EXPRESSIONS AND EQUATIONS

This unit is about simplifying expressions and solving equations. Techniques that will be explored are collecting like terms, multiplying a polynomial by a monomial, and applying the distributive property. This unit concludes with solving equations and writing equations for real-world problems.

Combining Like Terms Using Models

Combining Like Terms

Multiplying a Polynomial by a Monomial

Review of Basic Equations

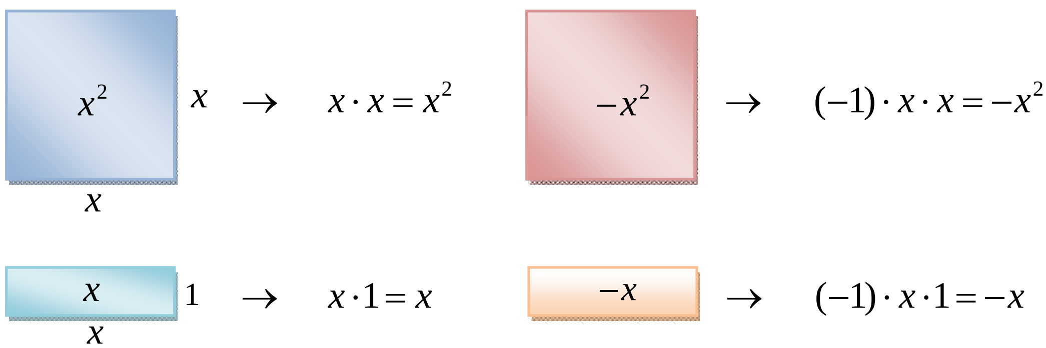
Solving Multi-Step Equations

Write an Equation for a Word Problem

Combining Like Terms Using Models

To combine like terms, it is helpful to see models of the problems to interpret the algebraic processes that take place.

First examine the meaning of the algebra tiles shown below based on the area of a rectangle.

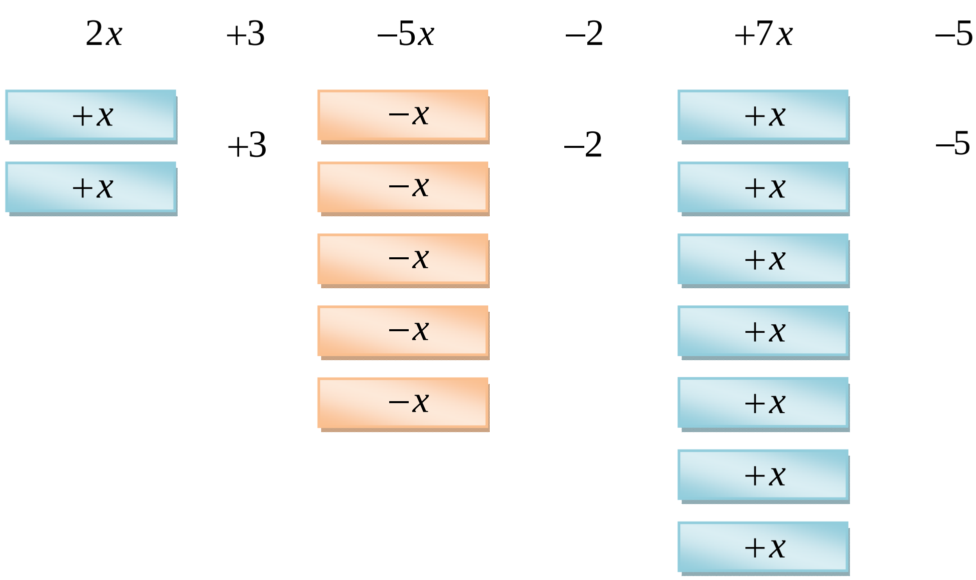


Example 1: Consider the following algebraic expression.

$$2x + 3 - 5x - 2 + 7x - 5$$

The expression may be simplified if we *collect* "like terms". The model below will help to understand "collecting like terms".

First model the expression using algebra tiles.



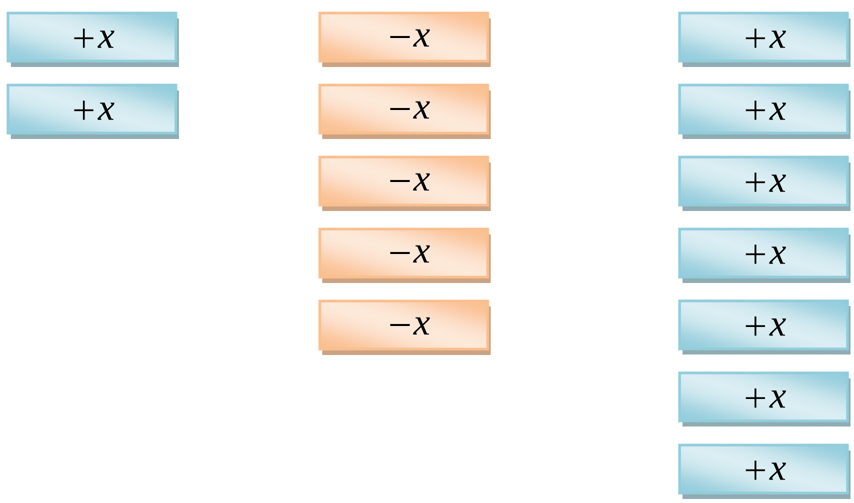
Look to find all of the terms that have x as a variable and combine their integer values. These terms are called “like” terms.

$$2x + 3 - 5x - 2 + 7x - 5$$

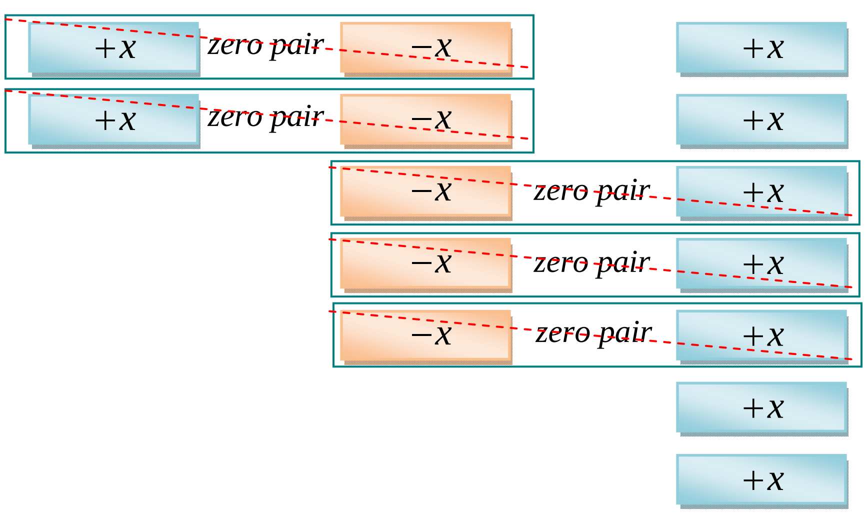
*Notice that the sign in front of the term determines whether the rectangles are positive or negative.

First, let’s just focus on the x-terms.

$2x$ $\quad$ $-5x$ $\quad$ $+7x$



Next, remove pairs of terms that make zero. A $+x$ and a $-x$ make zero.



Next, examine the terms that are left. Combine the remaining terms.

$$\left.\begin{array}{c} \boxed{+x} \\ \boxed{+x} \\ \boxed{+x} \\ \boxed{+x} \end{array}\right\} 4x$$

The x's simplify to $4x$. This is called “collecting like terms”.

Finally, examine all of the terms that are numeric and combine them also.

$$2x+3-5x-2+7x-5$$

*Notice that the sign in front of the numeric term is included with the term.

$$\begin{aligned} &+3-2-5= \\ &\quad +1\ \ -5= \\ &\quad\ \ -4 \end{aligned}$$

The numeric values combine to make -4.

Therefore, the expression simplifies to $4x-4$.

$$2x+3-5x-2+7x-5 \quad = \quad 4x-4$$

Seek to find the simplest form by combining all terms of the expression that have the same variables and/or have the same exponents.

Example 2: Simplify the given expression.

$$4n^2 + 3n + 2 - 2n^2 + n - 4$$

There are three types of like terms in this expression:

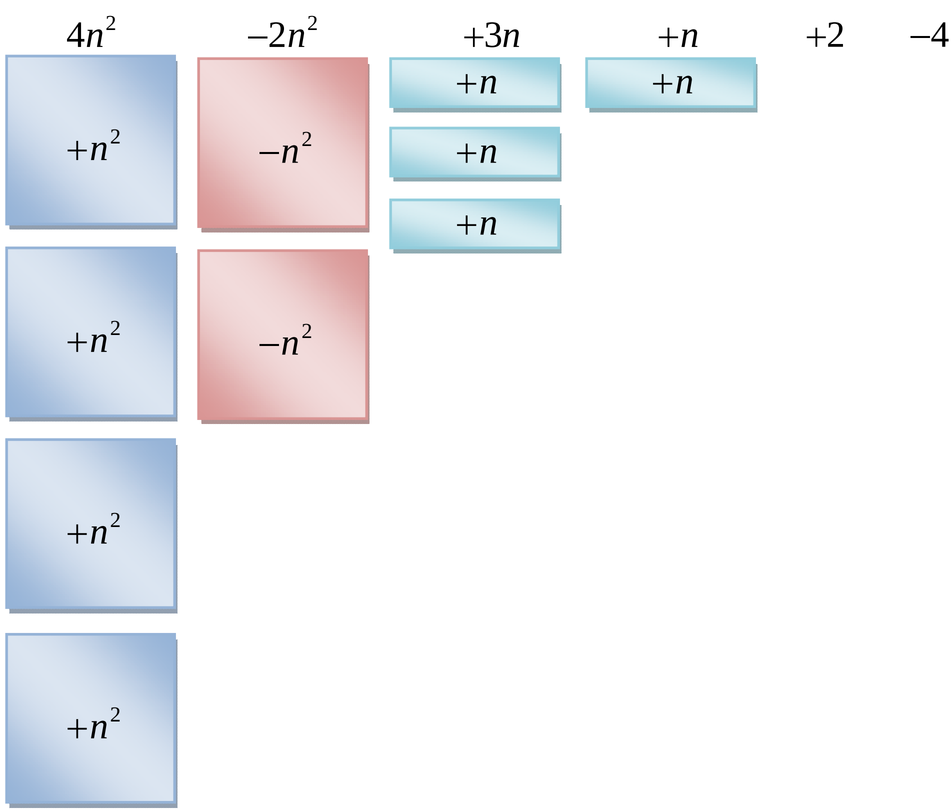
the n^2's, the n's, and the numeric values.

Rewrite the expression to prepare to collect like terms. Put all of the n^2's first, then all of the n's, and then the numeric values.

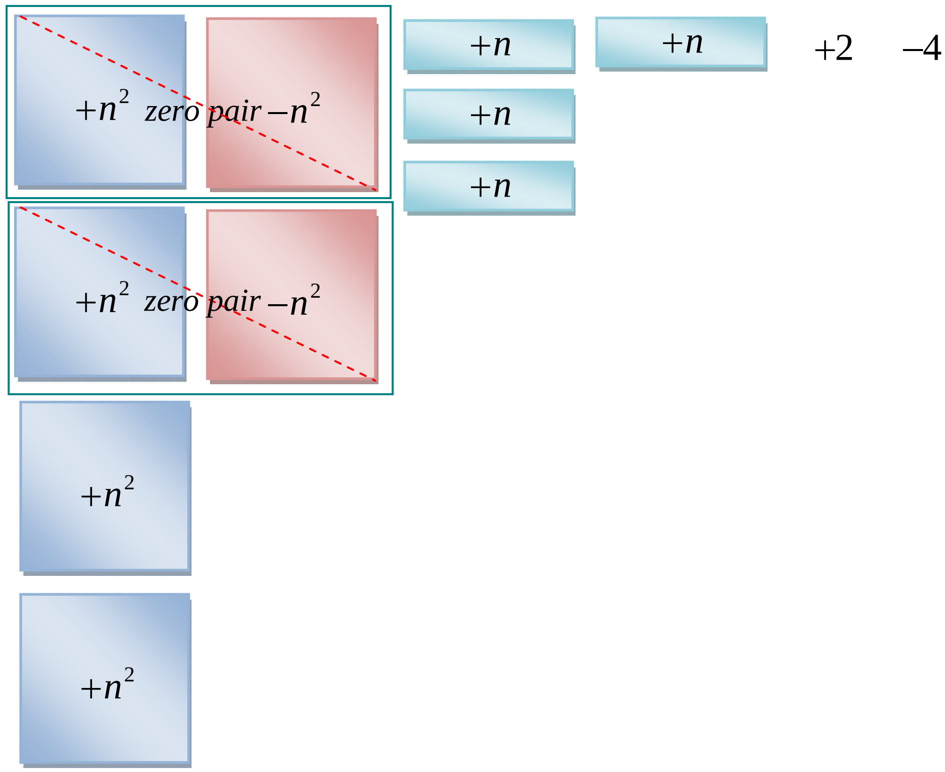
*Remember, the sign in front of the term goes with the term.

$$4n^2 + 3n + 2 - 2n^2 + n - 4 \quad \rightarrow \quad 4n^2 - 2n^2 + 3n + n + 2 - 4$$

Model the expression.



Remove the pairs of terms that make zero.



Collect the like terms.

$$\underbrace{4n^2 - 2n^2}_{2n^2} \underbrace{+\,3n + n}_{+4n} \underbrace{+\,2 - 4}_{-2}$$

Therefore, the expression simplifies to $2n^2 + 4n - 2$.

**Note: Use great care when negative values are involved.

Combining Like Terms

Let's take a look at "collecting like terms" without using models; but, first, take time to become familiar with terms associated with polynomials.

term – A term is a part of an algebraic expression that consists of a constant multiplier and one or more variables raised to a power.

polynomial – A polynomial is an algebraic expression that is a combination of two or more terms.

constant – A constant is a numeric term only.

coefficient – The coefficient of a variable term is the number that is in front of the variable. The sign in front of the number is included as part of the coefficient.

Example 1: Simplify the given expression.

$$-2x^2 + 3x - 5 - 4x^2 - 7x + 10$$

*Remember the sign of the term is the sign that is *in front* of it.

Rewrite the expression to prepare to collect like terms.

$$-2x^2 + 3x - 5 - 4x^2 - 7x + 10$$

$$-2x^2 - 4x^2 + 3x - 7x - 5 + 10$$

Collect the like terms.

The coefficients of the x^2-terms are -2 and -4.

$$-2-4=-2+(-4)=-6$$

The coefficients of the x-terms are $+3$ and -7.

$$+3-7=+3+(-7)=-4$$

$$\underbrace{-2x^2-4x^2}_{-6x^2}+\underbrace{3x-7x}_{-4x}\underbrace{-5+10}_{+5}$$

The expression simplifies to $-6x^2-4x+5$.

Example 2: Simplify the following polynomial expression.

$$5x^3-8+2x^2-9x+10x^3-3-6x^2$$

Rewrite the expression to prepare to collect like terms.

$$5x^3-8+2x^2-9x+10x^3-3-6x^2$$

$$5x^3+10x^3+2x^2-6x^2-9x-8-3$$

Collect the like terms.

The coefficients of the x^3-terms are 5 and $+10$.

$$5+10=15$$

The coefficients of the x^2-terms are $+2$ and -6.

$$+2-6=+2+(-6)=-4$$

The coefficient of the x-term is -9.

$$\underbrace{5x^3+10x^3}_{15x^3}+\underbrace{2x^2-6x^2}_{-4x^2}\underbrace{-9x}_{-9x}\underbrace{-8-3}_{-11}$$

The expression simplifies to $15x^3-4x^2-9x-11$.

Multiplying a Polynomial by a Monomial

term – A term is a part of an algebraic expression that consists of a constant multiplier and one or more variables raised to a power.

- There are three terms in the polynomial $4x^2+3x+1$.

polynomial – A polynomial is an algebraic expression that is a combination of two or more terms.

- Example: $4x^2+3x+1$

monomial – A monomial is an algebraic expression consisting of only one term.

- Example: $4x^2$

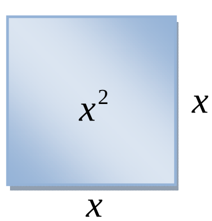
constant – A constant is a numeric term only.

- Example: 1

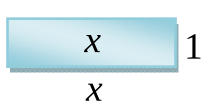
coefficient – The coefficient of a variable term is the number that is in front of the variable. The sign in front of the number is included as part of the coefficient.

- Example: In the polynomial $4x^2+3x+1$, the coefficient of x^2 is 4 and the coefficient of x is 3.

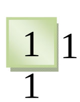
Examine the models and their meanings shown below.



$\rightarrow \quad x \bullet x = x^2$



$\rightarrow \quad x \bullet 1 = x$

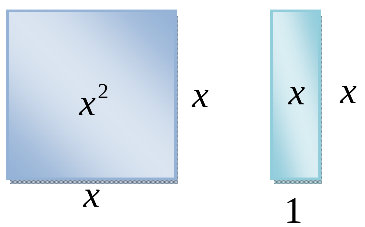


$\rightarrow \quad 1 \bullet 1 = 1$

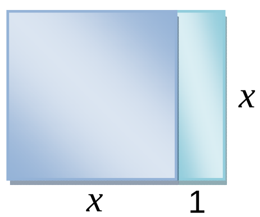
To multiply a polynomial by a monomial, each term of the polynomial is multiplied by the monomial by applying the distributive property.

First we'll use models to show that $x(x+1) = x^2 + x$.

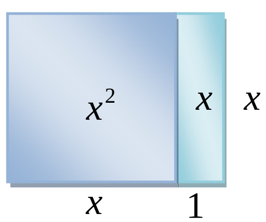
We will work backwards. We use an x^2-tile and an x-tile to represent $x^2 + x$.



Slide the tiles together to form a rectangle.

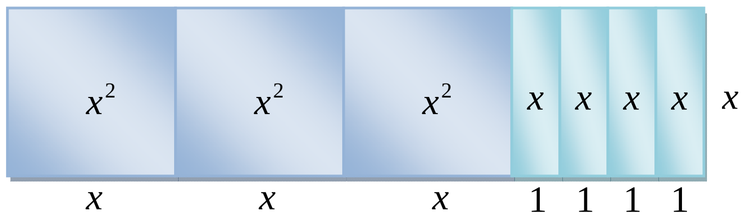


In a rectangle, *area* = *base* × *height*, thus $(x+1) \bullet x = x^2 + x$

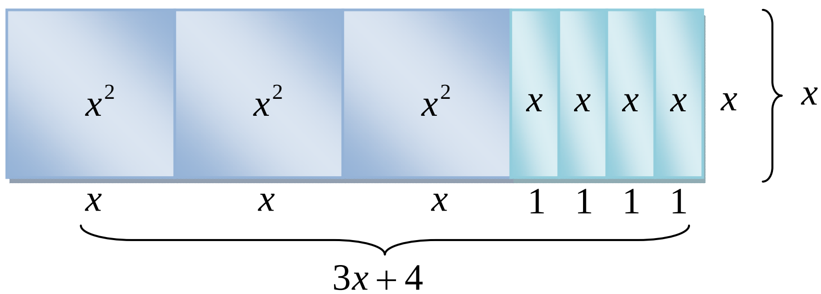


This expression can be written as $x(x+1) = x^2 + x$

Example 1: Write an algebraic statement that is represented by the model. Base the statement on the idea of *area* of a rectangle.



Write an expression to represent the length and width of the complete rectangle.



The base (b) of the rectangle is $3x+4$.

The height (h) of the rectangle is x.

The area (A) of the rectangle is

$$1x^2+1x^2+1x^2+1x+1x+1x+1x$$

which simplifies to $3x^2+4x$.

Thus, $(3x+4)\bullet x=3x^2+4x.$

This can also be written as follows: $x(3x+4)=3x^2+4x$.

Example 2: Without using models, multiply $5x(2x+3)$. Apply the distributive property.

$$\begin{aligned} 5x(2x+3) &= 5x(2x+3) \\ &= 5x(2x)+5x(3) \qquad *5\cdot 2\cdot x\cdot x+5\cdot 3\cdot x \\ &= 10x^2+15x \end{aligned}$$

Thus, $5x(2x+3)=10x^2+15x$

Example 3: Multiply $2z(3z^2-4)$.

Apply the distributive property.

$$\begin{aligned} 2z(3z^2-4) &= 2z(3z^2)-2z(4) \\ &= 2z(3z^2)-2z(4) \qquad *2\cdot 3\cdot z\cdot z\cdot z-2\cdot 4\cdot z \\ &= 6z^3-8z \end{aligned}$$

Thus, $2z(3z^2-4)=6z^3-8z$

More Examples of simplifying:

Example 4:

$3x + 4 + 8x$	$3x$ and $8x$ are like terms, combine them
$3x + 8x + 4$	
$(3 + 8)x + 4$	
$11x + 4$	This is in simplest form.

Example 5:

$6(b + 3) + 7b$	Apply the distributive property to eliminate the parentheses.
$6b + 6(3) + 7b$	Simplfy.
$6b + 18 + 7b$	
$6b + 7b + 18$	Combine 6b and 7b as they are like terms.
$(6 + 7)b + 18$	
$13b + 18$	

Example 6:

$2(x + y) + 3(2x + 3y)$	Apply the distributive property.
$2x + 2y + 3(2x) + 3(3y)$	Simplify.
$2x + 2y + 6x + 9y$	
$2x + 6x + 2y + 9y$	Combine like terms.
$(2 + 6)x + (2 + 9)y$	
$8x + 11y$	

Example 7:

$7(x^2 + 2y) - 5x^2$	Apply the distributive property.
$7x^2 + 7(2y) - 5x^2$	Simplify
$7x^2 + 14y - 5x^2$	
$7x^2 - 5x^2 + 14y$	Combine like terms.
$(7 - 5)x^2 + 14y$	
$2x^2 + 14y$	

*Notice that the exponent (2) did not change. When combining like terms the exponent stays the same.

Review of Basic Equations

This section of the unit is a review one-step and two-step equations.

Remember, to solve an equation involving more than one operation, perform the order of operations IN REVERSE to solve for the unknown variable.

Example #1: Solve $y-6=-21$ for *y*.

$$\begin{array}{l} y\cancel{-6}=-21 \\ \underline{\ \ \cancel{+6}\quad +6} \\ y\qquad=-15 \end{array}$$

Add 6 to both sides of the equation.

Therefore, $y=-15$.

✓Check the answer by replacing *y* with –15 in the original equation.

$y-6=-21$

$-15-6=-21$

$-21=-21$ *True*✓

Example #2: Solve $\frac{-m}{3}=5$ for *m*.

*Remember, to solve the equation, isolate *m* on one side.

$\frac{-m}{3}=5$

$\frac{-1m}{3}=5$ $\quad$ $-m$ can be written as $-1m$

$-\frac{1}{3}m=5$ $\quad$ $\frac{-1m}{3}$ can be written as $-\frac{1}{3}m$

$\cancel{-3}\left(-\frac{\cancel{1}}{\cancel{3}}\right)m=-3(5)$ $\quad$ Multiply both sides by -3, the reciprocal of $-\frac{1}{3}$.

$1m=-15$

$m=-15$ $\quad$ $1m$ is the same as just *m.*

Therefore, $m = -15$.

✓Check the answer by replacing *m* with –15 in the original equation.

$\frac{-m}{3}=5$

$\frac{-(-15)}{3}=5$

$\frac{15}{3}=5$

$5=5$ *True*✓

The next few examples will involve combining the two processes from above in one equation.

*Remember that to solve an equation, perform the order of operations in REVERSE ORDER, which means:

- Add or subtract first.
- Multiply or divide second.

Example #3: Solve $5x+6=31$ for *x*.

$$\begin{array}{ll} 5x\cancel{+6} & =31 \\ \underline{\quad\cancel{-6}} & \underline{-6} \\ 5x & =25 \\ x & =5 \end{array}$$

Subtract 6 from both sides of the equation.

Divide both sides by 5.

Therefore, *x* = 5.

✓Check the answer by replacing *x* with 5 in the original equation.

$5x+6=31$

$5(5)+6=31$

$25+6=31$

$31=31$ *True*✓

Example #4: Solve for $-6z-18=-132$ for *z*.

$$\begin{array}{ll} -6z\cancel{-18} & =-132 \\ \underline{\quad\cancel{+18}} & \underline{+18} \\ -6z & =-114 \\ z & =\ 19 \end{array}$$

Add 18 to both sides of the equation.

$(-132+18=-114)$

Divide both sides by -6.

$(-114\div-6=+19)$

Therefore, *z* = 19.

✓Check the answer by replacing *z* with 19 in the original equation.

$-6z-18=-132$

$-6(19)-18=-132$

$-114-18=-132$

$-132=-132$ *true*✓

Solving Multi-Step Equations

Now, let's take a look at solving equations with variables on both sides of the equals sign.

Example 1: Solve $8x + 5 = 2x – 16$ for x.

Step #1: Move the variables (with coefficients) to one side and the numbers (with no variable "attached") to the other side. Use algebra to justify the adjustments.

$$\begin{array}{l} 8x + 5 = \cancel{2x} - 16 \\ \underline{-2x \qquad \cancel{-2x}} \\ 6x \cancel{+5} = -16 \end{array}$$

Subtract 2x from both sides of the equation.
$(8x - 2x = 6x \qquad 2x - 2x = 0)$

$$\begin{array}{l} \underline{\quad \cancel{-5} \quad -5} \\ 6x = -21 \end{array}$$

Subtract 5 from both sides.
$(-16 - 5 = -16 + -5 = -21)$

Step #2: Divide both sides by 6 to solve for the unknown.

$$\frac{\cancel{6}x}{\cancel{6}} = \frac{-21}{6}$$

$$x = -3.5$$

Divide both sides by 6.

$$\frac{-21}{6} = \frac{-7}{2} = -3\frac{1}{2} = -3.5$$

Therefore, $x = –3.5$.

✓Check the answer by replacing x with –3.5 in the original equation.

$$8x + 5 = 2x - 16$$
$$8(-3.5) + 5 = 2(-3.5) - 16$$
$$-28 + 5 = -7 - 16$$
$$-23 = -23 \textit{ true}✓$$

To extend the process of solving equations, there may be times when the distributive property is used to eliminate any parentheses. Then, like terms may be combined and the equation can be solved.

Example #2: Solve $5(d+4)=7(d-2)$ for *d*.

Step #1: Eliminate the parentheses by using the distributive property on each of the quantities.

$$5(d+4)=7(d-2)$$
$$5d+20=7d-14$$

$$5(d+4)=5(d)+5(4)=5d+20$$

$$7(d-2)=7(d)-7(2)=7d-14$$

Step #2: Move the variables (with coefficients) to one side and the numbers (with no variables "attached") to the other side.

$$\begin{array}{rcl} \cancel{5d}+20 & = & 7d-14 \\ \cancel{-5d} & & -5d \\ \hline 20 & = & 2d\cancel{-14} \\ +14 & & \cancel{+14} \\ 34 & = & 2d \end{array}$$

Step #3: Divide both sides by 2 to solve for the unknown.

$$\frac{34}{2}=\frac{2d}{2}$$
$$17=d$$
$$d=17$$

Therefore, *d* = 17.

✓Check the answer by replacing d with 17 in the original equation.

$$\begin{aligned}&5(d+4)=7(d-2)\\&5(17+4)=7(17-2)\\&5(21)=7(15)\\&105=105\ \textit{true}✓\end{aligned}$$

Example #3: Solve $3(x-6)+2=4(x+2)-21$ for x.

Step #1: Eliminate the parentheses by using the distributive property on each of the quantities.

$$\begin{aligned}&3(x-6)+2=4(x+2)-21\\&3x-18+2=4x+8-21\end{aligned}$$

$$\begin{aligned}&3(x-6)=3(x)-3(6)=3x-18\\&4(x+2)=4(x)+4(2)=4x+8\end{aligned}$$

Step #2: Combine any like terms on either side of the equals sign. In this case, combine (–18 + 2) on the left and (+8 – 21) on the right.

$$\begin{aligned}&3x-18+2=4x+8-21\\&3x-16=4x-13\end{aligned}$$

$$-18+2=-16 \qquad +8-21=-13$$

Step #3: Move the variables (with coefficients) to one side and the numbers (with no variables "attached") to the other side.

$$\begin{aligned}\cancel{3x}-16&=4x-13\\ \underline{\cancel{-3x}\qquad\ }&\underline{\ -3x\qquad\ }\\ -16&=1x\cancel{-13}\\ +13\quad&\quad\ \cancel{+13}\\ -3&=1x\\ x&=-3\end{aligned}$$

Therefore, $x = –3$.

✓Check the answer by replacing x with –3 in the original equation.

$$
\begin{aligned}
&3(x-6)+2=4(x+2)-21 \\
&3(-3-6)+2=4(-3+2)-21 \\
&3(-9)+2=4(-1)-21 \\
&-27+2=-4-21 \\
&-25=-25 \ \textit{true}✓
\end{aligned}
$$

Write an Equation for a Word Problem

Example: Brooke and Brianna compared projects for the local county fair. Brooke raised five goats and Brianna raised rabbits. The number of rabbits Brianna raised divided by five is two less than the number of goats raised by Brooke. How many rabbits did Brianna raise?

What is given?

- Brooke raised five goats.
- The number of rabbits Brianna raised divided by five is two less than the number of goats Brooke raised.

What is asked?

- How many rabbits did Brianna raise?

Let r equal the number of rabbits Brianna raised.

Then, translate the problem into an equation using the variable r.

To solve:

The number of rabbits Brianna raised divided by five is two less than the number of goats raised by Brooke.

$$r \qquad \div 5 \qquad = \qquad 5-2$$

$\frac{r}{5}=5-2$	Note that $r \div 5$ can be expressed as $\frac{r}{5}$.
$\frac{r}{5}=3$	Simplify ... $5-2=3$
$\cancel{5}\times\frac{r}{\cancel{5}}=3\times 5$	Multiply each side by 5 to undo the division by 5.
$r=15$	Simplify ... $3\times 5=15$

Therefore, Brianna raised 15 rabbits.

To check, read the problem again to see if the answer makes sense.

The number of rabbits Brianna raised (15) *divided by five* (5) *is two less than the number of goats raised by Brooke* (5 – 2).

$$15 \div 5 = 5 - 2$$
$$3 = 3 \checkmark$$