PROBABILITY

In this unit, you will determine if events are inclusive or mutually exclusive. You will also calculate simple and compound probabilities of independent and dependent events.

Introduction to Probability

Theoretical Probability

Independent and Dependent Events

Inclusive and Mutually Exclusive Events

Random Numbers

Introduction to Probability

Probability is the likelihood of an event occurring.

Terminology (A coin is used for each of the examples.)	
Definition	**Example**
Trial: a systematic opportunity for an event to occur	tossing a coin in the air
Experiment: one or more trials	tossing a coin 6 times
Sample space: the set of all possible outcomes of an event	Head (***H***) or Tails (***T***)
Event: an individual outcome or any specified combination of outcomes	landing ***H*** or landing ***T***
Random sample: A sample in which every member of a population has an equal chance of being selected	a lottery machine makes a ticket with 6 numbers generated without bias

Probability is expressed as a number from 0 to 1. It is written as a fraction, decimal, or percent.

- An impossible event has a probability of 0.
- An event that must occur has a probability of 1.
- The sum of the probabilities of all outcomes in a sample space is 1.

The probability of an event can be assigned in two ways:

1) **experimentally**: approximated by performing trials and recording the ratio of the number of occurrences of the event to the number of trials (As the number of trials in an experiment increases, the approximation of the experimental probability increases.)

2) **theoretically:** based on the assumption that all outcomes in the sample space occur randomly

Theoretical Probability

If all outcomes in a sample space are equally likely, then the theoretical probability of event *B*, denoted *P(B)*, is defined by:

$$P(B) = \frac{\text{Number of outcomes in event } B \text{ (favorable outcomes)}}{\text{Number of outcomes in the sample space (possible outcomes)}}$$

Example: Find the probability of randomly selecting an orange marble out of a jar containing 3 blue, 3 red, and 2 orange marbles.

$$P(\text{1 orange}) = \frac{\textit{favorable}}{\textit{possible}} = \frac{\text{2 orange}}{\text{8 possible}}$$

$$= \frac{2}{8} = \frac{1}{4} \text{ or } 25\%$$

Theoretical Probability

Let's examine the toss of a coin. Let's say you are trying to determine who takes out the garbage, you or your brother. You decide to flip a coin to determine the winner. For this experiment, there are two possible outcomes – heads or tails. These results are said to be **equally likely outcomes** because it is just as likely for heads to come up as it is for tails.

A tossed coin will tend to come up heads half of the time and tails half of the time. We then say that the probability of getting heads is $\frac{1}{2}$ and the probability of getting tails is $\frac{1}{2}$.

The following notation is used for probability:

Probability of heads $\rightarrow P(\text{heads}) = \frac{1}{2}$

Probability of tails $\rightarrow P(\text{tails}) = \frac{1}{2}$

If you add the probabilities together, the result is 1.

The probability of an event occurring is referred to to $P = \left(\frac{\text{favorable outcomes}}{\text{possible outcomes}}\right)$.

Favorable outcomes doesn't necessarily mean good or bad. It refers to the outcome you want to happen.

In the coin toss experiment, you will notice that the probability of tossing heads is $\frac{1}{2}$ because the coin has heads on one side, but the possible outcome could be heads or tails, 2 possibilities.

The same concept can be explained using a single roll of a die. This time there are 6 possible outcomes – 1, 2, 3, 4, 5, or 6. Each of these possibilities has an equally likely outcome.

$P(1) = \frac{1}{6}$	$P(2) = \frac{1}{6}$	$P(3) = \frac{1}{6}$
$P(4) = \frac{1}{6}$	$P(5) = \frac{1}{6}$	$P(6) = \frac{1}{6}$

Again, if you add these probabilities, you will get 1.

Sample space: the set of all possible outcomes of a probability experiment

sample space for tossing a coin {H, T}

sample space for rolling a die {1, 2, 3 , 4, 5, 6}

Example 1: Find the sample space of tossing a coin three times.

First	Second	Third
H	H	H
H	H	T
H	T	H
H	T	T
T	T	T
T	T	H
T	H	T
T	H	H

The sample space for this experiment can be written as:

{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH}

Example 2: In the example above, what is the probability that all three tosses will be all heads or all tails?

P(3 heads or 3 tails) = ?

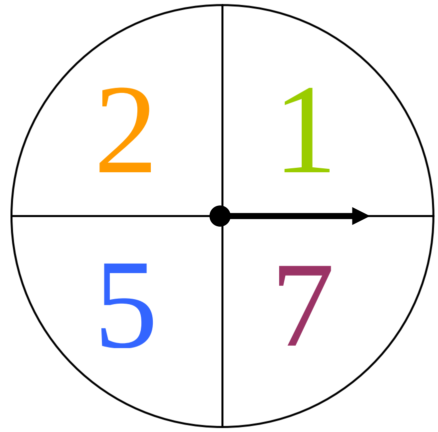
a) Since the sample space produced 2 possibilities of all heads or all tails, 2 is the “favorable” in our ratio.

b) The sample space produced 8 possible outcomes so 8 is our “possible” in our ratio.

P(3H or 3T) = $\frac{2}{8} = \frac{1}{4}$

The probability of tossing all heads or all tails in the 3 tosses is $\frac{1}{4}$ or 25%.

Example 3: The spinner at the right is spun two times.



a) List the sample space.

{(1, 1), (1, 7), (1, 5), (1, 2),

(7, 1), (7, 7), (7, 5), (7, 2),

(5, 1), (5, 7), (5, 5), (5, 2),

(2, 1), (2, 7), (2, 5), (2, 2)}

b) What is the probability that the sum of the spins will be greater than 10?

Look at the sample space.

{(1, 1), (1, 7), (1, 5), (1, 2),

(7, 1), (7, 7), (7, 5), (7, 2),

(5, 1), (5, 7), (5, 5), (5, 2),

(2, 1), (2, 7), (2, 5), (2, 2)}

Out of 16 possible out comes, there are 3 favorable outcomes.

Therefore: $\mathbf{P(sum > 10) = \dfrac{3}{16}}$

Independent and Dependent Events

Two events are **independent** if the occurrence (or non-occurrence) of one event has no effect on the likelihood of the occurrence of the other event. For example, rolling a die and choosing a marble out of a bag are independent events.

Probability of Independent Events

Events A and B are independent events if and only if

$$P(A \text{ and } B) = P(A) \times P(B)$$

The probability of two independent events can be found by multiplying the probability of the first event by the probability of the second event.

Example 1: Bag A contains 8 red buttons and 4 green buttons. Bag B contains 10 black buttons and 5 orange buttons. Find the probability of selecting one green button from bag A and one black button from bag B in one draw from each bag.

Solution:

Bag A

$$P(\text{green button}) = \frac{4}{8+4} = \frac{4}{12} = \frac{1}{3}$$

Bag B

$$P(\text{black button}) = \frac{10}{10+5} = \frac{10}{15} = \frac{2}{3}$$

The events are independent.

$$\text{P(green and black)} = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$$

The probability of selecting a green button from bag A and one black button from bag B in one draw from each bag is $0.2\overline{2} \approx 22\%$

*The formula for the probability of independent events can be extended to 3 or more events. For example, the probability of obtaining a roll of 2 in 3 rolls of a die is

$$\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216} \approx 0.46\%$$

Example 2: A bag contains 6 blue and 9 yellow marbles. Find the probability of randomly selecting a blue marble on the first draw and a yellow marble on the second draw when the marble is replaced after the first draw.

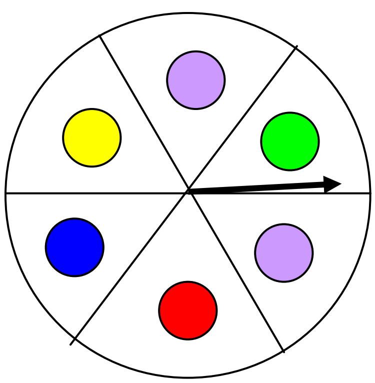
$$P(\text{blue}) = \frac{6}{6+9} = \frac{6}{15}$$

$$P(\text{yellow}) = \frac{9}{6+9} = \frac{9}{15}$$

$$P(\text{blue and yellow}) = \frac{6}{15} \times \frac{9}{15} = \frac{54}{225} \approx 24\%$$

The probability of randomly selecting a blue marble on the first draw and a yellow marble on the second draw when the marble is replaced after the first draw is approximately 24%.

Example 3: Emily rolls a number cube and spins the spinner below. What is the probability that she will roll a number less than 6 AND land on the purple section of the spinner?



*Because the roll of the number cube does not affect the result of the spinner, the events are **independent.**

a) Five of the numbers on the cube are less than 6.

$P(\text{rolling a number less than 6}) = \dfrac{5}{6}$

b) Two of the spaces on the spinner are purple.

$P(\text{landing on a purple space}) = \dfrac{2}{6} = \dfrac{1}{3}$

c) To find the probability of both happening, multiply the individual probabilities.

$$\frac{5}{6} \times \frac{1}{3} = \frac{5}{18} \approx 28\%$$

The probability Emily will roll a number less than 6 AND land on the purple section of the spinner is approximately 28%.

A **dependent event** is the probability of a second event happening depending on the outcome of the first event.

Probability of Dependent Events

If Events *A* and *B* are dependent events, then the probability of both events occurring is the product of the probability of *A* and the probability of *B* after *A* occurs.

$$P(A \text{ and } B) = P(A) \times P(B \text{ following } A)$$

Example 4: A bag contains 6 blue and 9 yellow marbles. Find the probability of randomly selecting a blue marble on the first draw and a yellow marble on the second draw if the marble is **not replaced** after the first draw.

$$\text{P(blue)} = \frac{6}{9+6} = \frac{6}{15}$$

$$\text{P(yellow)} = \frac{9}{8+6} = \frac{9}{14}$$

$$\text{P(blue and yellow)} = \frac{6}{15} \times \frac{9}{14} = \frac{54}{210} = \frac{9}{35} \approx 26\%$$

The probability of randomly selecting a blue marble on the first draw and a yellow marble on the second draw if the marble is NOT REPLACED after the first draw is approximately 26%.

Inclusive and Mutually Exclusive Events

Inclusive events are events that can occur at the same time. For example, a person can belong to more than one club at the same time.

Mutually exclusive events are events that cannot occur at the same time. For example, if you flip a coin, you cannot get both heads and tails.

Probability of A or B

Let A and B represent events in the same sample space.

If A and B are **mutually exclusive** events, then

$$P(A \text{ or } B) = P(A) + P(B)$$

If A and B are **inclusive** events, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

In a survey about a change in cafeteria food, 100 students were asked if they favor the change, oppose the change, or have no opinion about the change. The responses are indicated below.

	Boys	Girls	Total
Favor	**18**	**9**	**27**
Oppose	**12**	**25**	**37**
No Opinion	**20**	**16**	**36**
Total	**50**	**50**	**100**

Example 1: Find the probability that a randomly selected respondent to the survey opposes or has no opinion about the change in cafeteria food.

Solution:

The events “oppose” and “no opinion” are MUTUALLY EXCLUSIVE events.

P(oppose or no opinion) = P(oppose) + P(no opinion)

$$= \frac{37}{100} + \frac{36}{100}$$

$$= \frac{73}{100} \text{ or } 73\%$$

The probability that a respondent opposes the change or has no opinion is 73%.

Example 2: Find the probability that a randomly selected respondent to the survey is a boy or opposes the change in the policy.

Solution:

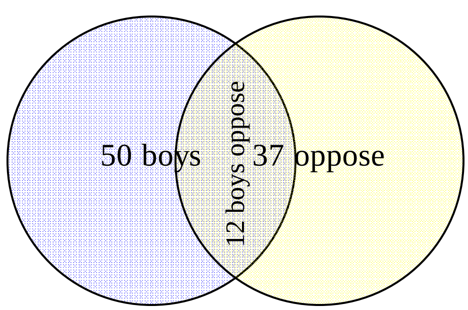
The events “boy” and “opposes” are INCLUSIVE EVENTS.

P(boy or opposes) = P(boy) + P(opposes) – P(boy and opposes)

$$= \frac{50}{100} + \frac{37}{100} - \frac{12}{100}$$

$$= \frac{75}{100} \text{ or } 75\%$$

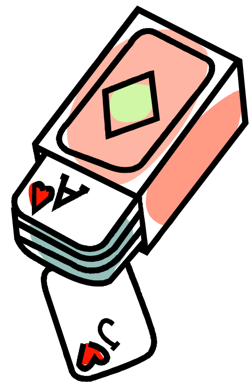
The Venn diagram below shows a visual interpretation of this problem and demonstrates that the two groups, “boys” and “oppose” overlap each other. Thus, these two groups are INCLUSIVE and the probability represented by the overlap must be subtracted from the probability of the two events.



The probability that a respondent is a boy or opposes the change in policy is 75%.

Here is another example of a mutually inclusive event that involves a standard deck of playing cards. INCLUSIVE EVENTS are events that can occur at the same time. If A and B are inclusive events, then P(A or B) = P(A) + P(B) – P(A and B).

Example 3: What is the probability of drawing a red card or an ace?



Ace through King = A(1), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (11), Queen (12), King (13) = 13 cards in one suit.

There are 4 suits in a standard deck of cards: Hearts, Diamonds, Clubs, and Spades

There are 2 Black Suits: Clubs and Spades.
There are 2 Red Suits: Hearts and Diamonds.

Again, we will use a Venn diagram to help visualize the relation of two sets, set A and set B. When considering A OR B, we think of the total given by the union of the sets. When considering A AND B, we think of the intersection of the sets.

When part of set A is in set B or vice versa, then we must subtract the intersection of the two sets. We subtract the intersection of A AND B because it is counted twice: once as part of set A and again as part of set B. It should only be counted once.

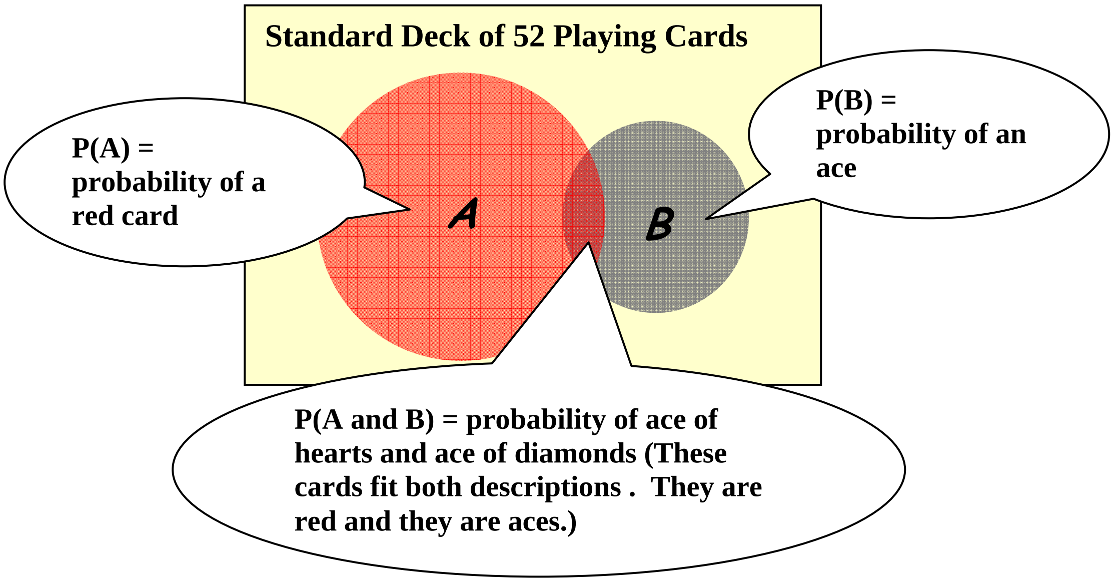
For this problem,

Set A will represent the probability of drawing a **red card**.

Set B will represent the probability of drawing an **ace**.

The Venn diagram below visually describes the probability of drawing a **red** card or an **ace** when one card is taken from a pack of 52. This is called an ***inclusive event***.

P(A or B) = P(A) + P(B) – P(A and B)



Mathematically, the Venn diagram illustrated above translates to the following.

Let P(A) = probability of a red card

$$\text{P(Red)} = \frac{26}{52} = \mathbf{\frac{1}{2}}$$

Let P(B) = probability of an ace

$$\text{P(Ace)} = \frac{4}{52} = \mathbf{\frac{1}{13}}$$

P(A and B) = probability of an ace and a red card

$$\text{P(Ace and Red)} = \frac{2}{52} = \mathbf{\frac{1}{26}}$$

$$\text{(Ace of Hearts, Ace of Diamonds)}$$

*Note: $\text{P(A and B)} = \text{P(A)} \times \text{P(B)} = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}.$

Now, to solve the problem given, that is, WHAT IS THE PROBABILITY OF DRAWING A RED CARD OR AN ACE?

P(A or B) = P(A) + P(B) – P(A and B) = probability of a red card + probability of an ace – probability of an ace and a red card

$$\begin{aligned}
P(\text{Red or Ace}) &= \left(\frac{1}{2} + \frac{1}{13}\right) - \frac{1}{26} \\
&= \left(\frac{13}{26} + \frac{2}{26}\right) - \frac{1}{26} \\
&= \frac{14}{26} \\
&= \frac{7}{13}
\end{aligned}$$

The probability of drawing a red card or an ace is $\mathbf{\frac{7}{13}}$.

Random Numbers

By definition, pi is the ratio of the circumference of a circle to its diameter. Pi is always the same number, no matter which circle you use to compute it.

Pi: π

For the sake of usefulness, people often need to approximate pi. For many purposes, you can use 3.14159, which is really pretty good, but if you want a better approximation you can use a computer to get it.

Here are the first twenty-one digits of pi: 3.14159265358979323846.

Pi is an **infinite decimal**. Unlike numbers such as 3, 9.876, and 4.5, which have finitely many nonzero numbers to the right of the decimal place, “pi” has infinitely many numbers to the right of the decimal point. If you write “pi” in decimal form, the **numbers to the right of the 0 never repeat in a pattern**. Hence, they are **random numbers**.

Many mathematicians have tried to find a pattern in “pi”; but, no repeating pattern for “pi” has been discovered. In fact, in 1768 Johann Lambert proved that there cannot be any such repeating pattern.

Source: http://mathforum.org/dr.math/faq/faq.pi.html

Let’s example some random numbers generated by “pi”.

Listed below are the first 1000 digits of “pi”.

$\pi = 3.$

1415926535897932384626433832795028841971693993751 0
5820974944592307816406286208998628034825342117067 9
8214808651328230664709384460955058223172535940812 8
4811174502841027019385211055596446229489549303819 6
4428810975665933446128475648233786783165271201909 1
4564856692346034861045432664821339360726024914127 3
7245870066063155881748815209209628292540917153643 6
7892590360011330530548820466521384146951941511609 4
3305727036575959195309218611738193261179310511854 8
0744623799627495673518857527248912279381830119491 2
9833673362440656643086021394946395224737190702179 8

60943702770539217176293176752384674818467669405132
00056812714526356082778577134275778960917363717872
14684409012249534301465495853710507922796892589235
42019956112129021960864034418159813629774771309960
51870721134999999837297804995105973173281609631859
50244594553469083026425223082533446850352619311881
71010003137838752886587533208381420617177669147303
59825349042875546873115956286388235378759375195778
18577805321712268066130019278766111959092164201989
….

We’ll use the first 80 digits to the right of the decimal point to solve some problems related to random numbers and probability.

Shown below are the first 80 digits of “pi” right of the decimal point. Set A is the first 20 digits, Set B is the next 20 digits, Set C is the next 20 digits, and Set D is the final 20 digits.

Set A: 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6

Set B: 2 6 4 3 3 8 3 2 7 9 5 0 2 8 8 4 1 9 7 1

Set C: 6 9 3 9 9 3 7 5 1 0 5 8 2 0 9 7 4 9 4 4

Set D: 5 9 2 3 0 7 8 1 6 4 0 6 2 8 6 2 0 8 9 9

Next, we’ll organize the digits into a chart. The chart shows how many times each of the ten single digits in our number system (0 through 9) occur in each of the four sets of numbers (A through D).

	Set A	Set B	Set C	Set D	Total
Zero's	0	1	2	3	6
One's	2	2	1	1	6
Two's	2	3	1	3	9
Three's	3	3	2	1	9
Four's	2	2	3	1	8
Five's	3	1	2	1	7
Six's	2	1	1	3	7
Seven's	1	2	2	1	6
Eight's	2	3	1	3	9
Nine's	3	2	5	3	13
Total:	20	20	20	20	80

*NOTE: Please keep in mind that the numbers occurring to the right of the decimal point "pi" π never repeat in a pattern and thus are RANDOM numbers.

Example 1: How many eights (8's) occurred in set B?

Read down the chart to "eight's", and then go across that row to the Set B column.

Three (3) eights occurred in set B.

Example 2: What is the probability that a digit in the first 80 digits of "pi" (after the decimal point) is an eight (8) or a nine (9)?

* The two events, P(8) or P(9) are **mutually exclusive**.

We can use the total column and read that 9 eights and 13 nines occurred.

Thus,

$$
\begin{aligned}
P(8 \text{ or } 9) &= \frac{9}{80} + \frac{13}{80} \\
&= \frac{20}{80} \\
&= \frac{1}{4}
\end{aligned}
$$

The probability that a digit of the first 80 digits of "pi" (after the decimal point) is an eight (8) or a nine (9) is 1/4.

Example 3: What is the probability that a digit in the first 80 digits of "pi" (after the decimal point) is in Set C or is a seven (7)?

* The two events, P(a digit in Set C) or P(7) are **inclusive**. This means there is an overlap in the two events and that overlap must be subtracted out.

$$\text{P(a digit in Set C)} = \frac{20}{80}$$

$$\text{P(7)} = \frac{6}{80}$$

$$\text{P(a digit in Set C and a 7)} = \frac{2}{80}$$

$$\begin{aligned}\text{P(a digit in Set C)} + \text{P(7)} - \text{P(a digit in Set C and a 7)} &= \frac{20}{80} + \frac{6}{80} - \frac{2}{80} \\ &= \frac{24}{80} \\ &= \frac{3}{10}\end{aligned}$$

The probability that a digit in the first 80 digits of "pi" (after the decimal point) is in Set C or is a seven (7) is 3/10.

Random number generators are used in many games of chance or lottery. As you can see in the examples above, generating digits of "pi" is one way to create a list of random numbers.