SYSTEMS OF EQUATIONS

This unit is about solving systems of equations by using two methods: graphing and substitution. In the first method, Graphing Systems of Equations, the system is solved by graphing the two linear equations, and then examining the graphs of the equations. If the two lines in the system intersect at one point, there is one solution (x, y). If the two lines graph on top of each other, there are many points of intersection; thus, there is an infinite number of solutions. If the lines are parallel, there is no solution because parallel lines never intersection. In the second method, Solving Systems by Substitution, algebraic manipulations are used to solve for x and y, if a solution exists.

Slope-Intercept Form

One way to graph the equation of a line is to use the slope-intercept form. Once an equation is placed in slope-intercept form, the slope and *y*-intercept are easily identified.

Slope-Intercept Form

$$y = mx + b$$

where *m* represents the slope and *b* represents the *y*-intercept, the point at which the graph crosses the *y*-axis (0, *b*).

Example 1: Identify the slope and *y*-intercept.

$$y = -\frac{2}{3}x - 4$$

slope $(m) = -\frac{2}{3}$ y-intercept $(b) = -4$ *or* $(0, -4)$

$$y = -\frac{2}{3}x - 4$$

The graphs of linear equations, that is, equations with graphs that are straight lines, can be graphed by using the slope and *y*-intercept.

Steps for graphing linear equations using the slope and *y*-intercept:

1) Arrange the equation into the form $y = mx + b$.
   (This means to solve the equation for *y*.)

2) Identify the slope (*m*) and *y*-intercept (*b*).

3) Plot the *y*-intercept, point (0, *b*).

4) Plot several additional points by using the slope ratio, $\frac{rise}{run}$.

5) Draw a line through the points with a straight edge. The straight line is the graph of the linear equation.

Example 2: Graph $-3x + 2y = -6$.

1) Put the equation in slope-intercept form by solving for *y*.

$$\begin{aligned} -3x + 2y &= -6 \\ +3x \quad\quad & \quad +3x \\ 2y &= 3x - 6 \end{aligned}$$

Add 3*x* to both sides.

$$\frac{2y}{2} = \frac{3x-6}{2}$$

Divide all terms by 2.

$$\frac{2}{2}y = \frac{3}{2}x - \frac{6}{2}$$

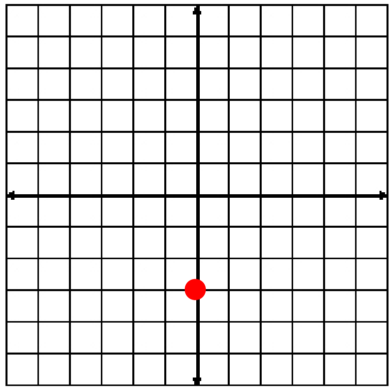
$$y = \frac{3}{2}x - 3$$

Simplify (Remember, 1*y* is just *y*.)

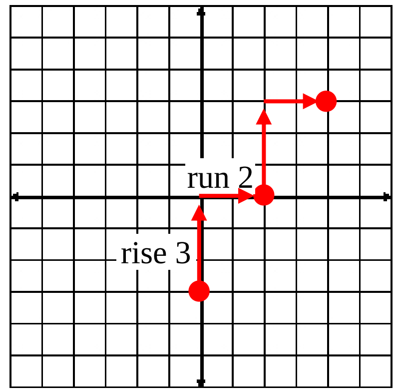
2) Identify the slope and *y*-intercept.

$$y = \frac{3}{2}x - 3 \qquad m = \frac{3}{2} \qquad y\text{-intercept} = -3 \quad (0, -3)$$

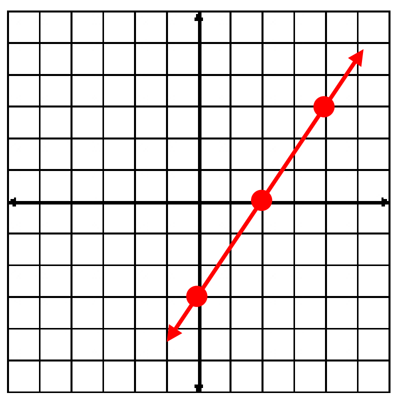
3) Plot the *y*-intercept, (0, –3).



4) Use the slope ratio $\frac{3}{2}$, $\frac{rise}{run}$, to plot more points.



5) Draw a line through the points with a straight edge. The straight line represents the graph of the equation $-3x + 2y = -6$.



$-3x + 2y = -6$

Graphing Systems of Equations

Two equations in two variables are called a system of equations. The solution to a system of equations is the point on a coordinate plane where the two equations intersect.

There are three possible solutions for a system of equations.

1) The lines may intersect at one point; therefore, the solution would be and ordered pair **(*x*, *y*)**.

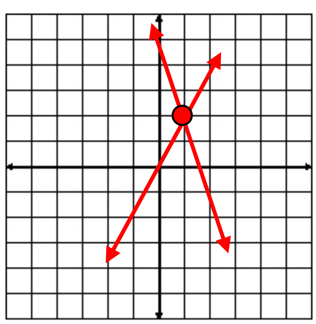
2) The lines may be parallel and not intersect at all; therefore, the solution would be **no solution**.

3) The lines may be identical and would lie on top of each other when graphed, so the solution would be **many solutions**.

Example 1: For each system of equations, identify if there is one solution, no solution, or many solutions.

$y = 2x + 0$

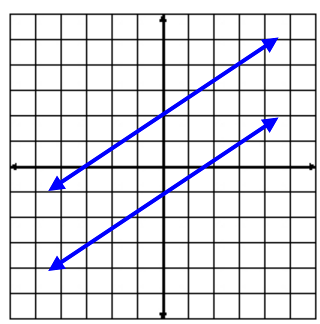
$y = -3x + 5$



(1, 2)

$y = \frac{2}{3}x + 2$

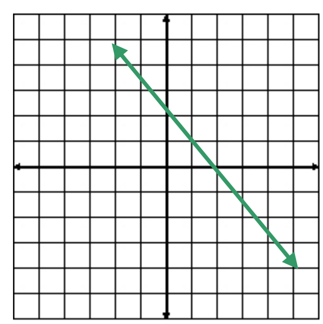
$y = \frac{2}{3}x - 1$



no solution

$y = -x + 2$

$y = -x + 2$



many solutions

To solve a system of equations by graphing:

(a) First solve each equation for y to make sure they are in the slope-intercept form ($y = mx + b$).

(b) Plot the y-intercept and use the slope ratio of $\frac{\text{rise}}{\text{run}}$ to plot more points for each linear equation.

(c) After graphing each equation, determine the point of intersection (x, y), if it exists.

Example 2: Solve the system of equations.

$$y - 3x = -1$$
$$y - 2x = 1$$

Step 1: Solve each equation for y.

$y - 3x = -1$ | $y - 2x = 1$

$y = 3x - 1$ | $y = 2x + 1$

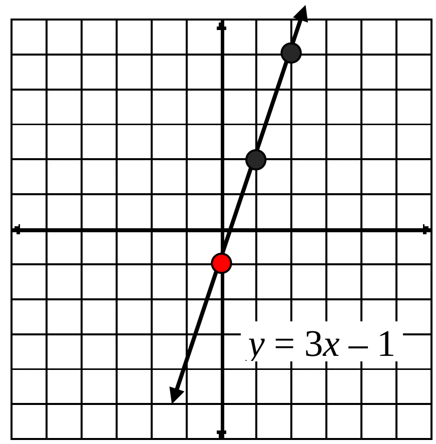
Step 2: Determine the y-intercept and the slope.

$y = 3x - 1$ | $y = 2x + 1$

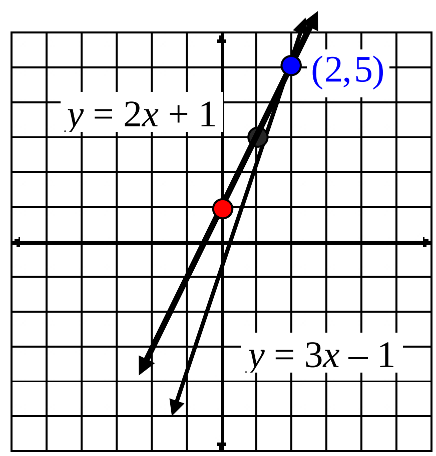
y-intercept = $(0, -1)$ | y-intercept = $(0, 1)$

slope = $3 = \frac{3}{1}$ | slope = $2 = \frac{2}{1}$

Step 3: Plot the y-intercept and graph additional points using rise/run for the first equation.



Step 4: Plot the y-intercept and graph additional points using rise/run for the second equation in the same coordinate plane as the first equation.



The solution to the system of equations is (2, 5) because this is one point of intersection between the two lines.

Example 3: Solve the system of equations.

$$3x + y = 4$$
$$2x - y = 6$$

Step 1: Solve each equation for *y*.

$3x + y = 4$

$y = -3x + 4$

$2x - y = 6$

$-y = -2x + 6$ (Multiply all terms by –1.)

$y = 2x - 6$

Step 2: Determine the *y*-intercept and the slope.

$y = -3x + 4$

y-intercept = (0, 4)

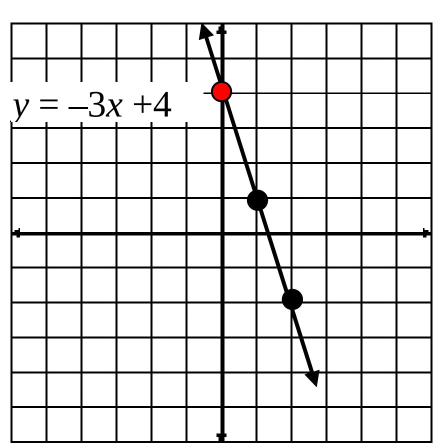
slope = $\frac{-3}{1}$

$y = 2x - 6$

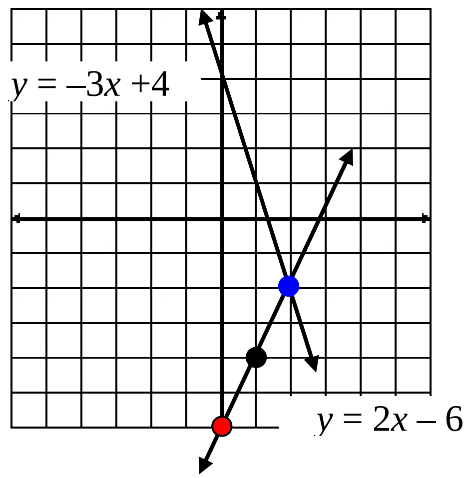
y-intercept = (0, –6)

slope = $\frac{2}{1}$

Step 3: Plot the *y*-intercept and graph additional points using rise/run for the first equation.



Step 4: Plot the *y*-intercept and graph additional points using rise/run for the second equation in the same coordinate plane as the first equation.



The solution to the system of equations is (2, –2) because this is the point at which the two lines intersect.

Solving Systems by Substitution

Another way to solve a system of equations is to solve the systems algebraically using substitution.

To solve a system of equations by substitution:

1) Solve one equation for a variable (it is much easier to solve an equation for a variable that has a coefficient of 1).

2) Substitute this value into the other equation and find the value of one variable.

3) Substitute the value found in step 2 into either of the equations to solve for the second variable.

Follow the example below.

Example 1: Solve the system by using substitution.

$x + y = 2$
$3x + y = 8$

In this particular case, there are two choices for solving for a variable. Solve the first equation for either x or y or solve the second equation for y. (Again it is easier to solve an equation for a variable that has a coefficient of 1).

1) Solve the first equation for x.

$x + y = 2$
$\quad - y \quad - y$

$x = 2 - y$

2) Substitute $2 - y$ into the second equation for x.

$3x + y = 8$

$3(2 - y) + y = 8$ (Distribute 3(2) – 3y)

$6 - 3y + y = 8$ Combine like terms and solve for y.

$6 - 2y = 8$
$-6 \qquad -6$

$-2y = 2$

$y = -1$

3) Substitute $y = -1$ back into either of the equations to solve for x.

$x + y = 2$

$x + (-1) = 2$

$x = 3$

The solution to the system is (3, –1).

Just as with graphing systems of equations, solving systems of equations through substitution has three possible outcomes: one solution, no solution, or many solutions.

First, let's take a look at an example where the system has no solution. Let's see what that looks like algebraically.

Example 2: Solve the system by using substitution.

$2x + y = 9$
$y + 2x = 7$

1) Solve the first equation for y.

$2x + y = 9$

$y = 9 - 2x$

2) Substitute $9 - 2x$ into the second equation for y.

$y + 2x = 7$

$(9 - 2x) + 2x = 7$

$9 = 7$ *False Statement*

Since this is not a true statement, this means the system does not have a solution. So, the solution is **no solution**.

Now, let's take a look at an example where the system has many solutions. Let's see what that looks like algebraically.

Example 3: Solve the system by using substitution.

$2x - y = -4$
$y - 2x = 4$

1) Solve the second equation for y.

$y - 2x = 4$

$y = 4 + 2x$

2) Substitute $4 + 2x$ into the second equation for y.

$2x - y = -4$

$2x - (4 + 2x) = -4$

$2x - 4 - 2x = -4$

$-4 = -4$ *True Statement*

Since this is a true statement this means that the solution is **many solutions**.

Applications of Systems of Equations

Some word problems may be set up using a system of equations. Let's take a look at an example.

Example: The sum of two numbers is 24. One of the numbers is 6 less than the other. What are the two numbers?

Step 1: Let x = the first number and y = the second number.

Step 2: Write an equation based on the first math statement.

"The sum of two numbers is 24."

$$\downarrow \qquad\qquad\qquad \downarrow \; \downarrow$$

$$x + y \qquad\qquad\qquad = 24$$

Step 3: Write an equation based on the second math phrase.

"One of the numbers is 6 less than the other."

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$x \qquad\qquad = \qquad\qquad y - 6$$

Step 4: Write the system of equations.

$$x + y = 24$$
$$x = y - 6$$

Step 5: Solve using substitution.

First, substitute to find *y*.

$x + y = 24$	$x = y - 6$
$y - 6 + y = 24$	Substitution.
$2y - 6 = 24$	Collect like terms.
$+6 \quad +6$	Add 6 to both sides.
$2y = 30$	Simplify.
$y = 15$	Divide.

Now, use the value $y = 15$, to solve for *x*.

$x = y - 6$	Solve for *x*.
$x = 15 - 6$	Substitution.
$x = 9$	Simplify.

The solution: $x = 9,\ y = 15$

Step 6: Check $x = 9$ and $y = 15$ in both equations to see if these values make true statements. If so, the values check.

Check 1:	Check 2:
$x + y = 24$	$x = y - 6$
$9 + 15 = 24$	$9 = 15 - 6$
$24 = 24$ ✓	$9 = 9$ ✓

Both values check in both equations. The solution is correct!