COUNTING AND PROBABILITY

In this unit, you will look at arrangements of letters, numbers, and words. You will first explore listing all possible outcomes through tables, grids, and tree diagrams. You will then connect the process of determining outcomes to the Fundamental Counting Principle, combinations, and permutations. You will also examine the "odds" of favorable and unfavorable outcomes.

Fundamental Counting Principle

Combinations

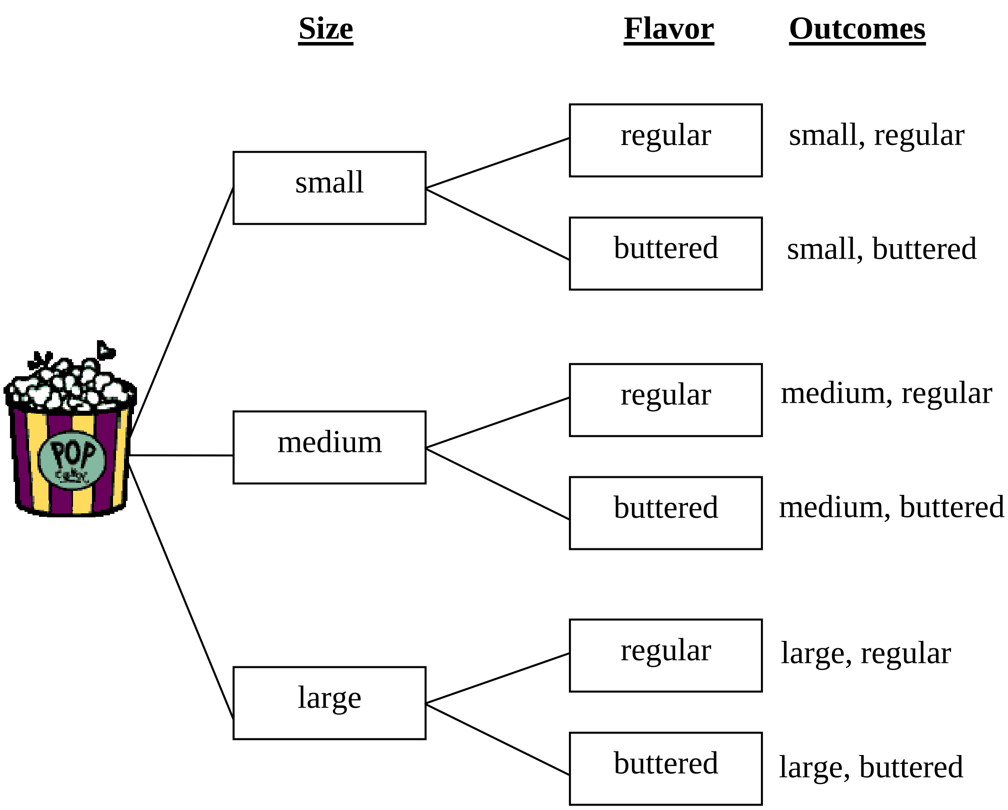
Permutations

Odds

Fundamental Counting Principle

> If there are m ways that one event can occur and n ways that another event can occur, then there are $\boldsymbol{m} \times \boldsymbol{n}$ ways that both events can occur.

Example 1: A movie theatre sells popcorn in small, medium, and large containers. Each size is also available in regular or buttered popcorn. How many options for buying popcorn does the movie theatre provide?



There are 6 possible options for buying popcorn at the movie theatre.

Example 2: How many options are there if the theatre adds three new flavors - caramel apple, jelly bean, and bacon cheddar?

For each size, you would add the three new flavors to the tree diagram. Applying the fundamental counting theorem, you would multiply 3×5 to get 15.

There would be a total of 15 options.

Example 3: Emily is choosing a password to gain access to the Internet. She decides not to use the digit 0 or the letters A, E, I, O, or U. Each letter or number may be used more than once. How many passwords of 3 letters followed by 2 digits are possible?

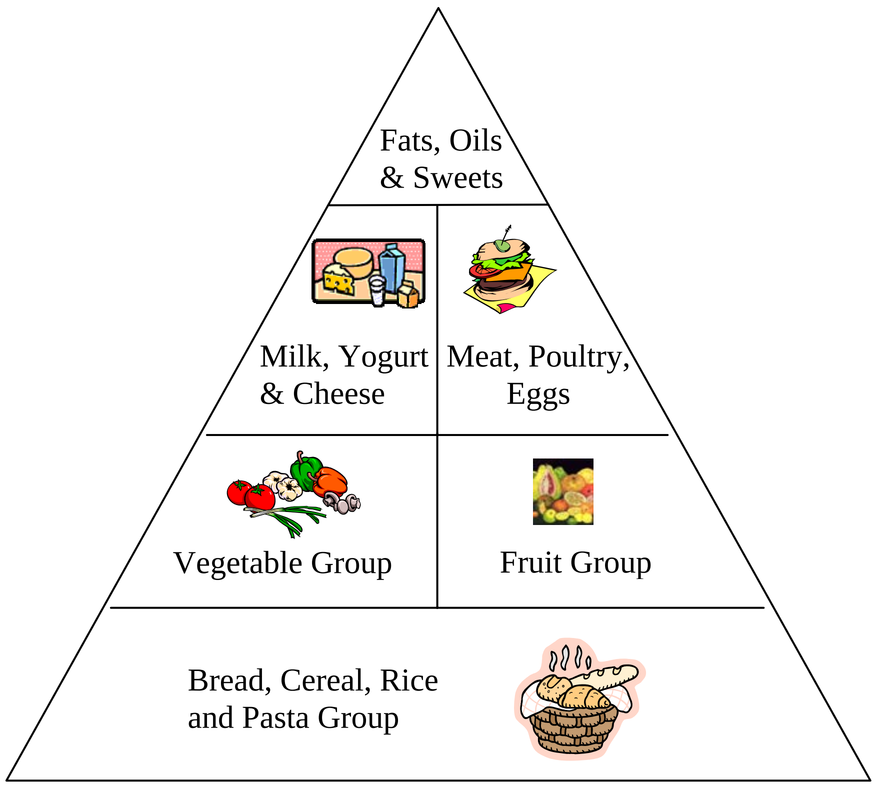
Use the fundamental counting principle. There are 21 possible letters and 9 possible digits.

first letter		second letter		third letter		first digit		second digit
?		?		?		?		?
21 choices	$\times$	21 choices	$\times$	21 choices	$\times$	9 choices	$\times$	9 choices

The number of possible passwords for Emily is $21^3 \times 9^2$ or 750,141.

Combinations

A well-planned meal or balanced diet gives you all the nutrients you need each day. To plan a balanced diet, you need to select foods from each of the main food groups. The food pyramid below is a practical tool to help you make food choices that are consistent with the dietary guidelines for Americans.



We are going to take a look at the different types of foods Hanna has for her friends and separate them into the food groups:

Meats: chicken and fish

Dairy: milk and cheese

Breads: spaghetti, brown rice, crackers, mixed nuts, dinner rolls

Fruits: mixed fruit, peaches

Vegetables: spaghetti sauce, lettuce

To determine the number of possible meals, Hanna will multiply the number of each type as shown below:

# of Meats	$\times$	# of Dairy	$\times$	# of Breads	$\times$	# of Fruits	$\times$	# of Vegetables	=	Total Possible Meals
2	$\times$	2	$\times$	5	$\times$	2	$\times$	2	=	80

Thus, Hanna could plan 80 different meals.

This example is an introduction to **combinations**, an arrangement of objects in which order is not important. Such situations may occur when choosing members for a committee, drawing numbers for bingo , or determining your chances of winning the lottery. The following formula for combinations is given below.

Combination of "*n* objects taken *r* at a time"

$$_{n}C_{r} = \frac{n!}{r!(n-r)!}$$

To understand this formula, we need to understand the symbolism.

The symbol "!" is a mathematical expression called "factorial".

This means that for an integer *n*, *n!* means to multiply all integers 1, 2, 3, 4, …, *n* together to produce a result called "*n factorial*". In combination problems, "*n* factorial" is written as "$n \ldots \times 4 \times 3 \times 2 \times 1$".

Example 1: $5! = 5\times4\times3\times2\times1$

$$= 120$$

Example 2: $\dfrac{9!}{5!} = \dfrac{9\times8\times7\times6\times\cancel{5\times4\times3\times2\times1}}{\cancel{5\times4\times3\times2\times1}}$

$$= 3024$$

Notice that 5!, which is, $5 \times 4 \times 3 \times 2 \times 1$, can be cancelled from 9!, so $\frac{9!}{5!}$ could actually be written as

$$\frac{9 \times 8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!}}$$

This now brings us to the formula, ${}_nC_r = \frac{n!}{r!(n-r)!}$. The n represents the number of things that are available, and the r represents the number of things you are choosing.

Example 3: A pizza parlor offers a selection of 8 different toppings. In how many ways can a pizza be made with 3 toppings?

8 represents *n*, the number of total toppings

3 represents *r*, the number of toppings you are choosing.

Replace these numbers in the formula and solve.

$${}_8C_3 = \frac{8!}{3!(8-3)!}$$

$${}_8C_3 = \frac{8!}{3!(5!)}$$

$${}_8C_3 = \frac{8 \times 7 \times 6 \times \cancel{5!}}{3!(\cancel{5!})}$$

$${}_8C_3 = \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$$

$${}_8C_3 = 56$$

There are 56 combinations of choosing 3 toppings from 8 selections.

Sometimes you will need to find combinations of more than one thing at a time. In this case, multiply the combinations together to find the total amount of combinations.

Let's go back to the pizza example and add different sizes to the list of choices.

Follow the example below.

Example 4: A pizza parlor offers a selection of 8 different toppings and 3 different sizes. In how many ways can a pizza be ordered with the following selections: 2 sizes and 4 toppings?

$$
\begin{array}{rccc}
 & \text{3 sizes} & \times & \text{8 toppings} \\
 & \text{choosing 2} & & \text{choosing 4} \\
 & {}_3C_2 & \times & {}_8C_4 \\
= & \dfrac{3!}{2!(3-2)!} & \times & \dfrac{8!}{4!(8-4)!} \\
= & \dfrac{3\times\cancel{2\times1}}{\cancel{2\times1}(1!)} & \times & \dfrac{\overset{2}{\cancel{8}}\times7\times\cancel{6}\times5\times\cancel{4!}}{\cancel{4}\times\cancel{3}\times\cancel{2}\times1(\cancel{4!})} \\
= & \dfrac{3}{1} & \times & \dfrac{2\times7\times5}{1} \\
= & 3 & \times & 70 \\
= & 210 & &
\end{array}
$$

Thus, there are 210 combinations of 2 sizes and 4 toppings.

Permutations

Another way to arrange objects is called permutations. A **permutation** is an arrangement of objects in a specific order. Such arrangements could include the batting order of a softball team, seat assignments in a classroom, or items displayed on a store shelf.

The following is the formula for finding the number of permutations of *n* objects taken *r* at a time.

$$\text{Permutation of "}n\text{ objects taken }r\text{ at a time"}$$
$$_nP_r = \frac{n!}{(n-r)!}$$

Example: Find the number of ways to listen to 5 CDs from a selection of 12 CDs.

$$_{12}P_5 = \frac{12!}{(12-5)!}$$

$$_{12}P_5 = \frac{12!}{7!}$$

$$_{12}P_5 = \frac{12\times11\times10\times9\times8\times\cancel{7!}}{\cancel{7!}}$$

$$_{12}P_5 = 95,040$$

There are 95,040 different ways to listen to 5 CDs from a selection of 12 CDs.

Odds

When all outcomes are equally likely, the ratio of the number of **favorable outcomes to** the number of **unfavorable outcomes** is called the **odds in favor** of an event.

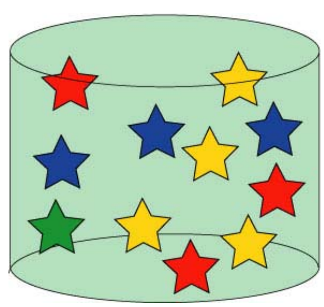
Odds in Favor of an Event

$$\text{Odds in favor} = \frac{\text{Number of favorable outcomes}}{\text{Number of unfavorable outcomes}} = a : b$$

a = number of favorable events
b = number of unfavorable events

$a + b$ = total number of outcomes

Consider selecting stars at random from the container shown below.



Example 1: What are the odds of selecting a gold star?

There are 4 gold stars (favorable) and 7 other stars (red, blue, and green) in the container (unfavorable).

4 : 7

The odds in favor of selecting a **gold star** are **4 : 7**.

Example 2: What are the odds of selecting a **black star**?

Since there are **no** black stars in the container, the odds are **0 : 11** or **no** chance of happening.

Example 3: What are the odds of selecting **any of the stars** in the container?

Since this includes the possibility of selecting **any** of the stars in the container, then the odds are **11 : 0,** a **sure thing** that it would happen.

Example 4: What are the odds in favor of drawing a **blue star** after **two gold stars** have already been drawn from the container?

After two gold stars are drawn from the container, there are 3 blue stars (favorable) and 6 other stars (red, yellow, and green) in the container (unfavorable).

The odds in favor of drawing a blue star after two gold stars are drawn are **3 : 6,** which simplifies to **1 : 2**.

When all outcomes are equally likely, the ratio of the number of **unfavorable outcomes to** the number of **favorable outcomes** is called the **odds against** an event.

Odds Against of an Event

$$\text{Odds Against} = \frac{\text{Number of unfavorable outcomes}}{\text{Number of favorable outcomes}} = b : a$$

b = number of unfavorable events
a = number of favorable events

$b + a$ = total number of outcomes

Consider the outcomes when rolling two dice.

36 Possible Outcomes											
1	1	2	1	3	1	4	1	5	1	6	1
1	2	2	2	3	2	4	2	5	2	6	2
1	3	2	3	3	3	4	3	5	3	6	3
1	4	2	4	3	4	4	4	5	4	6	4
1	5	2	5	3	5	4	5	5	5	6	5
1	6	2	6	3	6	4	6	5	6	6	6

Example 1: Find the odds **against** rolling a sum of six.

36 Possible Outcomes											
1	1	2	1	3	1	4	1	5	1	6	1
1	2	2	2	3	2	4	2	5	2	6	2
1	3	2	3	3	3	4	3	5	3	6	3
1	4	2	4	3	4	4	4	5	4	6	4
1	5	2	5	3	5	4	5	5	5	6	5
1	6	2	6	3	6	4	6	5	6	6	6

There are 31 possible ways to roll a sum that is **not** a sum of six (unfavorable outcome) and 5 possible ways to roll a sum of six which are highlighted in red in the table (favorable outcome).

The odds **against** rolling a sum of six are **31 : 5**.

Example 2: Find the odds **against** rolling an even sum.

36 Possible Outcomes											
1	1	2	1	3	1	4	1	5	1	6	1
1	2	2	2	3	2	4	2	5	2	6	2
1	3	2	3	3	3	4	3	5	3	6	3
1	4	2	4	3	4	4	4	5	4	6	4
1	5	2	5	3	5	4	5	5	5	6	5
1	6	2	6	3	6	4	6	5	6	6	6

The odds **against** rolling an even sum are **18 : 18**, which simplifies to **1 : 1**. There are 18 possible ways to roll a sum that is **not** an even sum (unfavorable outcome) and 18 remaining ways to roll a sum that **is** even (favorable outcome).

Sometimes odds will be presented in terms of percent. Eighteen (18) equals half of the numbers or 50% of the numbers. Therefore, this is an example of the odds being 50 to 50 (50% likely to 50% unlikely).