FACTORS, POWERS, AND SQUARE ROOTS

In this unit, you will begin with a review of factors, prime numbers, and prime factorization. You will then work through perfect squares and square roots. The perfect squares should be numbers that you commit to memory, as this memorization will lead you to quick answers later. You may wish to create flashcards with the answers on the back to assist in your memorization. You can keep the cards and practice anytime. You will also apply your knowledge of square roots and estimation to approximate non-perfect square roots. You will also multiply and divide expressions involving exponents with a common base, take a power to a power, and take positive rational numbers to whole-number powers.

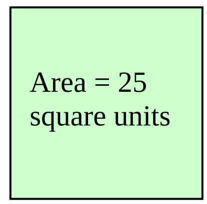
Factors and Powers

Squares

Five to the second power, 5^2, equals 5×5 and can be read "5-squared".

Example 1: Find the area of a square whose side is five (5) units long.

$$A = l \times w$$
$$A = 5 \times 5$$
$$A = 5^2 \text{ or } 25 \text{ square units}$$



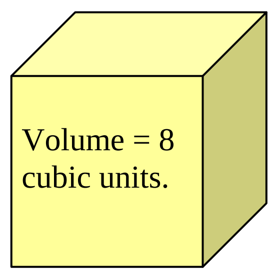
Side = 5 units

Cubes

Two to the third power, 2^3, equals $2 \times 2 \times 2$ and can be read "2-cubed".

Example 2: Find the volume of a cube whose side is two (2) units long.

$$V = l \times w \times h$$
$$V = 2 \times 2 \times 2$$
$$V = 2^3 \text{ or } 8 \text{ cubic units}$$



Side = 2 units in length

More Powers

Example 3: Evaluate 3^6.

Three to the sixth power, 3^6, equals $3 \times 3 \times 3 \times 3 \times 3 \times 3$ or 729.

$3^6 = 729$

Prime and Composite Numbers

A prime number is a number that has only **two factors, 1 and itself**.

> **One (1) is not** considered **a prime** since it has only one factor, 1, $(1 \times 1 = 1)$. Prime numbers have **two factors**.

Prime Numbers under 20 → {2, 3, 5, 7, 11, 13, 17, 19…}

Let's examine the first ten counting numbers and determine which ones are prime and why or why not.

Number	Is the number prime?	Reason
1	No	only one factor (1) $1 \times 1 = 1$
2	Yes	two factors (1 & 2) $1 \times 2 = 2$
3	Yes	two factors (1 & 3) $1 \times 3 = 3$
4	No	three factors (1, 2, & 4) $1 \times 4 = 4 \quad 2 \times 2 = 4$
5	Yes	two factors (1 & 5) $1 \times 5 = 5$
6	No	four factors (1, 2, 3, & 6) $1 \times 6 = 6 \quad 2 \times 3 = 6$
7	Yes	two factors (1 & 7) $1 \times 7 = 7$
8	No	four factors (1, 2, 4, & 8) $1 \times 8 = 8 \quad 2 \times 4 = 8$
9	No	three factors (1, 3, & 9) $1 \times 9 = 9 \quad 3 \times 3 = 9$
10	No	four factors (1, 2, 5, & 10) $1 \times 10 = 10 \quad 2 \times 5 = 10$

> **4, 6, 8, 9, and 10** are called **composite** numbers because they have more than two factors.

Prime Factorization

To find the prime factors of a composite number, write the number as a product of prime numbers.

Prime Numbers under 20 → {2, 3, 5, 7, 11, 13, 17, 19…}

A **prime** number has ***exactly*** two factors which are 1 and the number itself.

5 is a prime number. → $5 = 1 \times 5$

The factors of 5 are 1 and 5.

A **composite** number has ***more than*** two factors.

16 is a composite number → $16 = 1 \times 16$
$16 = 2 \times 8$
$16 = 4 \times 4$

The factors of 16 are 1, 2, 4, 8, and 16.

Note: Zero (0) and one (1) are neither prime nor composite.

To find the prime factorization of a number, express the number as a "product of prime numbers".

Example 1: Find the prime factorization of 40.

The steps for finding the prime factorization of 40:

$$40 = 2 \times 20$$
$$40 = 2 \times 2 \times 10 \qquad (20 = 2 \times 10)$$
$$40 = 2 \times 2 \times 2 \times 5 \qquad (10 = 2 \times 5)$$

When all of the factors are prime numbers, the prime factorization is complete.

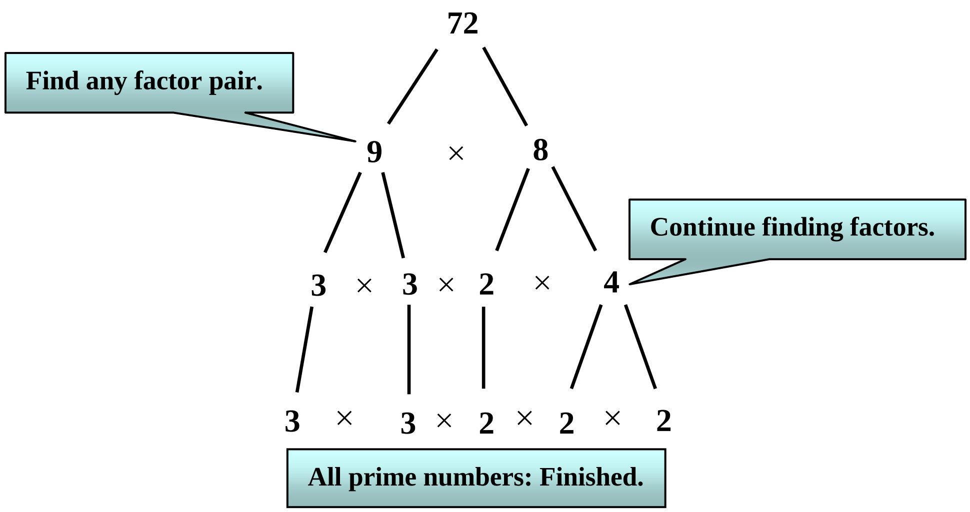
The prime factorization of 40 is $2 \times 2 \times 2 \times 5$ which can be expressed as $2^3 \times 5$.

Factor Tree

Another method to find the prime factorization for a number is creating a **factor tree.**

Example 2: Write the prime factorization for 72.

Solution: Create a factor tree.

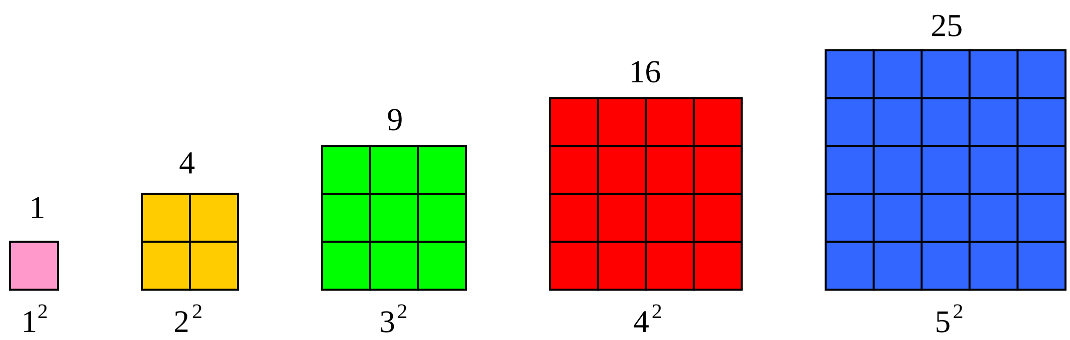


The prime factorization of $72 = 3 \times 3 \times 2 \times 2 \times 2$ or $3^2 \times 2^3$.

Perfect Squares and Square Roots

Perfect Squares

Perfect squares are numbers that are squares of integers.



Positive Integers	Perfect Square
1^2 (1-squared)	1
2^2 (2-squared)	4
3^2 (3-squared)	9
4^2 (4-squared)	16
5^2 (5-squared)	25

Negative Integers	Perfect Square
$(-1)^2 = (-1 \times -1)$	1
$(-2)^2 = (-2 \times -2)$	4
$(-3)^2 = (-3 \times -3)$	9
$(-4)^2 = (-4 \times -4)$	16
$(-5)^2 = (-5 \times -5)$	25

Square Roots

Square roots are the numbers that, when squared, make perfect squares.

Finding a square root can be as easy as guessing the solution to the following algebraic equation:

$$x^2 = 49$$

If we understand the meaning of the exponent "2", we know that a solution for *x* is 7.

$$7(7) = 49$$

We must also remember that if we include negative values, there is another solution, –7.

–7(–7) = 49

This guess and check system for finding values of this type is fine and will work for numbers like 49, 64, 81, 144, or even the value 1.

Actually, the set of values $\{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, ...\}$ all have "guessable" solutions.

This set of values is called "perfect squares" because the numbers that are used as double factors are **integral** values… "perfect".

The **square root of 25 is 5 or –5.**

To indicate which root is desired, we will use the following notation.

$\sqrt{25} = 5$ and $-\sqrt{25} = -5$

Examples:

a. $\sqrt{81} = 9$

b. $-\sqrt{81} = -9$

c. $\sqrt{0.49} = 0.7$

d. $\sqrt{\frac{25}{64}} = \frac{\sqrt{25}}{\sqrt{64}} = \frac{5}{8}$

Try These!

On paper, list the answers to the following problems. Look below for the correct answers.

Perfect Squares

1. List the perfect squares of the counting numbers 13 through 20.

2. What is the square of 30?

3. What is the square of 0.09?

Square Roots

4. What is $-\sqrt{144}$ **?**

5. What is $\sqrt{1.21}$?

6. What is $\sqrt{\frac{9}{100}}$?

Solutions

1. 169, 196, 225, 256, 289, 324, 361, 400

2. 900

3. 0.0081

4. –12

5. 1.1

6. $\frac{3}{10}$

Approximating Square Roots of Non-Perfect Squares

Consider solving this equation: $x^2 = 55$

Keep in mind that there is no integer that will give us a solution. However, the value for x will be between 7 and 8, and that number will be a rational number.

We will guess and check until we get an approximate answer…

Try 7.5, $\rightarrow$ $7.5^2 = 56.25$ Close, but greater than 55.

But can we get closer…

Try 7.4 $\rightarrow$ $7.4^2 = 54.76$ Close, but less than 55.

We will try to get just a little closer…

Try 7.45 $\rightarrow$ $7.45^2 = 55.5025$ Closer, but greater than 55.

Try a little lower…

Try 7.43 $\rightarrow$ $7.43^2 = 55.2049$ Getting closer, but still greater than 55.

Try a little lower…

Try 7.41 $\rightarrow$ $7.41^2 = 54.9081$ Getting closer, but lower than 55.

Try a little higher…

Try 7.42 $\rightarrow$ $7.42^2 = 55.0564$ Close enough!

Solution: $\sqrt{55} \approx 7.42$

In the previous estimation, we first guessed tenth values of numbers that were close to the answer, and then guessed hundredths values. We kept guessing closer until we were able to be accurate with the best hundredth value. Determining which place value for estimation will depend on the problem at hand.

Example: Find the square root of 115 to the nearest tenth: $\sqrt{115}$

Guess between 10 and 11, and since $10^2 = 100$ and $11^2 = 121$, guess closer to 11.

Try 10.8 $\rightarrow$ $10.8^2 = 116.64$ Close, but greater than 115.

Try 10.6 $\rightarrow$ $10.6^2 = 112.36$ Close, but less than 115.

Try 10.7 $\rightarrow$ $10.7^2 = 114.49$ Closest answer to nearest tenth.

Solution: $\sqrt{115} \approx 10.7$

Products of Powers

In the expression, 3^{100}, three is a product of itself 100 times. The number 3 represents the **base** and the 100 represents the **exponent**. The exponent tells how many times the base is used as a factor.

Example 1: $5^4 = 5 \cdot 5 \cdot 5 \cdot 5$

$= 25 \cdot 5 \cdot 5$

$= 125 \cdot 5$

$= 625$

Example 2: $3^3 = 3 \cdot 3 \cdot 3$

$= 9 \cdot 3$

$= 27$

Product of Powers

For all real numbers *a* and all integers *m* and *n*,

$$a^m \cdot a^n = a^{m+n}$$

The property above states that if you are multiplying like bases ("*a*" in the case above), then to simplify you will add the exponents.

Example 3: $2^3 \cdot 2^4 = 2^{3+4} = 2^7$

To prove this example, we will expand each term, count the bases, and rewrite the term using one base and one exponent.

$$\underbrace{2 \cdot 2 \cdot 2}_{2^3} \qquad \underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{2^4}$$

-count the number of bases and use this as your exponent

2^7 *This is the same answer we found when we used the product of powers rule.

Power to a Power

Sometimes in algebra it is necessary to raise a power to a power. In this case you would use the exponent rule called "Power of a Power" Rule.

Power of a Power

For all real numbers *a* and all integers *m* and *n*,

$$(a^m)^n = a^{m \times n}$$

As you can see by the rule stated above, if you have a power raised to another power, then you multiply the exponents.

Example 1: Simplify and evaluate $(3^2)^5$.

$$(3^2)^5 = 3^{2 \times 5} = 3^{10} = 59,049$$

Check: $(3 \cdot 3)^5 = (3 \cdot 3) \cdot (3 \cdot 3) \cdot (3 \cdot 3) \cdot (3 \cdot 3) \cdot (3 \cdot 3) = 59,049$

Example 2: Simplify $(1^3)^4$.

$$(1^3)^4 = 1^{3 \times 4} = 1^{12} = 1$$

Check: $(1 \cdot 1 \cdot 1)^4 = (1 \cdot 1 \cdot 1) \cdot (1 \cdot 1 \cdot 1) \cdot (1 \cdot 1 \cdot 1) \cdot (1 \cdot 1 \cdot 1) = 1$

Quotients of Powers

When simplifying quotients, you can do so by first expressing the powers in terms of their factors. Take a look at the example below and see if you can derive a rule on how to simplify quotients.

Example 1: $\frac{x^6}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = x^2$

Did you figure out a rule for dividing monomials? If you said that you can subtract the exponents, you are correct. Study the property below.

Quotient-of-Powers Property

For all nonzero real numbers *a* and all integers *m* and *n*,

$$\frac{a^m}{a^n} = a^{m-n}$$

Example 2: Use the property stated above to simplify the following quotients and evaluate.

a.) $\frac{2^7}{2^4} = 2^{7-4} = 2^3 = 8$

b.) $\frac{5^{100}}{5^{96}} = 5^{100-96} = 5^4 = 625$

Zero as an Exponent

Zero as an Exponent

For any nonzero number *a*,

$$a^0 = 1$$

The property illustrated above shows us that any number to the zero power is equal to one.

Example 1: $4^0 = 1$

Example 2: $7^{-3} \cdot 7^3 = 7^{-3+3} = 7^0 = 1$

Example 3: $\frac{8^4}{8^4} = 8^{4-4} = 8^0 = 1$

Example 4: $(9^3)^0 = 9^{3 \times 0} = 9^0 = 1$

Negative Exponents

Negative exponents are used to represent very small numbers. Study the property defining negative exponents below.

Negative Exponents

For all nonzero numbers *a* and all integers *n*,

$$a^{-n} = \frac{1}{a^n}$$

Example 1: Simplify using positive exponents.

$$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$$

*All properties from this unit and any previous units will also apply to negative exponents.

Example 2: $3^{-3} \cdot 3^2$ -like bases, add the exponents

$= 3^{-3+2}$

$= 3^{-1}$ -negative exponent

$= \frac{1}{3}$ -negative exponent property

Example 3: $\frac{12^{2}}{12^{-1}}$ -like bases, subtract the exponents

$=12^{2-(-1)}$

$=12^{3}$

$= 1728$

Example 4: $\frac{4^{5}}{4^{7}}$

$=4^{5-7}$

$=4^{-2}$ -negative exponent property

$=\frac{1}{4^{2}}=\frac{1}{16}$

Exponent Property Summary

Exponent Property Summary		
Product of Powers $a^m \cdot a^n = a^{m+n}$	**Quotient of Powers** $\frac{a^m}{a^n} = a^{m-n}$	**Power of a Power** $(a^m)^n = a^{m \times n}$
Zero Exponent $a^0 = 1$	**Negative Exponents** $a^{-m} = \frac{1}{a^m}$ or $\frac{1}{a^{-m}} = a^m$	