

FRACTIONS

In this unit, you will review greatest common factors (GCF) and least common multiples (LCM). You will also re-examine fractions by reviewing the four basic operations: add, subtract, multiply, and divide.

GCF

LCM

Addition of Fractions

Subtraction of Fractions

Mixed Fractions and Improper Fractions

Multiplication of Fractions

Reciprocals

Division of Fractions

GCF

GCF (Listing Factors)

The **Greatest Common Factor (GCF)** of two numbers is the largest factor that is the same when all the factors in a number have been listed. Follow the steps in this example to find the GCF.

Find the GCF of 18 and 24.

List all of the factors of 18.

$$\begin{array}{l} 1 \times 18 \\ 2 \times 9 \\ 3 \times 6 \end{array}$$

Write the factors in order from least to greatest.

{1, 2, 3, 6, 9, 18}

List all of the factors of 24.

$$\begin{array}{l} 1 \times 24 \\ 2 \times 12 \\ 3 \times 8 \\ 4 \times 6 \end{array}$$

Write the factors in order from least to greatest.

{1, 2, 3, 4, 6, 8, 12, 24}

To find the GCF, look for the largest factor that is the same in both lists.

{1, 2, 3, 6, 9, 18}
{1, 2, 3, 4, 6, 8, 12, 24}

The GCF is 6.

*Note: There are other common factors, 2 and 3, but the GREATEST common factor is 6.

You may use the GCF to reduce fractions.

Reduce $\frac{18}{24}$ to simplest form.

Reduce by the 6, the GCF.

$$\frac{18}{24} \div \frac{6}{6} = \frac{3}{4}$$

GCF (Prime Factorization)

To find the **Greatest Common Factor (GCF)** of two numbers, you can apply prime factorization by finding all the common factors in each prime factorization, and then multiplying them.

Example: Find the GCF of 36 and 54.

The steps for finding the prime factorization of 36 is:

$$\begin{aligned}36 &= 2 \times 18 \\36 &= 2 \times 2 \times 9 \\36 &= 2 \times 2 \times 3 \times 3\end{aligned}$$

The steps for finding the prime factorization of 54 is:

$$\begin{aligned}54 &= 2 \times 27 \\54 &= 2 \times 3 \times 9 \\54 &= 2 \times 3 \times 3 \times 3\end{aligned}$$

The GCF is the product of the common prime factors (those factors appearing in both factorizations).

$$\begin{aligned}36 &= 2 \times 2 \times 3 \times 3 \\54 &= 2 \times 3 \times 3 \times 3\end{aligned}$$

There is a two, a three, and a second three that is found in both prime factorizations.

Therefore, the GCF is $2 \times 3 \times 3 = 18$.

LCM

LCM (Listing Multiples)

To find the **Least Common Multiple (LCM)**:

1. List the multiples of both numbers.
2. Find the first multiple that is common to both sets of multiples.

Example 1: Find the LCM of 8 and 16.

List multiples of 16	$16 = \{16, 32, 48 \dots\}$
List multiples of 8	$8 = \{8, 16, 24, 32, 40, 48 \dots\}$

LCM is the first common multiple (“least”) in both sets. **LCM = 16**

*Note: There are other common multiples, 32 and 48, but 16 is the LEAST common multiple.

Example 2: Find the LCM of 8 and 6.

List multiples of 8	$8 = \{8, 16, 24, 32 \dots\}$
List multiples of 6	$6 = \{6, 12, 18, 24 \dots\}$

LCM is the first common multiple in both sets. **LCM = 24**

LCM (Prime Factorization)

To find the **Least Common Multiple (LCM)** of two numbers, you can apply prime factorization by finding all the prime factors in each prime factorization, and then multiplying the *highest occurrence of each different* factor.

Example 3: Find the LCM of 18 and 24.

The steps for finding the prime factorization of 18 is:

$$\begin{aligned}18 &= 2 \times 9 \\18 &= 2 \times 3 \times 3 \\18 &= 2 \times 3^2\end{aligned}$$

The steps for finding the prime factorization of 24 is:

$$\begin{aligned}24 &= 2 \times 12 \\24 &= 2 \times 2 \times 6 \\24 &= 2 \times 2 \times 2 \times 3 \\24 &= 2^3 \times 3\end{aligned}$$

The LCM is the product of the highest occurrence of each prime factor. When determining the LCM, you multiply each number that appears in each prime factorization the most times it occurs in any one factorization. (Example: If both prime factorizations have one 2, you include one 2. If one prime factorization has one 3 and the other has three 3's, you include three 3's.)

When you express the prime factorization in powers, and the factors occur in each prime factorization, just use the factor with the highest power.

$$18 \text{ is } 2 \times 3^2$$

$$24 \text{ is } 2^3 \times 3$$

The factor 2 occurs once in 18, but three times in 24, so use 2^3 in the LCM.

The factor 3 occurs twice in 18 and once in 24, so use 3^2 in the LCM.

Therefore, the LCM is $2^3 \times 3^2 = 72$.

Addition of Fractions

Adding Fractions with Like Denominators

Example 1: Find $4/9 + 2/9$. Simplify, if necessary.

Add the numerators* of the fractions and reduce.

$$\begin{array}{r} 4 \\ \frac{4}{9} \\ + \frac{2}{9} \\ \hline \frac{6}{9} = \frac{2}{3} \end{array}$$

Simplify.

$$\frac{6}{9} \div \frac{3}{3} = \frac{2}{3}$$

$$4/9 + 2/9 = 2/3$$

*Note: A numerator is the top part of a fraction. The bottom part is called the denominator

Example 2: Find $2 5/8 + 9 7/8$. Simplify, if necessary.

Add the whole numbers, and then add the numerators of the fractions.

$$\begin{array}{r} 2 \frac{5}{8} \\ + 9 \frac{7}{8} \\ \hline 11 \frac{12}{8} = 12 \frac{1}{2} \end{array}$$

$$\begin{aligned} & 11 + \frac{12}{8} \\ &= 11 + \frac{8}{8} + \frac{4}{8} \\ &= 11 + 1 + \frac{4}{8} \\ &= 12 + \frac{1}{2} \\ &= 12 \frac{1}{2} \end{aligned}$$

You may also divide an improper fraction's numerator by the denominator to find an equivalent mixed fraction.

$$\begin{array}{r} 1 \\ 8 \overline{)12} \\ \underline{8} \\ 4 \end{array} = 1 \frac{4}{8} = 1 \frac{1}{2}$$

$$11 + \frac{12}{8} = 11 + 1 \frac{1}{2} = 12 \frac{1}{2}$$

$$2 5/8 + 9 7/8 = 12 1/2$$

Example 3: Find $\frac{3}{4} + 8$. Simplify, if necessary.

$$\begin{array}{r} \frac{3}{4} \\ + 8 \\ \hline 8\frac{3}{4} \end{array}$$

Just bring down the 8 since $8 + 0 = 8$ and bring down the $\frac{3}{4}$ since $\frac{3}{4} + 0 = \frac{3}{4}$.

$$\frac{3}{4} + 8 = 8\frac{3}{4}$$

Example 4: Find $5\frac{1}{6} + 3\frac{5}{6}$. Simplify, if necessary.

Add the whole numbers, and then add the numerators of the fractions.

$$\begin{array}{r} 5\frac{1}{6} \\ + 3\frac{5}{6} \\ \hline 8\frac{6}{6} = 9 \end{array}$$

Simplify.

$$8\frac{6}{6} = 8 + \frac{6}{6} = 8 + 1 = 9$$

$$5\frac{1}{6} + 3\frac{5}{6} = 9$$

Adding Fractions with Unlike Denominators

Example 4: Find $7/8 + 5/16$. Simplify, if necessary.

First find the LCD of 8 and 16.

List multiples of 16. $16 = \{16, 32 \dots\}$

List multiples of 8. $8 = \{8, 16, 24 \dots\}$

LCD = 16

$$\begin{array}{r} \frac{7}{8} = \frac{14}{16} \\ + \frac{5}{16} = \frac{5}{16} \\ \hline \frac{19}{16} = 1\frac{3}{16} \end{array}$$

$\frac{7}{8} \times \frac{2}{2} = \frac{14}{16}$ or say 8 divides into 16, 2 times, $7 \times 2 = 14$.

$$\frac{19}{16} = \frac{16}{16} + \frac{3}{16} \text{ or } 16 \overline{)19} \begin{array}{r} 1 \\ \underline{16} \\ 3 \end{array} = 1\frac{3}{16}$$

$7/8 + 5/16 = 1\ 3/16$

Example 5: Find $3 \frac{7}{9} + 2 \frac{5}{6}$. Simplify, if necessary.

First find the LCD of 9 and 6.

List multiples of 9. $9 = \{9, 18, 27 \dots\}$

List multiples of 6. $6 = \{6, 12, 18, 24 \dots\}$

LCD = 18

$$\begin{array}{r} 3 \frac{7}{9} = 3 \frac{14}{18} \\ + 2 \frac{5}{6} = 2 \frac{15}{18} \\ \hline 5 \frac{29}{18} = 6 \frac{11}{18} \end{array}$$

$\frac{7}{9} \times \frac{2}{2} = \frac{14}{18}$ or say 9 divides into 18, 2 times, $7 \times 2 = 14$.

$\frac{5}{6} \times \frac{3}{3} = \frac{15}{18}$ or say 6 divides into 18, 3 times, $5 \times 3 = 15$.

$$5 \frac{29}{18} = 5 + \frac{18}{18} + \frac{11}{18} = 5 + 1 + \frac{11}{18} = 6 \frac{11}{18}$$

or

$$\frac{29}{18} = 1 \frac{11}{18}$$
$$5 \frac{29}{18} = 5 + 1 \frac{11}{18} = 6 \frac{11}{18}$$

$3 \frac{7}{9} + 2 \frac{5}{6} = 6 \frac{11}{18}$

Subtraction of Fractions

Subtracting Fractions with Like and Unlike Denominators

Example 1: Find $11/12 - 5/12$. Simplify, if necessary.

Subtract the numerators* of the fractions and reduce.

$$\begin{array}{r} 11 \\ - 5 \\ \hline 6 \\ \hline \end{array} \frac{1}{12} = \frac{1}{2}$$

$$\frac{6}{12} \div \frac{6}{6} = \frac{1}{2}$$

*Note: A numerator is the top part of a fraction. The bottom part is called the denominator

$$11/12 - 5/12 = 1/2$$

Example 2: Find $7/8 - 5/6$. Simplify, if necessary.

First find the LCD of 8 and 6.

List multiples of 8. $8 = \{8, 16, 24, 32 \dots\}$

List multiples of 6. $6 = \{6, 12, 18, 24, 30 \dots\}$

LCD = 24

$$\frac{7}{8} = \frac{21}{24}$$

$$\frac{7}{8} \times \frac{3}{3} = \frac{21}{24} \text{ or say 8 divides into 24, 3 times, } 7 \times 3 = 21.$$

$$- \frac{5}{6} = \frac{20}{24}$$

$$\frac{5}{6} \times \frac{4}{4} = \frac{20}{24} \text{ or say 6 divides into 24, 4 times, } 5 \times 4 = 20.$$

$$\begin{array}{r} 21 \\ - 20 \\ \hline 1 \\ \hline \end{array} \frac{1}{24}$$

$$7/8 - 5/6 = 1/24$$

Example 3: Find $8 \frac{14}{15} - 2 \frac{3}{5}$. Simplify, if necessary.

First find the LCD of 15 and 5.

List multiples of 15. $15 = \{15, 30, 45 \dots\}$

List multiples of 5. $5 = \{5, 10, 15, 20 \dots\}$

LCD = 15

$$8 \frac{14}{15} = 8 \frac{14}{15}$$

$$- 2 \frac{3}{5} = 2 \frac{9}{15}$$

$$\hline 6 \frac{5}{15} = 6 \frac{1}{3}$$

$$\frac{3}{5} \times \frac{3}{3} = \frac{9}{15}$$

or

say 5 divides into 15,
3 times, $3 \times 3 = 9$.

$$\frac{5}{15} \div \frac{5}{5} = \frac{1}{3}$$

$$8 \frac{14}{15} - 2 \frac{3}{5} = 6 \frac{1}{3}$$

Subtracting Fractions with Borrowing

Example 4: Find $7 - 2\frac{4}{13}$. Simplify, if necessary.

$$\begin{array}{r} 7 = 6\frac{13}{13} \\ - 2\frac{4}{13} \\ \hline 4\frac{9}{13} \end{array}$$

Borrow

$$7 = 6 + 1 =$$

$$6 + \frac{13}{13} = 6\frac{13}{13}$$

$$7 - 2\frac{4}{13} = 4\frac{9}{13}$$

Example 5: Find $9\frac{3}{8} - 4\frac{7}{8}$. Simplify, if necessary.

$$\begin{array}{r} 9\frac{3}{8} = 8\frac{11}{8} \\ - 4\frac{7}{8} = 4\frac{7}{8} \\ \hline 4\frac{4}{8} = 4\frac{1}{2} \end{array}$$

Borrow

$$9\frac{3}{8} = 8 + 1 + \frac{3}{8} = 8 + \frac{8}{8} + \frac{3}{8} = 8\frac{11}{8}$$

A shortcut to find the numerator, 11, is to add the 3 + 8, in $\frac{3}{8}$ and write it over 8 to get $\frac{11}{8}$.

$$9\frac{3}{8} - 4\frac{7}{8} = 4\frac{1}{2}$$

Example 6: Find $5 \frac{5}{12} - 2 \frac{11}{18}$. Simplify, if necessary.

First find the LCD of 12 and 18.

List multiples of 18. $18 = \{18, 36, 54 \dots\}$

List multiples of 12. $12 = \{12, 24, 36, 48 \dots\}$

LCD = 36

$$\frac{5}{12} \times \frac{3}{3} = \frac{15}{36} \text{ or say 12 divides into 36, 3 times, } 5 \times 3 = 15.$$

$$\begin{array}{r} 5 \frac{5}{12} = 5 \frac{15}{36} = 4 \frac{51}{36} \\ - 2 \frac{11}{18} = 2 \frac{22}{36} = 2 \frac{22}{36} \\ \hline 2 \frac{29}{36} \end{array}$$

$$\frac{11}{18} \times \frac{2}{2} = \frac{22}{36} \text{ or say 18 divides into 36, 2 times, } 11 \times 2 = 22.$$

Borrow

$$5 \frac{15}{36} = 4 + 1 + \frac{15}{36} = 4 + \frac{36}{36} + \frac{15}{36} = 4 \frac{51}{36}$$

A shortcut to find the numerator, 51, is to add the $15 + 36$, in $\frac{15}{36}$ and write it over 36 to get $\frac{51}{36}$.

$$5 \frac{5}{12} - 2 \frac{11}{18} = 2 \frac{29}{36}$$

Mixed Fractions and Improper Fractions

Changing a Mixed Fraction to an Improper Fraction

Example 1: Express $1\frac{1}{11}$ as an improper fraction.

$$1\frac{1}{11} = \frac{11}{11} + \frac{1}{11} = \frac{12}{11} \quad \text{or} \quad \frac{11 \times 1 + 1}{11} = \frac{12}{11}$$

Example 2: Express $2\frac{4}{9}$ as an improper fraction.

$$2\frac{4}{9} = \frac{18}{9} + \frac{4}{9} = \frac{22}{9} \quad \text{or} \quad \frac{9 \times 2 + 4}{9} = \frac{22}{9}$$

*To summarize, multiply the denominator by the whole number, and then add the numerator. Put that number over the denominator.

Changing an Improper Fraction to a Mixed Fraction

Example 3: Express $\frac{13}{10}$ as a mixed fraction.

Method 1

Express the fraction as two fractions where the first one is equal to the largest whole number within the fraction and the second fraction is the remaining fractional part.

In this problem, the whole number is 1 ($\frac{10}{10}$) and the remaining part is $\frac{3}{10}$.

$$\frac{13}{10} = \frac{10}{10} + \frac{3}{10} = 1\frac{3}{10}$$

Method 2

Divide the numerator by the denominator and express the remainder as a fraction.

$$\frac{13}{10} = \left. \begin{array}{r} 10 \overline{)13} \\ \underline{10} \\ 3 \end{array} \right\} = 1\frac{3}{10}$$

Thirteen-tenths equals one and three tenths.

Example 4: Express $\frac{15}{10}$ as a simplified mixed fraction.

$$\frac{15}{10} = \left. \begin{array}{r} 10 \overline{)15} \\ \underline{10} \\ 5 \end{array} \right\} = 1\frac{5}{10} \text{ which reduces to } 1\frac{1}{2}$$

Fifteen tenths equals one and one-half.

Multiplication of Fractions

Multiplying Fractions and Canceling

To multiply fractions, just multiply the numerators and multiply the denominations.

$$\text{Multiply: } \frac{2}{3} \times \frac{8}{9}$$

$$\frac{2}{3} \times \frac{8}{9} = \frac{2 \times 8}{3 \times 9} = \frac{16}{27}$$

Multiply the numerators.
Multiply the denominators.

$$\mathbf{\frac{2}{3} \times \frac{8}{9} = \frac{16}{27}}$$

Multiplication of fractions may be made easier by using canceling, and then multiplying the numerators and denominators. Let's take a look at how to use canceling to make our work easier.

Canceling: Look for a numerator and a denominator that will simplify to lower terms.

Multiply: $\frac{3}{4} \times \frac{8}{11}$

$$\frac{3}{\cancel{4}^1} \times \frac{\cancel{8}^2}{11}$$

$$\frac{3 \times 2}{1 \times 11} = \frac{6}{11}$$

Cancel the 4 and 8 by dividing 4 into 4 to get 1 and dividing 4 into 8 to get 2.

$$\mathbf{\frac{3}{4} \times \frac{8}{11} = \frac{6}{11}}$$

In canceling, one number must be in the numerator and the other number must be in the denominator.

Multiply: $\frac{2}{3}$ of 9

$$\frac{2}{\cancel{3}^1} \times \frac{\cancel{9}^3}{1}$$

$$\frac{2 \times 3}{1 \times 1} = \frac{6}{1} = 6$$

First make the 9 a fraction by placing it over 1. Notice that “of” means to multiply.

Cancel the 3 and 9 by dividing 3 into 3 to get 1 and dividing 3 into 9 to get 3.

$$\mathbf{\frac{2}{3} \text{ of } 9 = 6}$$

Multiplying Mixed Fractions

Change the mixed numbers into improper fractions.

$$1\frac{1}{11} = \frac{11 \times 1 + 1}{11} = \frac{12}{11}$$
$$2\frac{4}{9} = \frac{9 \times 2 + 4}{9} = \frac{22}{9}$$

Multiply: $1\frac{1}{11} \times 2\frac{4}{9}$

$$\frac{12}{11} \times \frac{22}{9}$$
$$4\frac{12}{11} \times \frac{22}{9}$$

Cancel the 12 and 9 by 3 to get 4 and 3.
Cancel the 11 and 22 by 11 to get 1 and 2.

$$\frac{4 \times 2}{1 \times 3} = \frac{8}{3} = 2\frac{2}{3}$$

$$1\frac{1}{11} \times 2\frac{4}{9} = 2\frac{2}{3}$$

Multiplying Multiple Fractions

When multiplying multiple fractions, cancel any numerator with any denominator, and then multiply.

Multiply: $\frac{9}{20} \times \frac{5}{8} \times \frac{2}{3}$

$$\frac{\cancel{9}^3}{20} \times \frac{5}{8} \times \frac{2}{\cancel{3}_1} \quad \left[\text{First we'll cancel the 9 and 3.} \right]$$

$$\frac{\cancel{9}^3}{\cancel{20}_4} \times \frac{\cancel{5}^1}{8} \times \frac{2}{\cancel{3}_1} \quad \left[\text{Now we'll cancel the 5 and 20.} \right]$$

$$\frac{\cancel{9}^3}{\cancel{20}_4} \times \frac{\cancel{5}^1}{8} \times \frac{\cancel{2}^1}{\cancel{3}_1} \quad \left[\text{Last we'll cancel the 2 and 8.} \right]$$

$$\frac{\cancel{9}^3}{\cancel{20}_4} \times \frac{\cancel{5}^1}{8} \times \frac{\cancel{2}^1}{\cancel{3}_1} = \frac{3}{16} \quad \left[\text{Multiply the cancelled numerators and denominators.} \right]$$

$$\mathbf{\frac{9}{20} \times \frac{5}{8} \times \frac{2}{3} = \frac{3}{16}}$$

Reciprocals

The **reciprocal** (inverse) of a fraction is the fraction that is multiplied by the original fraction to get 1.

For example, the reciprocal of $\frac{1}{2}$ is $\frac{2}{1}$ because $\frac{1}{2} \times \frac{2}{1} = 1$. Just flip the fraction to find the reciprocal.

The **reciprocal** of $\frac{1}{2}$ is $\frac{2}{1}$.

To find the **reciprocal** of a mixed number, first write it as an improper fraction and then write the reciprocal.

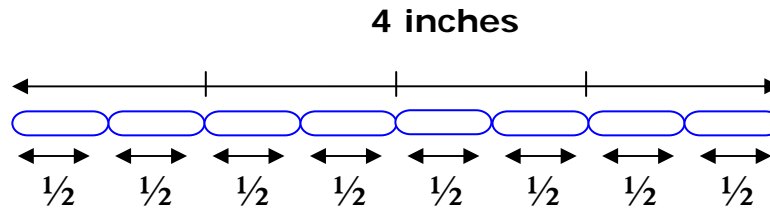
For example, to find the reciprocal of $7\frac{2}{3}$, first write it as $\frac{23}{3}$, and then flip the improper fraction.

The **reciprocal** of $7\frac{2}{3}$ is $\frac{3}{23}$.

Division of Fractions

Introduction to Division of Fractions

A child's bracelet is 4 inches long. Each of the links is $\frac{1}{2}$ inch long. How many links are in the chain?



$$4 \div \frac{1}{2} = 8$$

It takes eight $\frac{1}{2}$ -inch links to make up a bracelet 4 inches long.

To divide fractions, multiply by the inverse. The inverse is sometimes referred to as the reciprocal. Just flip the fraction to find the reciprocal. The reciprocal of $\frac{1}{2}$ is $\frac{2}{1}$.

Remember, the inverse (reciprocal) of a fraction is the fraction that is multiplied by the original fraction to get 1. For example, the inverse of $\frac{1}{2}$ is $\frac{2}{1}$ because $\frac{1}{2} \times \frac{2}{1} = 1$.

Here are the steps to do this problem mathematically.

Steps

$$4 \div \frac{1}{2} =$$

$$4 \times \frac{2}{1} =$$

$$\frac{4}{1} \times \frac{2}{1} =$$

$$\frac{4}{1} \times \frac{2}{1} = \frac{8}{1} = 8$$

$$4 \div \frac{1}{2} = 8$$

Reasons

Given

Multiply by the reciprocal.

Write 4 as a fraction by putting it over 1.

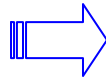
Multiply the numerators and denominators, and then simplify.

Dividing Fractions

To divide fractions, multiply the first fraction by the reciprocal (inverse) of the second fraction. Study the four examples shown below.

Divide: $\frac{3}{4} \div \frac{4}{5}$

Multiply by the **reciprocal** of the second number.



$$\frac{3}{4} \div \frac{4}{5} = \frac{3}{4} \times \frac{5}{4} = \frac{3 \times 5}{4 \times 4} = \frac{15}{16}$$

$$\frac{3}{4} \div \frac{4}{5} = \frac{15}{16}$$

Divide $\frac{2}{3} \div \frac{8}{9}$

$$\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{3}{\cancel{9}}}{\underset{4}{\cancel{8}}}$$

$$\frac{1 \times 3}{1 \times 4} = \frac{3}{4}$$

Divide $12 \div \frac{3}{5}$

$$\frac{\overset{4}{\cancel{12}}}{\underset{1}{\cancel{1}}} \times \frac{\overset{5}{\cancel{5}}}{\underset{3}{\cancel{1}}}$$

$$\frac{4 \times 5}{1 \times 1} = \frac{20}{1} = 20$$

12 is a whole number and can be written in fraction form as $\frac{12}{1}$.

Reciprocal of second fraction

Divide $2\frac{2}{7} \div 6\frac{2}{3}$

$$\frac{\overset{4}{\cancel{16}}}{\underset{7}{\cancel{20}}} \times \frac{\overset{3}{\cancel{3}}}{\underset{5}{\cancel{5}}}$$

$$\frac{4 \times 3}{7 \times 5} = \frac{12}{35}$$

$$\left(2\frac{2}{7} = \frac{7 \times 2 + 2}{7} = \frac{16}{7} \right)$$

$$\left(6\frac{2}{3} = \frac{3 \times 6 + 2}{3} = \frac{20}{3} \right)$$

$$\frac{2}{3} \div \frac{8}{9} = \frac{3}{4}$$

$$2\frac{2}{7} \div 6\frac{2}{3} = \frac{12}{35}$$

$$12 \div \frac{3}{5} = 20$$