COUNTING AND PROBABILITY

This unit is about various counting techniques to calculate probability and the number of outcomes. The Fundamental Counting Principal is the underlying principle for determining the number of possible outcomes. There are two types of counting arrangements: permutations and combinations. A permutation is an arrangement of objects in which the order of the arrangement is important to the number of outcomes. A combination is an arrangement of objects where order is not taken into account and results in fewer outcomes than permutations. The second part of this unit is about probability. Simple probability and sample spaces is reviewed, and then computing the probability of independent and independent is examined closely.

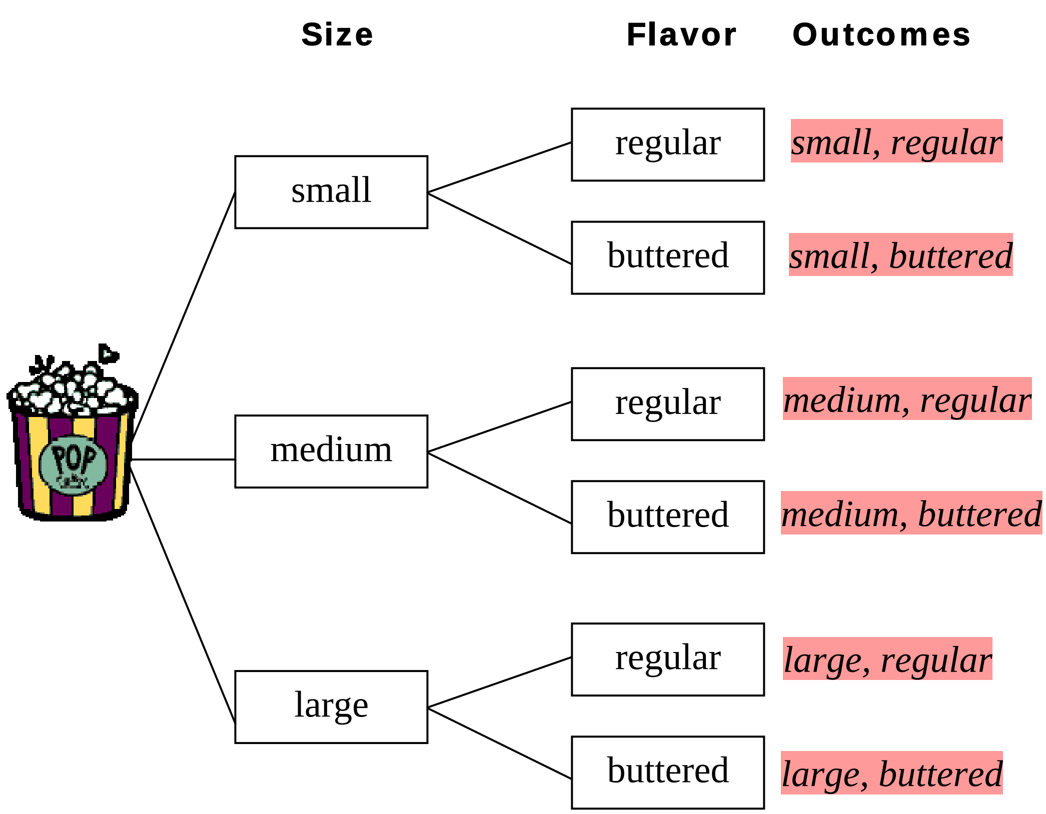
Fundamental Counting Principle

> **Fundamental Counting Principle**
>
> If there are *m* ways that one event can occur and *n* ways that another event can occur, then there are ***m x n*** ways that both events can occur.

Example 1: A movie theater sells popcorn in small, medium, or large containers. Each size is also available in regular or buttered popcorn. How many options for buying popcorn does the movie theatre provide?

First consider the three options for size: small, medium, and large. Then consider that each size of popcorn can be the regular flavor or the buttered flavor.

In the figure below, the diagram first shows the three sizes and then the two possible flavors for each size. The resulting combinations are listed under the "outcomes". Study the diagram to determine the total number of outcomes.



There are ***six*** possible options for buying popcorn at the movie theatre. They are small and regular, small and buttered, medium and regular, medium and buttered, large and regular, and large and buttered.

The Fundamental Theorem can be used to determine the number of outcomes quickly.

$$\text{size} \times \text{flavor} = \text{number of outcomes}$$

$$3 \times 2 = 6$$

Example 2: If the theater in the previous problem adds three new flavors, caramel apple, jelly bean, and bacon cheddar, to the popcorn choices, how many options will the customers have?

Apply the Fundamental Theorem to determine the number of outcomes.

$$\text{size} \times \text{flavor} = \text{number of outcomes}$$

$$3 \times 5 = 15$$

There are a total of 15 options.

Example 3: Emily is choosing a password for access to her Internet account. She decides to use a combination of digits and letters; but she doesn't want to use the digit 0 or the letters A, E, I, O, or U. Each letter or number may be used more than once. The password must be 3 letters followed by 2 digits. How many choices does Emily have?

Consider there are 26 letters in the alphabet, but Emily is choosing not to use five of them (A, E, I O, and U); thus, there are 21 possible letters from which to choose.

Also consider there are 10 single digit numbers (0 through 9), but Emily is choosing not to use one of them (0); thus, there are 9 possible digits from which to choose.

Apply the fundamental counting principle to determine the number of choices Emily will have in determining the password. There are 21 possible letters and 9 possible digits.

first letter		*second letter*		*third letter*		*first digit*		*second digit*
?	×	?	×	?	×	?	×	?
21 choices		21 choices		21 choices		9 choices		9 choices

$21 \times 21 \times 21 \times 9 \times 9 = 750{,}141$

There are 750,141 passwords that Emily could choose!

Factorial Numbers

When considering the arrangement of objects we use permutations or combinations. Before we look at permutations and combinations, we need to understand factorial numbers. Let's take a look.

$n!$ is read "n factorial"

For example, $6!$ means $6 \times 5 \times 4 \times 3 \times 2 \times 1$.

This is the product of all the whole numbers starting with 6 and going down to 1.

$6 \times 5 = 30$

$30 \times 4 = 120$

$120 \times 3 = 360$

$360 \times 2 = 720$

$720 \times 1 = 720$

$6! = 720$

Example: Evaluate $10!$

$10!$ means $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$10! = 3{,}628{,}800$

Permutations

A **permutation** is an arrangement of objects in a specific order. When objects are arranged in a row, the permutation is called a **linear permutation**.

First, let's consider an arrangement of a given number of items and arranging ALL of them in as many ways as possible.

Permutations of *n* Objects

The number of permutations of n objects is given by $n!$
(! Is called factorial and means to multiply all consecutive integers starting with *n*.)

Example 1: On a baseball team, nine players are designated as the starting lineup. Before a game, the coach announces the order in which the nine players will bat. How many different batting orders are possible?

Think: First there are 9 player choices, and then once a player is chosen, there are then 8 players left from which to choose. Once that player is chosen, there are 7 choices left, and so on.

Mathematically, this looks like: $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.

Using factorial notation:

$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 362,880 \text{ possible batting orders.}$$

When choosing from nine players, there are 362,880 possible batting orders.
*To use a scientific calculator, find the factorial key (!) or (n!). Just press the number 9, and then the factorial key.

Now, let's consider an arrangement of a given number of items and arranging a specified number of these items in as many ways as possible.

Permutations of *n* Objects Taken *r* at a Time

The number of permutations of *n* objects taken *r* at a time, denoted by $P(n, r)$ or ${}_nP_r$ is given by:

$$P(n,\ r) = {}_nP_r = \frac{n!}{(n-r)!},\ \text{where } r \leq n$$

Example 2: There are 8 finalists in a spelling bee. Each student is given a number to determine the order in which he or she will compete. How many ways can the spelling order of the first, second, and third positions be assigned?

In this problem, order is important. The students will be arranged in a specific order. We are determining how many arrangements of 8 students can be made in the first, second, and third positions. Thus, this problem is a permutation of 8 items taken 3 at a time.

$${}_nP_r = \frac{n!}{(n-r)!}$$

$${}_8P_3 = \frac{8!}{(8-3)!}$$

$$= \frac{8!}{5!} = \frac{8\cdot 7\cdot 6\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1} = \frac{8\cdot 7\cdot 6}{1}$$

$$= 336$$

There are 336 ways the eight students could be assigned first, second, and third places.

Now, let's revisit arranging a given number of items and arranging ALL of them in as many ways as possible. This time we'll use the formula.

Example 3: How many ways can the letters of the word "random" be arranged?

R A N D O M

Order is important in arranging the letters and we are arranging ALL of them. Thus, this problem is a permutation of 6 items taken 6 at a time.

$$
\begin{aligned}
{}_nP_r &= \frac{n!}{(n-r)!} \\
{}_6P_6 &= \frac{6!}{(6-6)!} \\
&= \frac{6!}{0!} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{1} \\
&= 720
\end{aligned}
$$

*0! is accepted as equal to 1.

*Sometimes, the formula for this special case of *n* items take *n* at a time is written as follows:

$$
\begin{aligned}
{}_nP_n &= n! \\
{}_6P_6 &= 6! = 720
\end{aligned}
$$

There are 720 ways the letters in the word "random" may be arranged.

Example 4: If we look at arranging letters in the word "success", we need to realize that when an s or c is selected, it does not matter which is which. So there are less ways to select the arrangement. How many ways can the letters of the word "success" be arranged?

S U C C E S S

This problem is an example of a permutation with repetition. The formula for this problem is:

$P = \dfrac{n!}{a!b!}$ where "*a*" and "*b*" are repeating letters.

We use all 7 letters but the "s" is repeated 3 times and the "c" is repeated twice.

$$n = 7 \qquad a = 3 \qquad b = 2$$

$$P = \frac{7!}{3!2!}$$

$$P = \frac{7\times 6\times 5\times 4\times 3\times 2\times 1}{3\times 2\times 1\times 2\times 1}$$

$$P = \frac{7\times 6\times 5\times \not{4}^{2}\times \not{3}\times \not{2}\times 1}{\not{3}\times \not{2}\times 1\times \not{2}\times 1}$$

$$P = \frac{7\times\ 6\times\ 5\times\ 2}{1}$$

$$P = 420$$

The letters in the word "success" may be arranged 420 different ways.

Combinations

An arrangement of objects in which order is **NOT** important is called a combination.

Combinations of *n* Objects Taken *r* at a Time

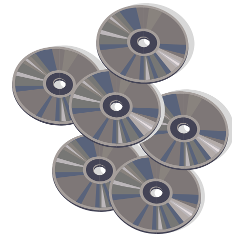
The number of combinations of n objects taken r at a time is given by:

$$C(n,r) = {}_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}, \text{ where } 0 \leq r \leq n$$

$C(n,r)$, ${}_nC_r$ and $\binom{n}{r}$ have the same meaning.

All are read "*n* choose *r*".

Example: Find the number of ways to choose 6 different CD's from a selection of 18 CD's.



We have 18 CD's that can be chosen in any order. So, order does NOT matter in this problem; thus, the arrangement in this problem is a combination.

$n = 18 \qquad r = 6$

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

$${}_{18}C_6 = \frac{18!}{6!(18-6)!}$$

$$= \frac{18!}{6!12!}$$

$$= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= \frac{\cancel{18}^{3} \cdot 17 \cdot \cancel{16}^{4} \cdot \cancel{15}^{\cancel{5}^{1}} \cdot \cancel{14}^{7} \cdot 13 \cdot \cancel{12} \cdot \cancel{11} \cdot \cancel{10} \cdot \cancel{9} \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}{\cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{12} \cdot \cancel{11} \cdot \cancel{10} \cdot \cancel{9} \cdot \cancel{8} \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1}$$

$$= 3 \cdot 17 \cdot 4 \cdot 7 \cdot 13$$

$$= 18{,}564$$

There are 18,564 different ways that 6 CD's can be chose from a selection of 18 CD's.

The solution to this problem can be written in a simpler way.

$$_{18}C_6 = \frac{18!}{6!(18-6)!}$$

$$= \frac{18!}{6!12!}$$

$$= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 12!}$$

$$= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot \cancel{12!}}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{12!}}$$

$$= \frac{\cancel{18}^{3} \cdot 17 \cdot \cancel{16}^{4} \cdot \cancel{15}^{\cancel{5}^{1}} \cdot \cancel{14}^{7} \cdot 13 \cdot \cancel{12!}}{\cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{12!}}$$

$$= 3 \cdot 17 \cdot 4 \cdot 7 \cdot 13$$

$$= 18,564$$

Introduction to Probability

Probability is the likelihood of an event occurring.

Terminology (Note: The tossing of a coin is used for each of the examples.)	
Definition	Example
Trial: a systematic opportunity for an event to occur.	tossing a coin in the air
Experiment: one or more trials.	tossing a coin 6 times
Sample space: the set of all possible outcomes of an event.	H or T
Event: an individual outcome or any specified combination of outcomes.	landing H or landing T

Probability is expressed as a number from 0 to 1. It is written as a fraction, decimal, or percent.

Consider the following comments.

- An impossible event has a probability of 0.

- An event that must occur has a probability of 1.

- The sum of the probabilities of all outcomes in a sample space is 1.

The probability of an event can be assigned in two ways:

1) **experimentally**: approximated by performing trials and recording the ratio of the number of occurrences of the event to the number of trials. (As the number of trials in an experiment increases, the approximation of the experimental probability increases.)

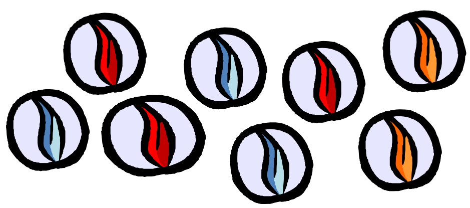
2) **theoretically:** based on the assumption that all outcomes in the sample space occur randomly.

Theoretical Probability

If all outcomes in a sample space are equally likely, then the theoretical probability of event B, denoted P(B), is defined by:

$$P(B) = \frac{\text{Number of outcomes in event B (favorable outcomes)}}{\text{Number of outcomes in the sample space (possible outcomes)}}$$

Example: Find the probability of randomly selecting an orange marble out of a jar containing 3 blue marbles, 3 red marbles, and 2 orange marbles.



$$P(\text{1 orange}) = \frac{\text{favorable}}{\text{possible}} = \frac{\text{2 orange}}{\text{8 possible}} = \frac{2}{8} = \frac{1}{4} \text{ or } 25\%$$

The probability of selecting an orange marble is 1/4 or 25%.

Independent and Dependent Events

Independent Events

Two events are **independent** if the occurrence (or non-occurrence) of one event has no effect on the likelihood of the occurrence of the other event. For example, rolling a die and choosing a marble out of a bag are examples of independent events.

Probability of Independent Events

Events A and B are independent events, if and only if,

$$P(A \text{ and } B) = P(A) \times P(B)$$

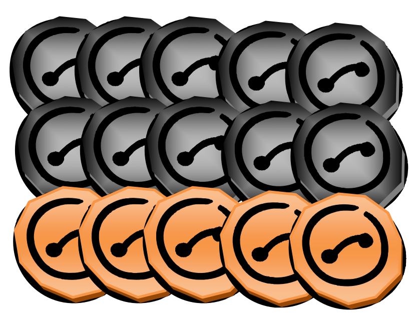
The probability of two independent events can be found by multiplying the probability of the first event by the probability of the second event.

Example 1: Bag A contains 8 red buttons and 4 green buttons. Bag B contains 10 black buttons and 5 orange buttons.

Find the probability of selecting one green button from bag A and one black button from bag B in one draw from each bag.

Bag A

Bag B



Bag A **Bag B**

$$\text{P(Green)} = \frac{4}{8+4} = \frac{4}{12} = \frac{1}{3} \qquad \text{P(Black)} = \frac{10}{10+5} = \frac{10}{15} = \frac{2}{3}$$

The events are independent, so multiply the first event times the second event.

$$\text{P(Green and Black)} = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9}$$

The probability of selecting a green button from bag A and a black button from bag B in one draw from each bag is 2/9 or $.2\overline{2}$ or about 22%.

*The formula for the probability of independent events can be extended to 3 or more events.

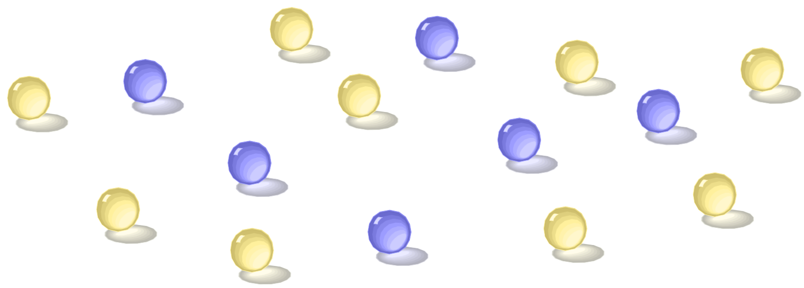
Example 2: What is the probability of rolling a two on a die in 3 rolls of a die?

$P(2) = \frac{1}{6}$ on each roll.

So, $P(2)$ on three rolls is $\frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{216} \approx 0.0046 \approx 0.46\%$

Thus, the probability of obtaining a roll of two in 3 rolls of a die is 1/216 or less than 1 percent, about 0.46%.

Example 3: A bag contains 6 blue and 9 yellow marbles. Find the probability of randomly selecting a blue marble on the first draw and a yellow marble on the second draw when the marble IS replaced after the first draw.

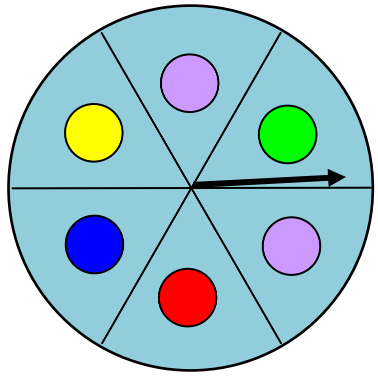


$$\text{P(blue)} = \frac{6}{6+9} = \frac{6}{15} \qquad \text{P(yellow)} = \frac{9}{6+9} = \frac{9}{15}$$

$$\begin{aligned} \text{P(blue and yellow)} &= \frac{6}{15} \times \frac{9}{15} \\ &= \frac{\cancel{6}^{2}}{\cancel{15}_{5}} \times \frac{\cancel{9}^{3}}{\cancel{15}_{5}} \\ &= \frac{6}{25} = 24\% \end{aligned}$$

Thus, the probability of selecting a blue marble on the first draw and a yellow marble on the second draw when the blue marble is replaced is 6/25 or 24%.

Example 4: Emily rolls a number cube and spins the spinner below. What is the probability she will roll a number less than 5 AND land on a purple section of the spinner?



*Because the roll of the number cube does not affect the result of the spinner, the events are **independent.**

a) Four of the numbers on the cube (1, 2, 3, and 4) are less than 5.

P(rolling a number less than 5) $= \frac{4}{6} = \frac{2}{3}$

b) Two of the spaces on the spinner are purple.

P(landing on a purple space) $= \frac{2}{6} = \frac{1}{3}$

c) To find the probability of both happening, multiply the individual probabilities.

$$\frac{2}{3} \times \frac{1}{3} = \frac{2}{9} \approx 22\%$$

The probability of rolling a number on the die that is less than five AND landing on a purple section of the spinner is 2/9 or about 22%.

Dependent Events

A **dependent event** is the probability of a second event happening, depending on the outcome of the first event.

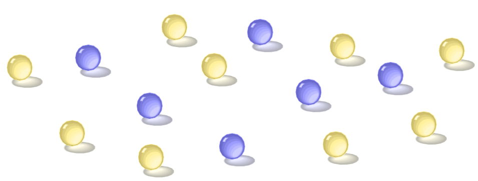
Probability of Dependent Events

If Events A and B are dependent events, then the probability of both events occurring is the product of the probability of A and the probability of B after A occurs.

$$P(\text{A and B}) = P(\text{A}) \times P(\text{B following A})$$

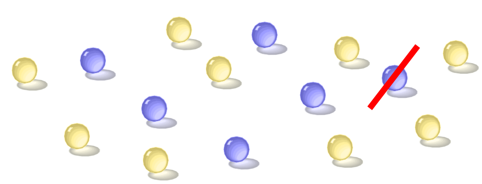
Example: A bag contains 6 blue and 9 yellow marbles. Find the probability of randomly selecting a blue marble on the first draw and a yellow marble on the second draw if the marble is **not replaced** after the first draw.

Event A
P(blue)



$$P(\text{blue}) = \frac{6}{9+6} = \frac{6}{15}$$

Event B
P(yellow following blue)



$$P(\text{yellow following blue}) = \frac{9}{9+5} = \frac{9}{14}$$

$$\begin{aligned} P(\text{blue and yellow}) &= \frac{6}{15} \times \frac{9}{14} \\ &= \frac{\cancel{6}^{3}}{\cancel{15}^{5}} \times \frac{\cancel{9}^{3}}{\cancel{14}^{7}} \\ &= \frac{9}{35} \approx 26\% \end{aligned}$$

The probability of drawing a blue marble and then a yellow marble if the blue marble is not replaced is 6/35 or about 26%.