1 1 =</p> <p>7 ÷ 8 =</p>
<p><b>Irrational numbers</b> are numbers that <b>cannot</b> be expressed as a fraction where both the numerator and the denominator are integers.</p>	<p><b>Do not repeat or terminate</b></p> <p>First 12 digits of <math>\pi</math> (<math>\pi</math>) =  <b>3.14159265359</b></p> <p>First 7 digits of  <math>\sqrt{2} = 1.414213</math></p>	<p>Not all calculators use the same symbols, so the “pi” and “square root” buttons may look a little different.</p> <p><math>\pi</math></p> <p><math>\sqrt{\quad}</math> 2</p>

\* Note: Since division by zero is undefined, the denominator of a fraction cannot equal 0.

Using a calculator, we can explore irrational and rational numbers.

$$\frac{2}{3} = 2 \div 3 = 0.6666666 - \text{repeating decimal} - \text{rational}$$

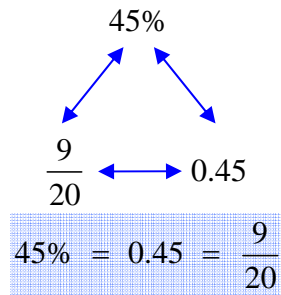
$$\sqrt{50} = \text{square root } (\sqrt{\quad}) \text{ of } 50 = 7.0710678 - \text{non-repeating decimal} - \text{irrational}$$

$$\frac{4}{5} = 4 \div 5 = 0.8 - \text{terminating decimal} - \text{rational}$$

$\sqrt{64} = 8$  or  $8.0$  – terminating decimal – **rational**

## Comparing Real Numbers

Percents, fractions, and decimals may all be used to represent the same quantity. Let's take a look at how to apply this connection.



*Example 1:* Rearrange the given numbers in order from least to greatest.

$$\frac{3}{4}, 15 \text{ out of } 16, 0.075, 79\%$$

To solve, express each number as a decimal and then compare.

$$\begin{aligned}\frac{3}{4} &= && 0.75 \\ 15 \text{ out of } 16 &= 15 \div 16 = && 0.9375 \\ 0.075 &&& \text{No change} \\ 79\% &= && 0.79\end{aligned}$$

List the decimals and add enough zeros to compare.

$$\begin{aligned}0.7500 \\ 0.9375 \\ 0.0750 \\ 0.7900\end{aligned}$$

Least to greatest as decimals: 0.0750, 0.7500, 0.7900, 0.9375

Least to greatest as original numbers: 0.075,  $\frac{3}{4}$ , 79%, 15 out of 16

Now, let's look at another example that includes negative numbers.

*Example 2:* List the numbers shown below in order from greatest to least:

$$\sqrt{19}, 65\%, -4\frac{2}{3}, -7.5 \times 10^{-1}$$

To make a comparison, change each number to a decimal and then line up the decimal points.

$$\begin{aligned}\sqrt{19} &= 4.3588989435\dots \\ 65\% &= 0.65 \text{ (65\% means } \frac{65}{100}\text{)} \\ -4\frac{2}{3} &= -4.6666\dots \\ -7.5 \times 10^{-1} &= -0.75\end{aligned}$$

1) There are two negative numbers. Negative numbers are smaller than positive numbers.

2)  $-4.666\dots$  is further left on the number line than  $-0.75$ ; thus,  $-4.666$  is smaller than  $-0.75$ .

$$-4.666\dots < -0.75 \text{ or } -0.75 > -4.666\dots$$

3) For the two positive numbers, look at each whole number before the decimal. You can see that 4 is greater than 0; therefore  $4.35\dots$  is greater than  $0.65$ .

$$4.35\dots > 0.65$$

4) Next, use the decimal numbers and line them up from greatest to least:

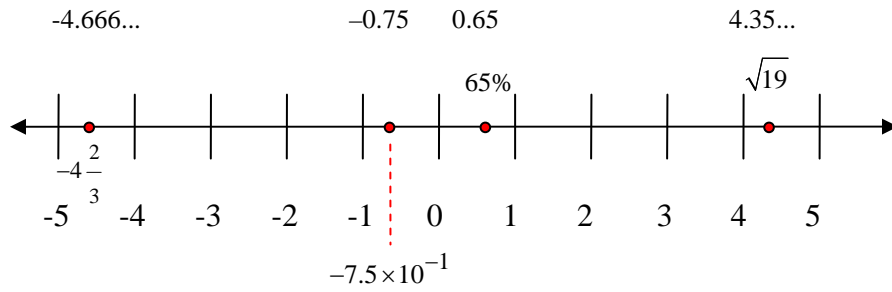
$$4.35, 0.65, -0.75, -4.666\dots$$

5) Now, refer back to the original numbers to state the final answer.

$$\sqrt{19}, 0.65, -7.5 \times 10^{-1}, -4\frac{2}{3}$$



6) Look at the number line below to find the location of each number. As you go right on the number line, the numbers get larger; and as you go left on the number line, the numbers get smaller.



## Properties of Real Numbers

In the previous section you learned how to translate verbal phrases into algebraic sentences. You were able to do this because of mathematical properties. In this section you will study these properties that you will use in future units to solve equations.

### Commutative Properties of Addition and Multiplication

The **order** in which numbers are added does not make a difference in the sum.

$$6 + 4 = 4 + 6$$

For any numbers  $x$  and  $y$ ,

$$x + y = y + x$$

The **order** in which numbers are multiplied does not make a difference in the product.

$$6 \times 4 = 4 \times 6$$

For any numbers  $x$  and  $y$ ,

$$xy = yx$$

### Associative Properties of Addition and Multiplication

The way in which numbers are **grouped** does not change the sum.

$$(2 + 3) + 4 = 2 + (3 + 4)$$

For any numbers  $x$ ,  $y$ , and  $z$ ,

$$(x + y) + z = x + (y + z)$$

The way in which numbers are **grouped** does not change the product.

$$(2 \times 3) \times 4 = 2 \times (3 \times 4)$$

For any numbers  $x$ ,  $y$ , and  $z$ ,

$$(xy)z = x(yz)$$

### Identity Properties of Addition and Multiplication

The sum of a number and zero is that number.

$$3 + 0 = 3$$

For any number  $n$ ,  $n + 0 = n$

The product of a number and one is the number.

$$5 \cdot 1 = 5$$

For any number  $n$ ,  $n \cdot 1 = n$

## Multiplicative Property of Zero

The product of a number and zero is zero.

$$7 \cdot 0 = 0$$

$$\text{For any number } n, n \cdot 0 = 0$$

These properties will be helpful when you start solving equations later on in the course. Right now you will be asked in the assignment to identify properties illustrated by algebraic expressions, so let's practice deciding which property is shown.

*Examples:*

- |                                |                                     |
|--------------------------------|-------------------------------------|
| 1) $4 + (9 + 5) = (4 + 9) + 5$ | Associative Property of Addition    |
| 2) $y + 14 = 14 + y$           | Commutative Property of Addition    |
| 3) $(9 + n) + 6 = 6 + (9 + n)$ | Commutative Property of Addition    |
| 4) $5m \cdot 0 = 0$            | Multiplicative Property of Zero     |
| 5) $0 + 17 = 17$               | Identity Property of Addition       |
| 6) $8xyz \cdot 1 = 8xyz$       | Identity Property of Multiplication |

## Order of Operations

In order to find the numerical value (**evaluate**) of any combination of numbers and operations (**expression**) correctly, mathematicians have established the order of operations which tells us which operations to do first in any mathematical problem.

**P** (parentheses)

This “saying” may be used to help remember the order of operations.

**E** (exponents – powers)

**M** (multiply) }  
**D** (divide) } work left to right

**A** (add) }  
**S** (subtract) } work left to right



\*Note: Multiplication and division are at the same level, meaning multiplication does NOT take priority over division. Work these two operations as they occur, left to right. The same is true about addition and subtraction. Work the two operations as they occur, left to right.

Thus, if “multiplication and division” or “addition and subtraction” are the only two operations in the expression, work the problem from left to right!

*Examples:*

1)  $6 \times 4 + 2$

1) Multiply  $6 \times 4$

$24 + 2$

2) Add  $24 + 2$

$26$

2)  $4(6+3) - 5 \cdot 2$

1) Parentheses  $(6 + 3)$

$4(9) - 5 \cdot 2$

2) Multiply  $4(9)$

$36 - 5 \cdot 2$

3) Multiply  $5 \cdot 2$

$36 - 10$

4) Subtract

$26$

3) $5[(3+12)-2(4)]$	1) Work within [ ]
	a. Multiply $2 \cdot 4$
$5[(3+12)-8]$	b. Add $3 + 12$
$5[15-8]$	c. Subtract $15 - 8$
$5[7]$	2) Multiply $5[7]$
35	

4) $4[3(3+2)^2]$	1) Work within [ ]
	a. Parenthesis $(3 + 2)$
$4[3(5)^2]$	b. Powers $(5)^2$
$4[3 \cdot 25]$	c. Multiply $3 \cdot 25$
$4[75]$	2) Multiply $4[75]$
300	

5) $4 \cdot 5 - 18 \div 6 + 2 \cdot 3$	1) Multiply and divide left to right
$20 - 3 + 6$	2) Add and subtract left to right
$17 + 6$	3) Add
23	

You will continue to use the **order of operations** throughout the remainder of this unit and throughout any other mathematics courses you continue to take.

## Introduction to Variables and Expressions

### Did you know?

Most countries in the world use the Celsius scale to measure temperature. Two critical temperatures on the Celsius scale are  $0^\circ$  freezing and  $100^\circ$  boiling. On the other hand, in the United States we use the Fahrenheit scale most of the time. The same two critical temperatures on the Fahrenheit scale measure  $32^\circ$  freezing and  $212^\circ$  boiling. It is possible to convert between temperature scales by using algebra.

If the Celsius temperature is multiplied by  $\frac{9}{5}$  and then added to 32, the Fahrenheit temperature can be determined.

Algebra can be thought of as a language of symbols. For example, we already know the symbols for addition (+) and multiplication ( $\times$  or  $\cdot$ ) so we could write the temperature relationship from above as follows:

$$\frac{9}{5} \cdot \text{Celsius} + 32$$

In arithmetic we could write the same expression as:

$$\frac{9}{5} \cdot \square + 32$$

where  $\square$  represents the Celsius temperature and is serving as a place holder.

In algebra when a problem has missing or “unknown” information, the place holders used are called **variables**. Variables are letters such as  $x$ ,  $n$ , or  $a$  that are used to represent the unknown value. (You may use any letter as a variable; these were just a few examples.)

**\*When choosing a variable to represent an unknown value, make sure not to use the letter “o” because it could be mistaken for the number zero.**

Let's take a look at how to represent an expression using algebra and a variable.

$$\frac{9}{5} \cdot \text{Celsius} + 32 \quad \text{Words and symbols}$$

$$\frac{9}{5} \cdot \square + 32 \quad \text{Arithmetic}$$

$$\frac{9}{5}C + 32 \quad \text{Algebra (variable)}$$

$\frac{9}{5} \cdot C + 32$  is called an **algebraic expression** because it contains a combination of variables, numbers and at least one operation.

Algebraic expressions can be **evaluated** by replacing the variable with numbers.

For example, if given the expression  $a + b - 24$  and asked to evaluate it for the given values  $a = 19$  and  $b = 20$ , you would:

- a) replace  $a$  and  $b$  with the given values
- b) evaluate the expression using the order of operations

Let's try the example above. You will be given this type of problem in the following form.

*Example 1:* Evaluate  $a + b - 24$  if  $a = 19$  and  $b = 20$ .

$$a + b - 24$$

$$19 + 20 - 24$$

$$39 - 24$$

$$15$$

1) replace  $a$  and  $b$  with the given values

2) evaluate using the order of operations

*Example 2:* Evaluate  $5a + bc - c$  if  $a = 4$ ,  $b = 2$ , and  $c = 3$ .

$$5a + bc - c$$

$$5(4) + (2)(3) - 3 \quad 1) \text{ replace } a, b, \text{ and } c \text{ with the given values}$$

$$20 + (2)(3) - 3 \quad 2) \text{ multiply } 5(4)$$

$$20 + 6 - 3 \quad 3) \text{ multiply } 2(3)$$

$$26 - 3 \quad 4) \text{ add } 20 + 6$$

$$23 \quad 5) \text{ subtract } 26 - 3$$

As you become more familiar with evaluating expressions, you will be able to perform more than one operation per step. For example, in the example above it would have been okay in step 2 to also multiply 2 and 3. Again you will be able to do this after more practice.

Throughout this course and any of the more advanced mathematics courses you will take, it will be necessary to interpret verbal sentences into algebraic sentences. For this you will need to know the words and phrases that suggest the operation to use.

The chart below lists some of the most common phrases that will be used.

<b>Addition</b>	<b>Subtraction</b>	<b>Multiplication</b>	<b>Division</b>
sum	difference	product	quotient
plus	minus	times	divided by
increased by	decreased by	of	ratio
more than	less than	twice ( $\times 2$ )	per
total	subtract	multiplied by	average



To translate verbal phrases into algebraic expressions:

- a.) chose a variable to represent the unknown (if it is not given)
- b.) determine what operation will be used based on the phrase
- c.) write the algebraic expression

*Example 3:* “eight more points than Rachel’s score”

- a.) Let  $r$  represent Rachel’s score
- b.) “More than” suggests addition
- c.) The algebraic expression is  $r + 8$  or  $8 + r$

*Example 4:* “four times as much money as Pete”

- a.) Let  $p$  represent Pete’s money
- b.) “Times” suggests multiplication
- c.) The algebraic expression is  $p \times 4$ ,  $4 \times p$ , or more commonly seen in algebra as  $4p$ .

\*At this point it should be noted that anytime a variable or variables are multiplied with a number, it will be written with the number first and then the variable or variables following. Study the examples given below.

$6x$  means 6 times  $x$  OR  $x$  times 6.

$7y$  means 7 times  $y$  OR  $y$  times 7.

$3xyz$  means 3 times  $x$  times  $y$  times  $z$  in any order.

## Solving One-Step Equations

A mathematical sentence such as  $374 + x = 795$  is called an **equation** because it contains an equal sign.

The **solution** to an equation is the value of the variable that results in a true statement. The process of finding this solution is called **solving the equation** which uses opposite operations in order to isolate the variable. When we say opposite operations, we mean the opposite of adding is subtracting, the opposite of multiplying is dividing, and visa versa.

At this point it should be mentioned that an equation must be balanced. This means that whatever is done on one side of the equation must also be done on the other side of the equal sign.

Let's take a look at a few examples of solving equations.

*Example 1:* Solve  $x + 4 = 17$

-Since we want  $x$  by itself, we have to perform the opposite of  $+ 4$  on both sides of the equal sign, opposite of  $+ 4$  is  $- 4$ .

-Performing opposite operations can be done different ways; below you will see that this can be done vertically or horizontally; you make the choice.

$$\begin{array}{r} x + 4 = 17 \\ \underline{-4 \quad -4} \\ x + 0 = 13 \\ x = 13 \end{array} \quad \text{or} \quad \begin{array}{r} x + 4 - 4 = 17 - 4 \\ x + 0 = 13 \\ x = 13 \end{array}$$

-check the solution by replacing  $x$  in the equation with the solution 13.

$$\begin{array}{l} \text{Check: } 13 + 4 = 17 \\ \quad \quad 17 = 17 \end{array}$$

Since this is true, **13** is the correct solution.

Example 2: Solve  $\frac{n}{4} = 7$

-The opposite of dividing by 4 is multiplying by 4, multiply both sides by 4.

$$4\left(\frac{n}{4}\right) = (7)4$$

$$\frac{4n}{\cancel{4}} = 28 \quad \text{Cancel the 4's to isolate the variable.}$$
$$n = 28$$

$$\text{Check: } \frac{28}{4} = 7$$
$$7 = 7$$

Since this is true, **28** is the correct solution.

Example 3: Solve  $y - 26 = 38$

-The opposite of  $-26$  is  $+26$ , add 26 to both sides of the equation.

$$\begin{array}{r} y - 26 = 38 \\ +26 \quad +26 \\ \hline y = 64 \end{array}$$

$$\text{Check: } 64 - 26 = 38$$
$$38 = 38$$

Example 4: Solve  $5n = 75$

-The opposite of multiplying by 5 is dividing by 5.

$$\frac{5n}{5} = \frac{75}{5}$$

$$n = 15$$

$$\text{Check: } 5(15) = 75$$
$$75 = 75$$

## Solving Equations with Rational Numbers

When solving equations involving rational numbers, you will use the same principle of opposite operations as you did when solving equations with integers. One exception is that when you divide by a rational fraction, you need to remember that means to multiply by the reciprocal.

Let's try a few examples:

$$\text{Example 1: } x - \frac{5}{6} = \frac{7}{8}$$

*Step 1:* Add  $\frac{5}{6}$  to both sides

$$x - \frac{5}{6} + \frac{5}{6} = \frac{7}{8} + \frac{5}{6}$$

*Step 2:* Determine the common denominator (24)

$$x = \frac{21}{24} + \frac{20}{24}$$

*Step 3:* Add the rational numbers

$$x = \frac{41}{24} \quad \text{or} \quad 1\frac{17}{24}$$

Example 2:  $\frac{5}{6} = \frac{-2}{3}x$

Step 1: Multiply both sides by the reciprocal of  $\frac{-2}{3}$  which is  $\frac{-3}{2}$ .

$$\frac{-3}{2}\left(\frac{5}{6}\right) = \left(\frac{-2}{3}x\right)\frac{-3}{2}$$

Step 2: Simplify.

$$\frac{\cancel{-3}^1}{2}\left(\frac{5}{\cancel{6}_2}\right) = \left(\frac{\cancel{-2}}{\cancel{3}}x\right)\frac{\cancel{-3}}{\cancel{2}}$$

Step 3: Multiply.

$$\frac{-1 \cdot 5}{2 \cdot 2} = x$$

$$x = \frac{-5}{4} \quad \text{or} \quad x = -1\frac{1}{4}$$