LINEAR FUNCTIONS

This unit is a study of linear functions which are the most basic algebraic functions. When working with a set of ordered pairs in a linear function, the y-values will be dependent on the corresponding x-values. There are various ways to graph linear equations such as making a table of x and y values or plotting two points connected by the slope of the line. Slope will be examined closely by comparing the steepness of lines and also by calculating the slope using the slope formula. The graphs of linear equations will be predicted based on keeping some values constant while changing others. The unit concludes with examining rates of change and direct variation.

Graphing Linear Equations

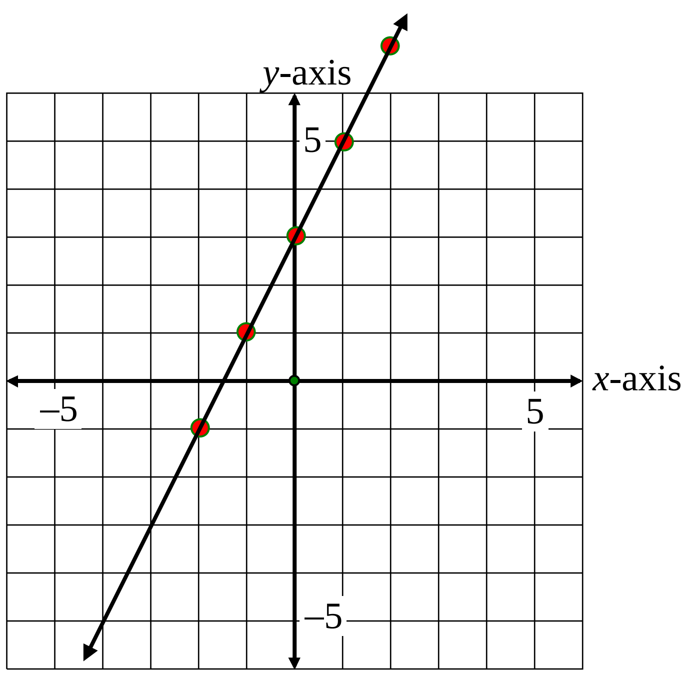
The graph of a linear equation is a straight line.

Example 1: Graph $y = 2x + 3$.

- First organize the data in a table.
- Choose several values for x. In this example, we chose –2 through 2.
- Substitute the values for x in the equation to find corresponding y-values.
- Make ordered pairs with the results.
- Graph the points.
- Draw a straight line through the points.

$y=2x+3$			
x	Substitution	y	Ordered Pairs $(x\ y)$
-2	$y=2\times(-2)+3$	-1	$(-2,-1)$
-1	$y=2\times(-1)+3$	1	$(-1, 1)$
0	$y=2\times(0)+3$	3	$(0, 3)$
1	$y=2\times(1)+3$	5	$(1, 5)$
2	$y=2\times(2)+3$	7	$(2, 7)$

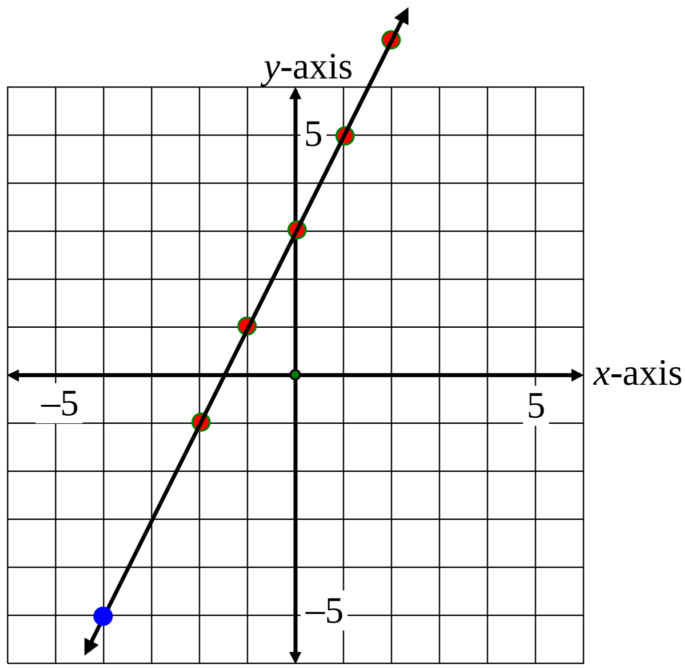
Use the ordered pairs to graph the function.



✓Check the linear graph to see if it matches the equation:

- Chose another point on the line that was NOT generated from the table of values.
- Substitute the x and y values of that point in the equation to see if they test true.

Let's choose (–4, –5). It is highlighted in blue in the graph.



$$y = 2x + 3$$

$$-5 = 2(-4) + 3 \qquad (-4,-5) \rightarrow x = -4,\ y = -5$$

$$-5 = -8 + 3$$

$$-5 = -5\ \checkmark$$

The point tests true and is one of the ordered pairs for the equation $y = 2x + 3$.

Graphing a Line Using the Intercepts

Another way to graph linear equations is by using the x-intercept and the y-intercept.

- The x-intercept is the point at which the line crosses the x-axis.
- The y-intercept is the point at which a line crosses the y-axis.

To find the intercepts:

1) To locate the ***y*-intercept** $(0, y)$, replace ***x* with 0** in the equation and solve for y.

2) To locate the ***x*-intercept** $(x, 0)$, replace ***y* with 0** in the equation and solve for x.

3) Plot the two points and connect them with a straight edge.

*Remember: To find the ***x*-intercept, let *y* = 0** and to find the ***y*-intercept, let *x* = 0**.

Example: Graph $2x - 3y = 6$ by using the x- and y-intercepts.

1) To find y-intercept, let $x = 0$.

$2x - 3y = 6$

$2(0) - 3y = 6$

$-3y = 6$

$y = -2$

y-intercept = $(0, -2)$

2) To find x-intercept, let $y = 0$.

$2x - 3y = 6$

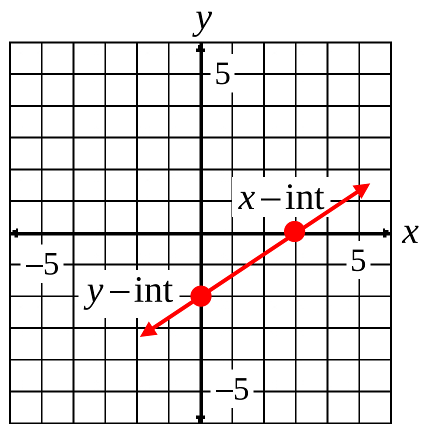
$2x - 3(0) = 6$

$2x = 6$

$x = 3$

x-intercept = $(3, 0)$

The line crosses the y-axis at (0, –2) and it crosses the x-axis at (3, 0).



The graph of the equation, $2x - 3y = 6$, is a straight line that passes through points (0, –2) and (3, 0).

Slope

The slope of a line describes the steepness of the line. The slope is the ratio of vertical rise to horizontal run.

To find the slope of a line graphed on a coordinate plane

-Identify a point on the line.

-From that point move up or down until you are directly across from the next point.

-Move left or right to the next point.

Example 1: From the graph below determine the slope of the line.

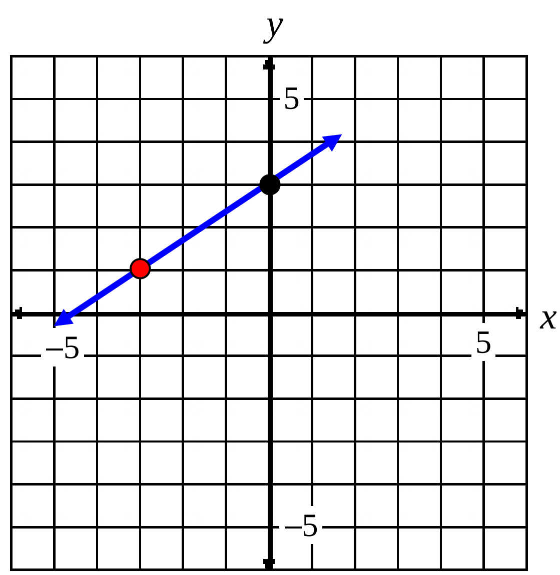
-Put your pencil on the red point.

-Move straight up (vertical rise) until your pencil is in the same line as the black point, (2 units)

-Move right (horizontal run) until you reach the black point. (3 units)

-Slope = $\dfrac{\text{vertical rise}}{\text{horizontal run}}$

You have now determined the slope of the line to be $\dfrac{2}{3}$.



*Just remember: slope $= \dfrac{\text{rise}}{\text{run}}$

Example #2: From the graph below determine the slope of the line.

-Put your pencil on the red point.

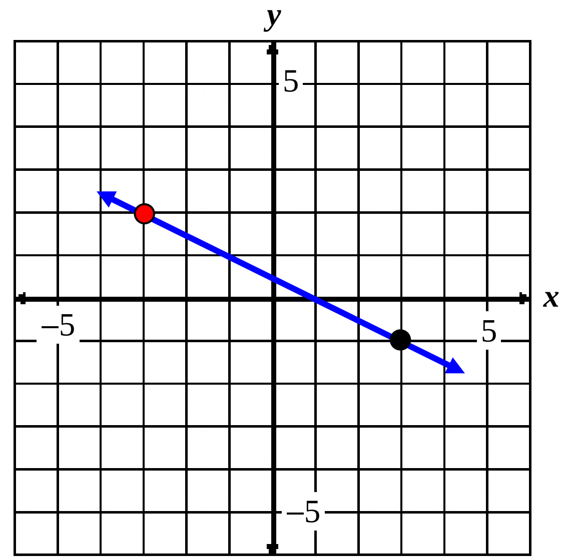
-Move straight down (vertical rise) until your pencil is in the same line as the black point, (–3 units)

-Move right (horizontal run) until you reach the black point. (6 units)

-Slope = $\frac{\text{rise}}{\text{run}}$

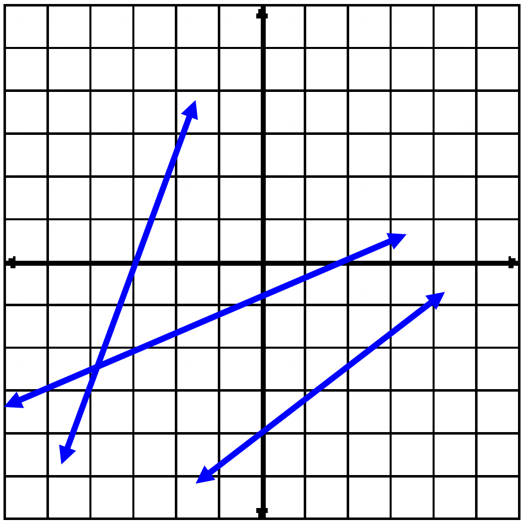
You have now determined the slope of the line to be $\frac{-3}{6}$.

This slope can be reduced to $-\frac{1}{2}$.

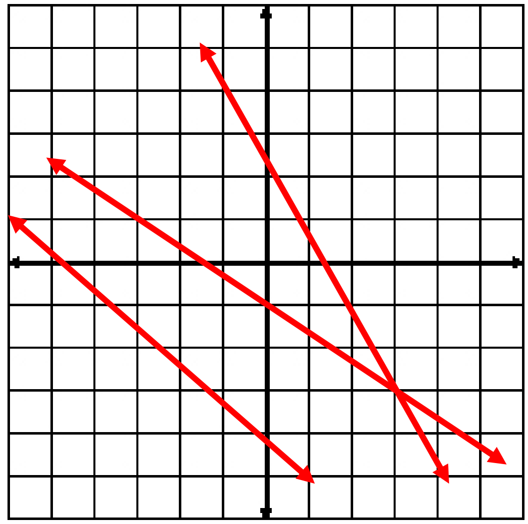


*Notice that if you count down one unit (rise) from the red point and right two units (run), you will be on a point of the line.

On a coordinate plane there are lines that have positive slopes and lines that have negative slopes. Below is an illustration of both.



Lines with positive slopes rise to the right.



Lines with negative slopes rise to the left.

At this point we are going to learn how to find the slope of a line by using two points that lie on the line.

The definition of slope (m) states that given two points (x_1, y_1) and (x_2, y_2), the formula for finding the slope of a line containing these points is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

*Notice, this formula represents the vertical change over the horizontal change.

Example #3: Find the slope of the line containing point A(–2, –6) and B(3, 5).

Let point B = (x_2, y_2) and point A = (x_1, y_1).

$(x_2, y_2) = (3,5) \quad \rightarrow \quad x_2 = 3, \quad y_2 = 5$

$(x_1, y_1) = (-2,-6) \quad \rightarrow \quad x_1 = -2, \quad y_1 = -6$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{5-(-6)}{3-(-2)}$$

$$m = \frac{11}{5}$$

The slope (m) of the line is $\frac{11}{5}$.

Graphing a Line Using a Point and the Slope

Another way to graph a line is by using the **slope-intercept form**. The slope intercept form of a line is written as ***y* = *mx* + *b*** where ***m*** is the slope and ***b*** is the *y*-intercept.

Slope-Intercept Form

$$y = mx + b$$

where ***m*** represents the **slope** and ***b*** represents the ***y*-intercept**, the point at which the graph crosses the *y*-axis.

The slope-intercept form of a line is where *m* represents the slope and *b* represents the *y*-intercept.

Example 1: The equation, $y=\frac{1}{2}x+2$, is in slope-intercept form. Graph $y=\frac{1}{2}x+2$ using the slope-intercept method.

a) Identify the *y*-intercept (*b*) and plot this point. Since the equation is in slope-intercept form, the *y*-intercept (*b*) is 2.

$$y = mx + b$$
$$y=\frac{1}{2}x+2$$
$$b=2$$
$$y\text{-intercept}=(0,2)$$

Check: The *y*-intercept is where the line crosses the *y*-axis. At that point, $x = 0$.

$$y = \frac{1}{2}x + 2$$
$$y = \frac{1}{2}(0) + 2$$
$$y = 2$$

The *y*-intercept is (0, 2).

b) Identify the slope and count out the rise/run.

Since the equation is in slope-intercept form, the slope (*m*) is 1/2.

$$y = mx + b$$
$$y = \frac{1}{2}x + 2$$
$$m = \frac{1}{2}$$

To graph the linear equation using the slope and *y*-intercept, follow the steps below.

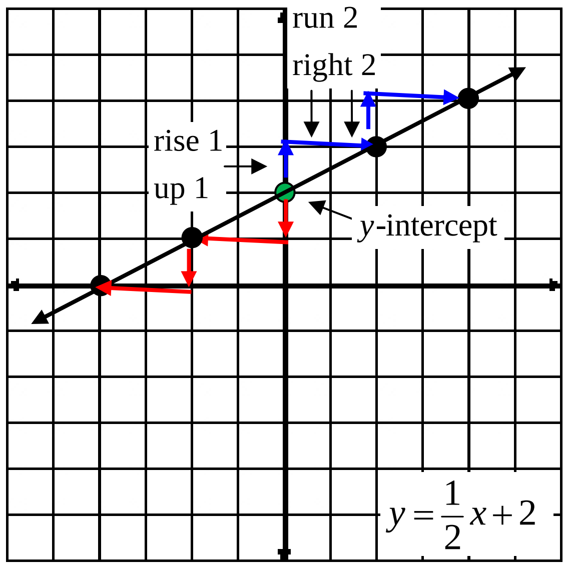
a) Solve the equation for "*y*" and write it in the slope-intercept form of $y = mx + b$. (This step is completed above.)

b) Identify the *y*- intercept "*b*" and plot the point on the *y*-axis (0, *b*).

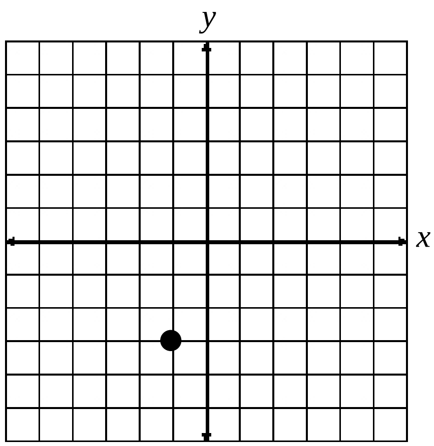
c) Use the slope ratio of $\frac{\text{rise}}{\text{run}}$ to plot more points.

From the *y*-intercept (0, 2) use the slope ratio (1/2) to plot more points.

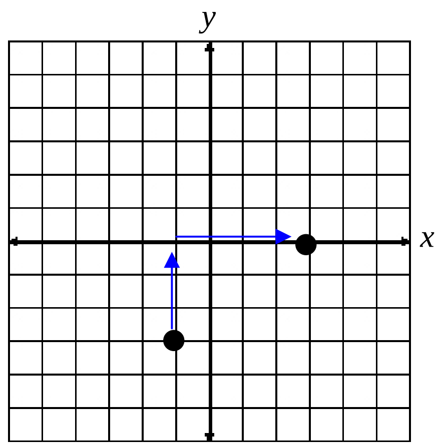
$\left(\dfrac{\text{rise}}{\text{run}} = \dfrac{1}{2}\right)$ This means to move up 1 unit and move right 2 units, and then continue on to plot more points, up 1, 2 right, etc.
(OR do the opposite which is down 1, left 2, down 1, left 2.)



Example 2: Graph the line containing the point (–1, –3) and having a slope of $m = \frac{3}{4}$.

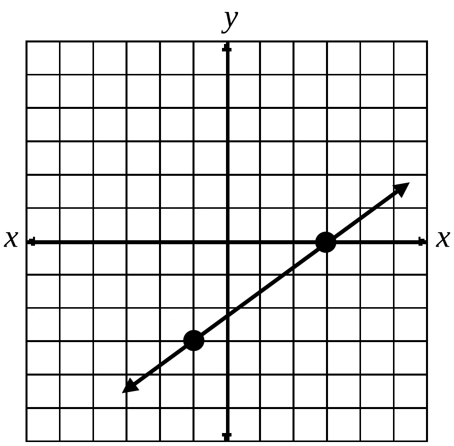


1. Plot the point (–1, –3).



2. Use the rise (3 units) over run (4 units) ratio for slope to plot a second point.

Remember: $m = \frac{3}{4}$



3. Draw a line through the points with a straightedge.

Example 3: Graph the line for the equation, $y = \frac{3}{2}x - 3$.

1) Identify the *y*-intercept and the slope.

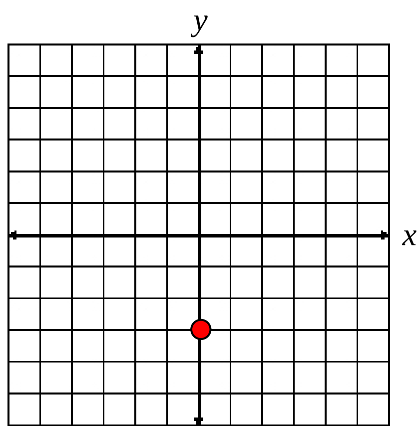
$y = \frac{3}{2}x - 3$

y-intercept (*b*) = (0, –3)

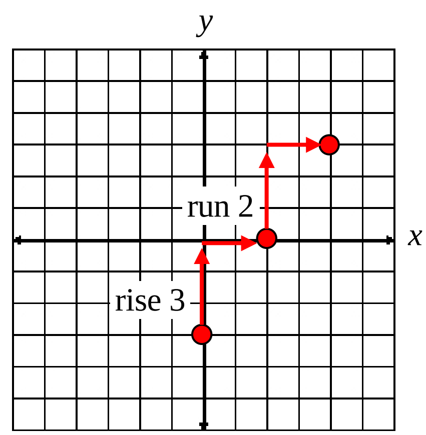
$y = \frac{3}{2}x - 3$

slope (*m*) = $\frac{3}{2}$

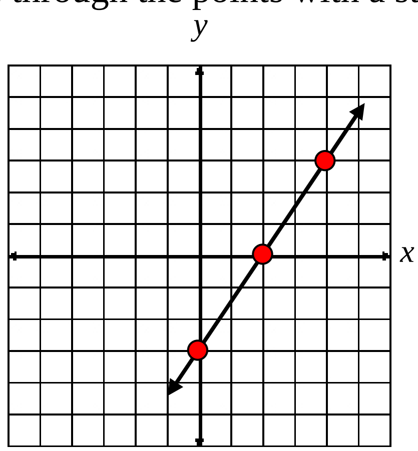
2) Plot the *y*-intercept, (0, –3).



3) Use the ratio of $\frac{rise}{run}$ and the slope $\frac{3}{2}$ to plot more points.



4) Draw a line through the points with a straight edge.



Using the slope-intercept method to plot equations in slope-intercept form is a very useful method of graphing linear equations.

Compare How Changes in an Equation Affect the Related Graph

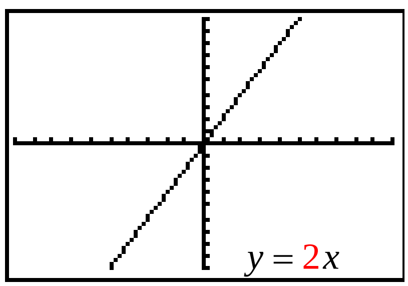
Once we know what to expect from certain relations that are defined by equations, we can experiment with changing the equations and predicting how those changes will appear "graphically".

First, we'll examine changes of the coefficient of *x* in the equation $y = 2x$.

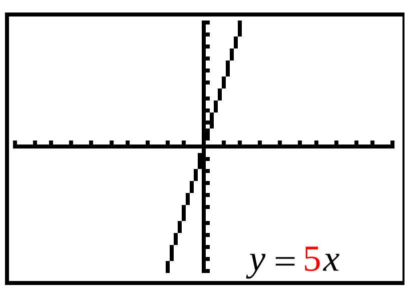
coefficient - Coefficient is the term used to refer to the number before *x* where it is multiplied with *x*.

*Note: The graphs shown below were generated using a graphing calculator. The graphing calculator displays digitized images made up of pixels; thus, the lines appear to be "jagged" rather than smooth and straight. However, for all of the lines that represent equations in this unit, assume the lines are smooth and straight.

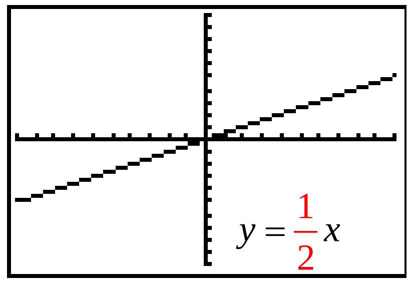
Let's begin with the graph, $y = 2x$.



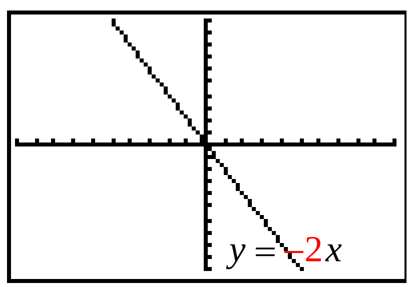
Now, let's increase the coefficient of *x* from 2 to 5, $y = 5x$. The graph of this equation is a *steeper* line.



Next, let's decrease the coefficient of *x* from 5 to 0.5, $y = \frac{1}{2}x$. The graph of this equation is a *flatter* (less steep) line.



Finally, let's change the coefficient from 0.5 to –2, $y = -2x$. The graph of this equation is a line that changes direction, that is, up and to the left. (All the other lines were up and to the right.)



This demonstrates the effect on a line called "slope". We can see larger values greater than one will make steep lines while small fractional values make flatter lines. Negative values will point the line in the opposite direction. All of these changes occurred as we changed the coefficient of *x*.

Now let's compare four equations as the coefficient of "*x*" increases.

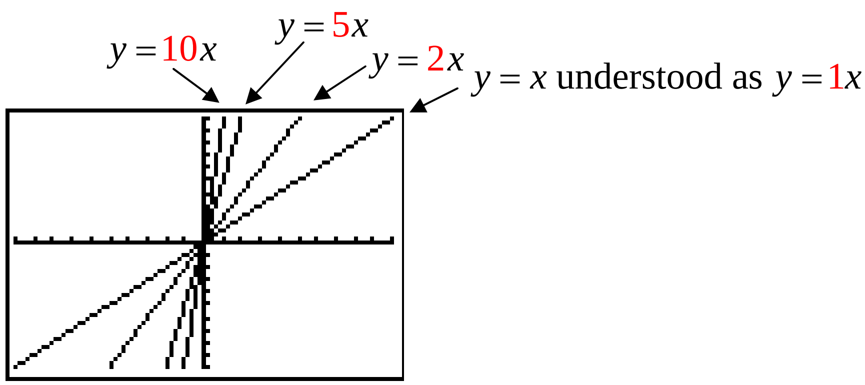
Example 1: Graph the following equations and make a statement about the change in the graphs when the coefficient of x changes: $y = x, y = 2x, y = 5x, y = 10x$.

$y = x$

$y = 2x$

$y = 5x$

$y = 10x$



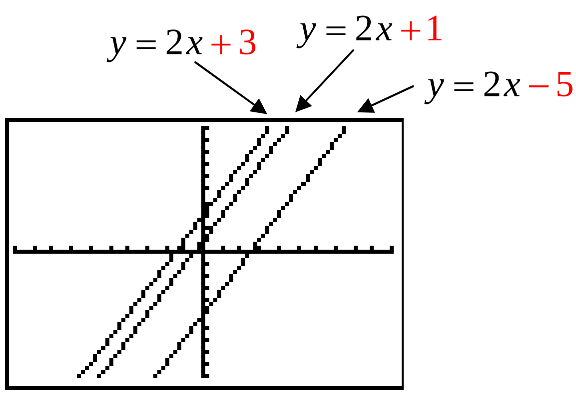
The graphs of these equations show that the lines appear to grow steeper as the coefficient of x increases.

Example 2: Graph the following equations and make a statement about the change in the graphs when the constant that is added to or subtracted from $2x$ changes: $y = 2x+1, y = 2x+3, y = 2x-5$.

$y = 2x+1$

$y = 2x+3$

$y = 2x-5$



Observe that the lines are parallel and appear to have shifted in position along the y-axis.

All of the equations have the same slope of two, but the line slides up and down the y-axis. The graphs of the equations show that adjusting the constant value will translate the line up or down the y-axis.

Notice that $y = 2x + 3$ is passes through (0, 3) on the *y*-axis.
Notice that $y = 2x + 1$ is passes through (0, 1) on the *y*-axis.
Notice that $y = 2x - 5$ is passes through (0, –5) on the *y*-axis.

Now practice observing changes in linear graphs. If a graphing calculator is available, it would be best to enter the equations and watch as they are graphed. The first equation will be graphed first, and so on. If a graphing calculator is not available, then create tables of values and make the graphs on graph paper.

*Note: An explanation of how to enter equations in a graphing calculator is provided in the content link to "Graphing Linear Equations with a Graphing Calculator".

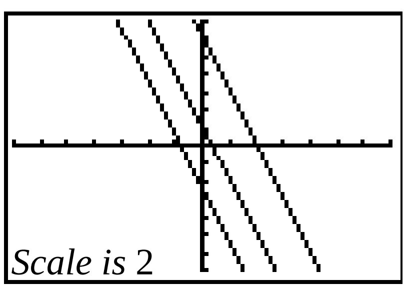
Two problems are given below as practice. The answers are shown at the bottom of the page.

Problem 1: Graph the following equations:
$y = -3x + 2, y = -3x - 5, y = -3x + 12.$

*Note: The scale on the graphing calculator is set to two (instead of one) for both the *x*-axis and the *y*-axis.

- What remains the same in the graphs of all three equations?
- What changes occur in the graphs of the three equations?

$y = -3x + 2$
$y = -3x - 5$
$y = -3x + 12$



Problem 2: Graph the following equations:

$$y = x - 1, y = 2x - 1, y = -12x - 1.$$

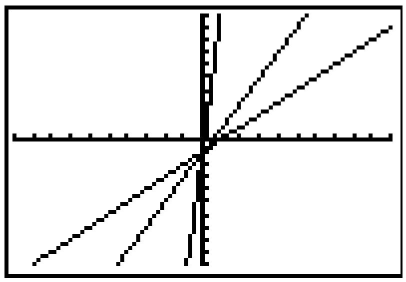
*Note: The scale on the graphing calculator is set to one for both the *x*-axis and the *y*-axis.

- What remains the same in the graphs of all three equations?
- What changes occur in the graphs of the three equations?

$y = x - 1$

$y = 2x - 1$

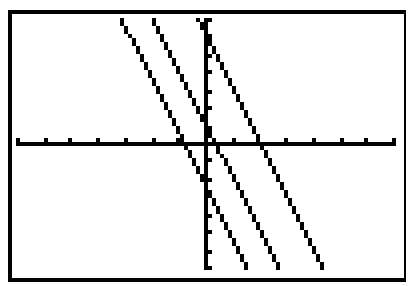
$y = 12x - 1$



Answers

Problem 1: The slope is the same, –3. The graph moves up and down the *y*-axis depending on the constant value.

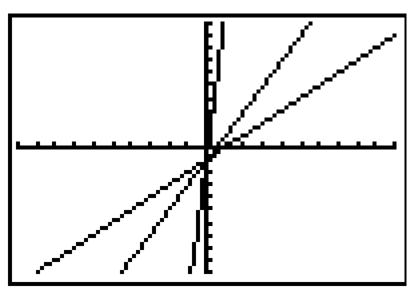
$y = -3x + 12$ crosses the *y*-axis at (0, 12).
$y = -3x + 2$ crosses the *y*-axis at (0, 2).
$y = -3x - 5$ crosses the *y*-axis at (0,–5).



Problem 2: The *y*-intercept is the same for all three equations, (0,–1). As the coefficient of *x* (slope) increases, so does the steepness of the line.

$y = x - 1$ is the less steep of the three equations.

$y = 12x - 1$ is the most steep.



*Note: All of the equations used in this unit were written in the slope intercept form:

$$y = mx + b$$

where m represents the slope, the steepness of the line

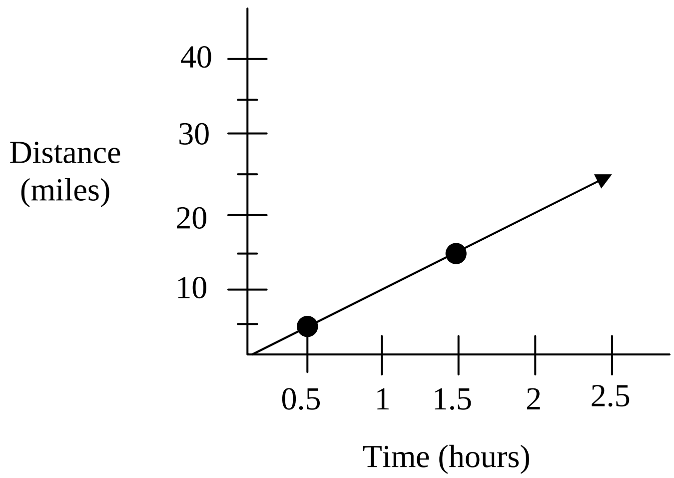
and b represents the y-intercept, the point where the line crosses (intercepts) the y-axis.

Rate of Change

Rate of change is directly related to slope and can be found using the following:

$$\text{rate of change} = \frac{\text{change in distance}}{\text{change in time}}$$

Example: The graph shows the distance a bicyclist travels at a constant speed. Find the speed of the bicyclist.



Select two points and use the slope formula.

Let $(x_1, y_1) = (0.5, 5)$

Let $(x_2, y_2) = (1.5, 15)$

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{15-5}{1.5-0.5} = \frac{10 \text{ miles}}{1 \text{ hour}}$$

The bicyclist travels 10 miles for every hour he travels.

Constant Rates of Change and Predicting Solutions

In the problem below, we will examine a constant rate of change and use that rate to make a prediction.

On a trip across country, the Wilson family was able to travel an average of 50 miles per hour for several days. They drove nine hours per day and spent the other 15 hours per day sightseeing and sleeping. Establish a table to determine how many miles, days, and hours were spent on the two week trip.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Hours driving	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
Miles (d=50 h)	450	900	1350	1800	2250	2700	3150	3600	4050	4500	4950	5400	5850	6300	6750

Based on data in the table, we can write relationships and equations about the data.

To find the number of hours (H) spent driving, we can multiply the number of days (D) times 9. A formula to represent this relationship is:

$$H = 9D$$

To find the miles traveled (M), we can multiply 9 hours per day times the number of days (D) time 50, the average rate of speed. A formula to represent this relationship is:

$$M = 9D(50)$$

or

$$M = 9(50)D$$

Using this chart and a little algebra, we can predict much about a long trip or a similar situation like this one.

Example 1: Answer the follow questions about the table and relationships discussed about the Wilson's family trip.

a) Predict how many hours the Wilson family would drive after 20 days?

$$H = 9d$$
$$H = 9(20)$$
$$H = 180$$

The Wilson family would drive 180 hours in 20 days.

b) Predict how many miles the Wilson family would travel in 20 days.

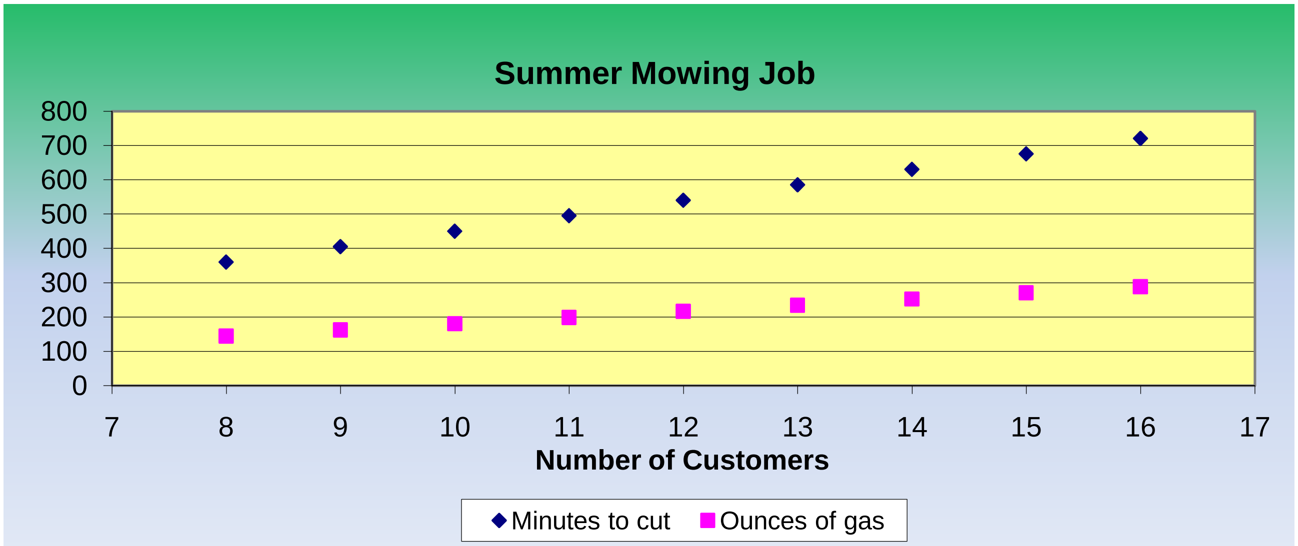
$$M = 9(50)d$$
$$M = 9(50)(20)$$
$$M = 9000$$

The Wilson family would travel 9000 miles in 20 days.

Example 2: Set up a chart to analyze Joe's work on his summer job. Joe mows lawns for customers in his neighborhood. He will spend 45 minutes mowing each lawn and use about 18 ounces of gasoline per lawn. Joe prepared a chart and graph of his work schedule and load for 8 through 16 customers.

Customers	8	9	10	11	12	13	14	15	16
Minutes to cut	360	405	450	495	540	585	630	675	720
Ounces of gas	144	162	180	198	216	234	252	270	288

Joe can now use the chart and graph to decide how many customers to service during the summer.



Notice how the values increase as the number of customers increase. This prediction is a positive correlation as well as a positive slope.

If Joe had 22 customers, (a) how many minutes would he spend mowing grass? (b) How much gasoline would he need?

(a) Since Joe is allowing 45 minutes per customer, write the following relationship: Number of minutes (M) = 45 x number of customers (C).

$$M = 45 \times C$$
$$M = 45\,(22)$$
$$M = 990$$

Joe would spend 990 minutes mowing for 22 customers.

(b) Since Joe is allotting 18 ounces of gasoline per lawn, he could write the following relationship: number of gallons of gasoline (G) = 18 x number of customers (C).

$$G = 18 \times C$$
$$G = 18 \times 22$$
$$G = 396$$

Joe would use 396 ounces of gasoline for 22 customers.

Direct Variation

The variable y varies directly as x if there is a nonzero constant k such that $y = kx$. The equation $\boldsymbol{y = kx}$ is called a direct variation equation and the number k is called the constant of variation.

*There are many situations in which one quantity varies directly as another:

- an employee's wages vary directly to the number of hours worked
- the amount of sales tax varies directly to the total price of the merchandise

y varies directly to x

To find the constant of variation (k) and the direct-variation equation, use the following steps.

1. Replace the x and y with the given values.
2. Solve for k.
3. Replace k in the direct variation equation.

Example 1: Find the constant of variation (k), and the direct-variation equation, if y varies directly as x and y = –72 when x = –18.

Step #1: Replace x and y with the given values.

$$y = kx$$

$$-72 = k(-18) \qquad y = -72, \qquad x = -18$$

Step #2: Solve for k.

$$\frac{-72}{-18} = \frac{k\,\cancel{(-18)}}{\cancel{-18}} \qquad \text{Divide both sides by } -18.$$

$$4 = k$$

Step #3: Replace k in the direct variation equation.

$$y = kx$$

$$y = 4x$$

Example 2: Each day Michael roller blades for exercise. When traveling at a constant rate, he travels 4 miles in about 20 minutes. At this rate, how long will it take Michael to travel seven miles?

To solve:

First, find a direct variation equation that models Michael's distance as it varies with time using $d = rt$.

Distance (d) varies directly as (t) and rate (r) is the constant of variation.

$$y = kx \rightarrow \quad d = rt$$

Step 1: Find the constant of variation (r).

$$d = rt$$

$$4 \text{ miles} = r(20 \text{ minutes}) \qquad d = 4 \text{ miles}, \; t = 20 \text{ minutes}$$

$$\frac{4 \text{ mi}}{20 \text{ min}} = \frac{r\,(\cancel{20 \text{ min}})}{\cancel{20 \text{ min}}}$$

Divide both sides by 20 minutes.

$$r = \frac{4 \text{ mi}}{20\,\text{min}} = \frac{1 \text{ mi}}{5\,\text{min}} \text{ or } \frac{1}{5} \text{ mile per minute}$$

Step 2: Write the direct variation equation.

$$d = rt \qquad r = \frac{1}{5}$$

$$d = \frac{1}{5}t$$

Now, use the direct variation equation to solve the problem.

Step 3: Apply the direct variation equation.

$$d = \frac{1}{5}t$$

$$7 = \frac{1}{5}t \qquad \text{Substitution } (d = 7 \text{ miles})$$

$$(5)7 = \cancel{(5)}\frac{1}{\cancel{5}}t \qquad \text{Multiply both sides by 5.}$$

$$35 = t$$

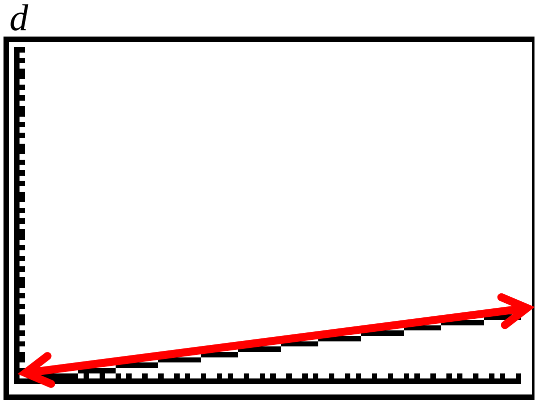
Thus, at the rate of 4 miles in 20 minutes, it will take Michael 35 minutes to travel seven miles.

The function above is graphed below. Study the graph carefully. As ***t* increases, *d* increases**.

For example:

When t is 5, d is 1.
When t increases to 10, d increases to 2.
When t increases to 35, d increases to 7.

d



t

*Note: the graph is **linear**.

*This graph was created on a graphing calculator, causing pixelation of the straight line. Thus, a **straight red line** was added to show the true appearance of the linear graph.