INTEGERS AND COORDINATE GEOMETRY

This unit begins with multiplying and dividing integers. Next, graphing in the coordinate plane is reviewed and expanded to include various types of transformations. Four types of translations will be examined including translations (slides), reflections (flips), rotations (turns), and dilations which are enlargements or reductions of a given figure. The lesson ends with learning how to solve problems by working backwards, a very useful problem-solving strategy.

Multiply and Divide Integers

Graphing in the Coordinate Plane

Analyze Shapes Using Coordinate Geometry

Types of Transformations

Translations

Reflections and Symmetry

Rotations

Dilations

Problem Solving – Working Backwards

Multiply and Divide Integers

Introduction to Multiplying and Dividng Integers

To multiply integers, we will first look at a number pattern.

7 ×	4	= 28
7 ×	3	= 21
7 ×	2	= 14
7 ×	1	= 7
7 ×	0	= 0
7 ×	(−1)	= −7
7 ×	(−2)	= −14
7 ×	(−3)	= −21

In the pattern, each time 7 is multiplied by one less number the resulting answer is 7 less.

At 7×0, 0 the product is 7 less than 7.
At $7 \times (-1)$, −7 the product is 7 less than 0.
At $7 \times (-2)$, −14 the product is 7 less than −7.

…a*nd the pattern continues on.*

When multiplying integers with different signs, multiply the numbers and the answer will be negative.

$$7 \times (-3) = -21 \qquad -2 \times 7 = -14$$

Now we will examine a pattern that shows how a "negative number times a negative number" becomes a positive number.

−3 × 7	=	−21
−3 × 6	=	−18
−3 × 5	=	−15
−3 × 4	=	−12
−3 × 3	=	−9
−3 × 2	=	−6
−3 × 1	=	−3
−3 × 0	=	0
−3 × (−1)	=	3
−3 × (−2)	=	6
−3 × (−3)	=	9
−3 × (−4)	=	12
−3 × (−5)	=	15

In the pattern, each time 3 is multiplied by one less number the resulting answer is 3 more.

At -3×0, 0 the product is 3 more than −3.
At $-3 \times (-1)$, 3 the product is 3 more than 0.
At $-3 \times (-2)$, 6 the product is 3 more than 3.
At $-3 \times (-3)$, 9 the product is 3 more than 6.

…a*nd the pattern continues on.*

> **When multiplying integers with same signs, multiply and the answer will be positive.**
>
> $$-3 \times -5 = 15 \qquad\qquad 4 \times 3 = 12$$

Introduction to Dividing Integers

To divide integers, we will look at the multiplication, and then write a **related** division statement.

$$-7 \times 8 = -56, \quad \text{therefore } -56 \div 8 = -7$$
$$-7 \times -8 = 56, \quad \text{therefore } 56 \div (-8) = -7$$

> **When dividing integers with different signs, divide and the answer will be negative.**
>
> $$48 \div (-6) = -8 \qquad -72 \div 9 = -8$$

$$7 \times 8 = 56, \quad \text{therefore } 56 \div 8 = 7$$
$$7 \times -8 = -56, \quad \text{therefore } -56 \div (-8) = 7$$

> **When dividing integers with same signs, divide and the answer will be positive.**
>
> $$-64 \div (-8) = 8 \qquad 70 \div 10 = 7$$

Multiplying and Dividing Integers

The rules for multiplying and dividing integers are a little easier to remember because there are only two rules.

Multiplying & Dividing

Same sign: If the numbers have the same sign, the answer will be **positive**.

Example 1: $(-4)(-5) = 20$ $(-35) \div (-7) = 5$

Example 2: $(6)(3) = 18$ $(60) \div (10) = 6$

Different signs: If the numbers have different signs, the answer will be **negative**.

Example 3: $(-4)(8) = -32$ $(-99) \div (9) = -11$

*When multiplying more than one number, it may be helpful to remember the following rule in determining the sign of your answer:

Even number of negative signs results in a **positive** answer.

Odd number of negative signs results in a **negative** answer.

Example 4: (–4)(–3)(5)(–2)

Since there are 3 negative signs, the answer to this will be negative.

Let's check it out by multiplying from left to right.

(–4)(–3)(5)(–2) $\left[(-4)(-3) = 12 \right]$

(12)(5)(–2) $\left[(12)(5) = 60 \right]$

(60)(–2)

–120

Example 5: (–2)(–4)(–3)(–2)

Since there are 4 negative signs, the answer to this will be positive.

Let's check it out by multiplying from left to right.

(–2)(–4)(–3)(–2) $\left[(-2)(-4) = 8 \right]$

(8)(–3)(–2) $\left[(8)(-3) = -24 \right]$

(–24)(–2)

48

Graphing in the Coordinate Plane

Quadrant I

Ordered pair - An ordered pair is a pair of numbers that represent the location in a grid. In the coordinate plane, an ordered pair is the x and y-coordinate of a point represented as (x, y).

Origin - The origin is the beginning point in the coordinate plane. It is the point where the x-axis and the y-axis intersect. The coordinates of the origin are $(0,0)$.

Quadrants - Quadrants are the four regions of the coordinate plane. The x and y-axis divide the coordinate plane into four quadrants.

Axes - Axes is the plural of axis. There are two axes in a coordinate plane. The x-axis is the horizontal axis and is a number line. The y-axis is the vertical axis and is a vertical number line.

Coordinates - Coordinates are the components of an ordered pair. In an ordered pair, the first number is called the x-coordinate and the second number is called the y-coordinate.

Coordinate plane - The coordinate plane is a numbered grid system that has a horizontal and a vertical number line in the center. These lines are perpendicular to each other and meet at the origin, the point considered the starting point of the system. The origin is numbered as $(0, 0)$.

In a **coordinate plane**, points may be located by **plotting** them. The coordinate plane is divided into **four quadrants** by the **x–axis** and the **y-axis**. The starting point, the **origin**, is the center, or point where the x and y-axis intersect (cross).
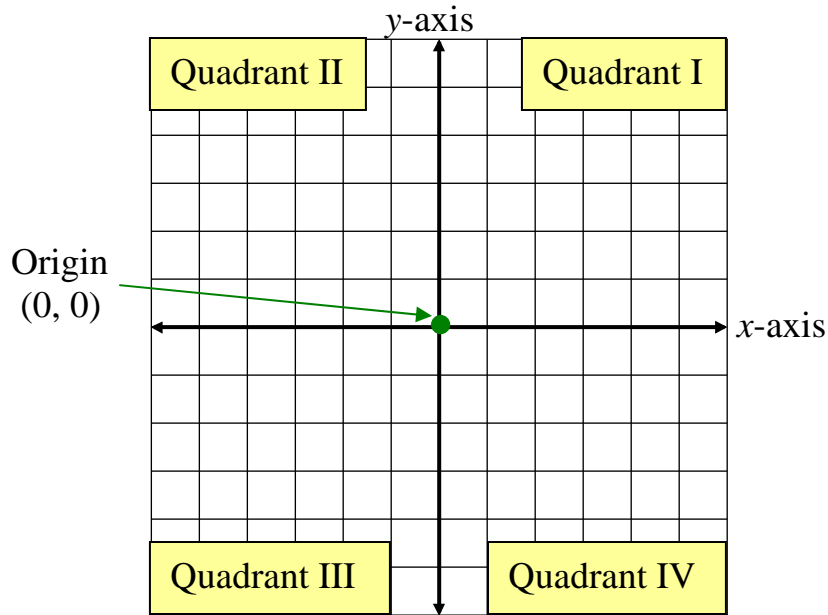
A point is designated by both an ***x*-coordinate** and a ***y*-coordinate.** The origin's coordinates are (0, 0). The *x*-coordinate is the first number and the *y*-coordinate is the second number.

The ***x*-coordinate** is how far you count **right or left** of the origin. The ***y*-coordinate** is how far you then count **up or down**. A point's location is written as an **ordered pair (*x, y*)**.

In this grid, each space represents one unit.

Plot (5, 4)

When plotting points, start at the origin. Count right if the *x*-coordinate is positive, left if it is negative. Then count up if the *y*-coordinate is positive, count down if it is negative.

To plot (5,4) start at the origin, count 5 units to the right, and then count 4 units up.

y-axis

Quadrant I

x-axis
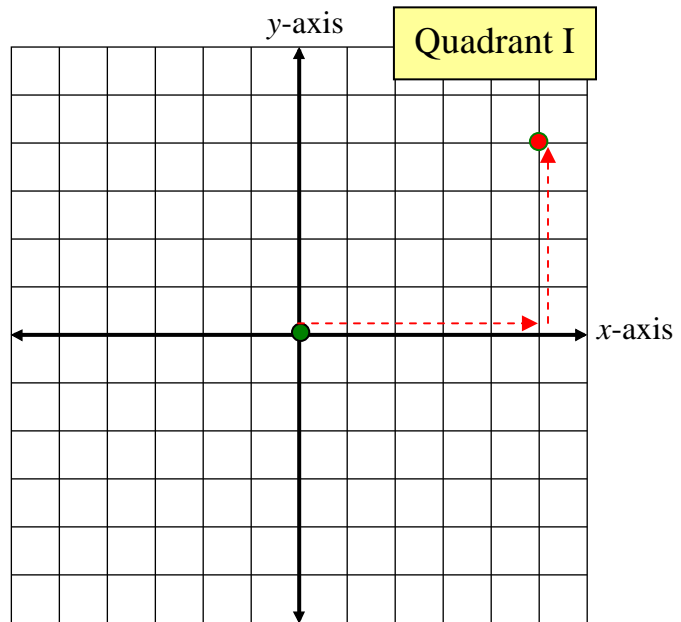
Quadrants II, III, and IV

In these grids each space represents one unit. The starting point is the origin (0,0).

Plot (–4, 2)

When plotting points, start at the origin. Count right if the *x*-coordinate is positive, left if it is negative. Then count up if the *y*-coordinate is positive, count down if it is negative.

To plot (–4, 2) starting at the origin, count 4 units to the left, and then count 2 units up.

Quadrant II

y-axis

x-axis

(0, 0)
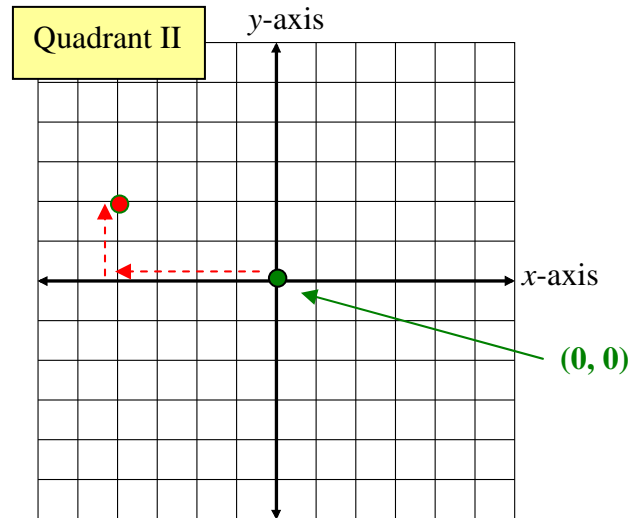
Plot (–5, –3)

To plot (–5, 3), start at the origin, count 5 units to the left, and then count 3 units down.

y-axis

(0, 0)

x-axis

Quadrant III
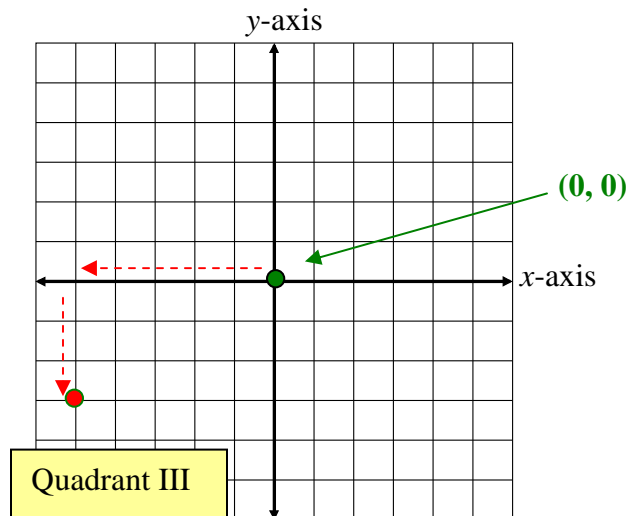
Plot (1, –3)

To plot (1, –3), start at the origin, count 1 unit to the right, and then count 3 units down.

y-axis

(0, 0)

x-axis

Quadrant IV
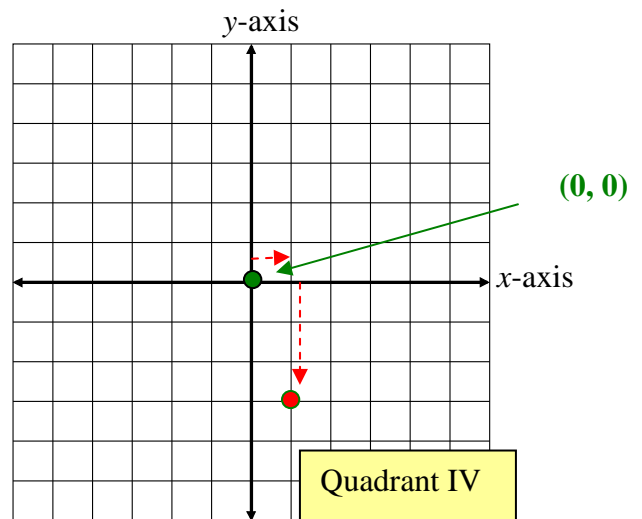
Analyze Shapes Using Coordinate Geometry

Now let's consider some ideas about graphing in the coordinate plane that are more closely related to geometry.

First we will review a few shapes and their properties.

Square - 4 sides congruent, 4 right angles
Rectangle - 4 right angles, 2 pairs of parallel and congruent sides.
Parallelogram - opposite sides are parallel and congruent.
Trapezoid - one pair of parallel sides.
Equilateral triangle - all three sides congruent, all angles 60˚.

If we take certain points on a coordinate plane and connect them to a geometric idea, we should be able to predict a missing point.

Example 1: Consider a rectangle: $A(1,2), B(1,6), C(3,2),$ and $D(x,y)$. Find the coordinates for point D that completes the shape into a rectangle.

Plot the points and take a look. We can find the missing ordered pair!
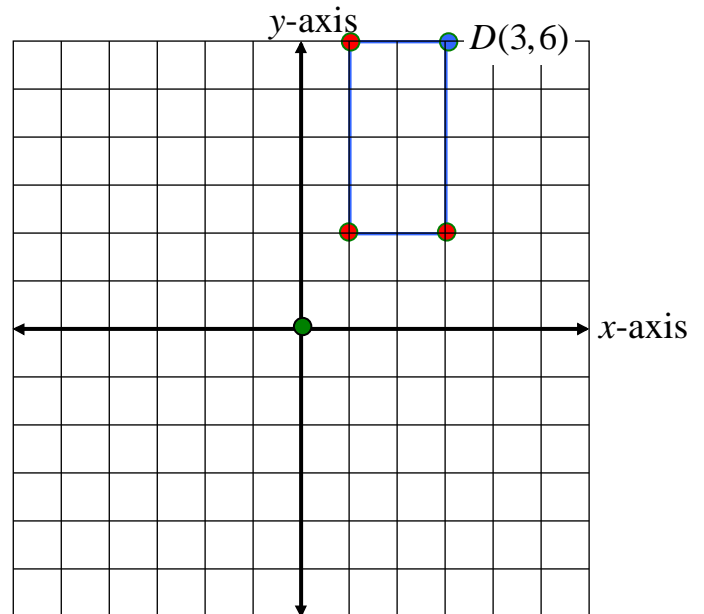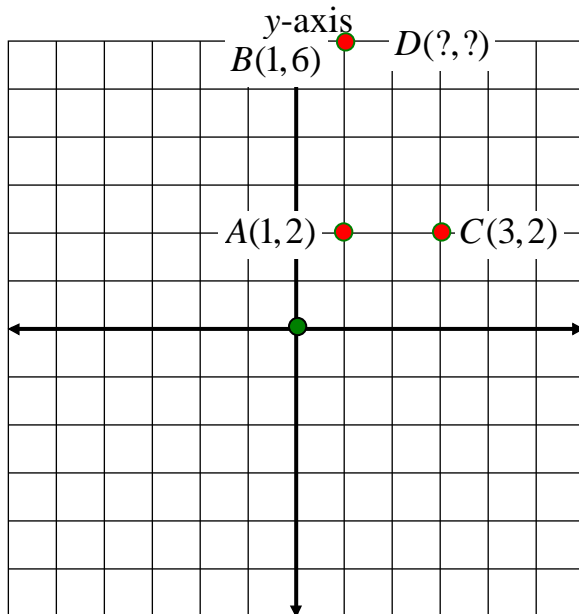
Make observations keeping the properties of a rectangle in mind.

-Since point D is located directly above point C, both points will have the same **x-coordinate**, **3**.

-Since point D is located directly across from point B, both points will have the same **y-coordinate**, **6**.

Thus, the missing point must be (3, 6).

Example 2: Consider the parallelogram $E(-1,-3), F(0,2), G(3,-3),$ and $H(x, y)$.
Find the coordinates for a point that completes the shape into a parallelogram.

Plot the points and take a look to find the missing ordered pair.
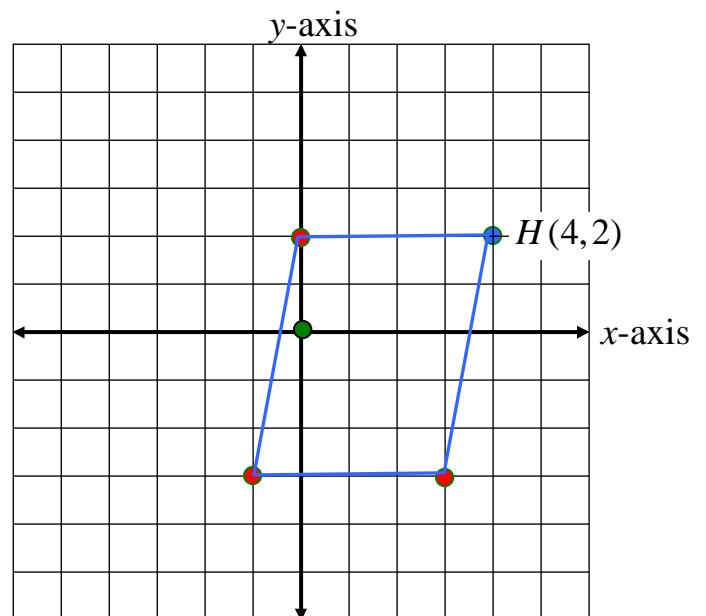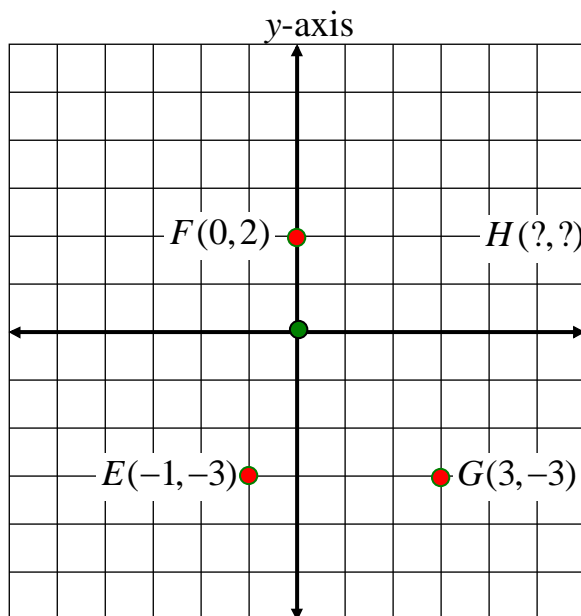
Make observations keeping the properties of a parallelogram in mind.

-Since point F is located one unit to the right of point E, the **x-coordinate** of point H will be in a similar position with point G; that is, one unit to the right of point G, $(3+1)$, **4**.

-Since point H will be located on the same line as point F, the **y-coordinate** will be the same as point F, **2**.
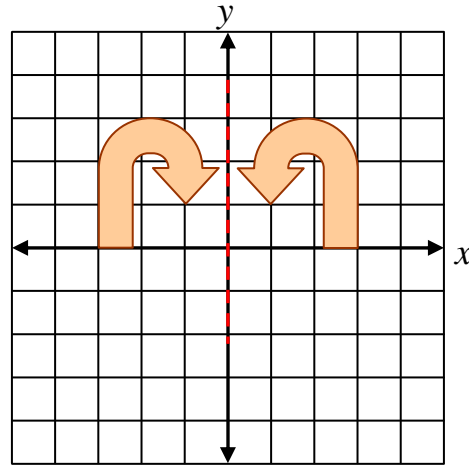
The missing point must be (4, 2).

Types of Transformations

A *transformation* is a change made to the size or location of a figure. The new figure formed by the translation is called an *image*.
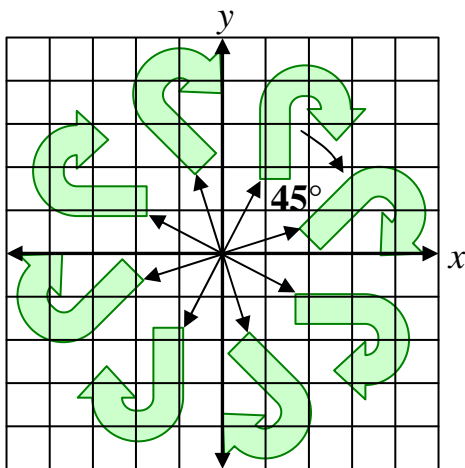
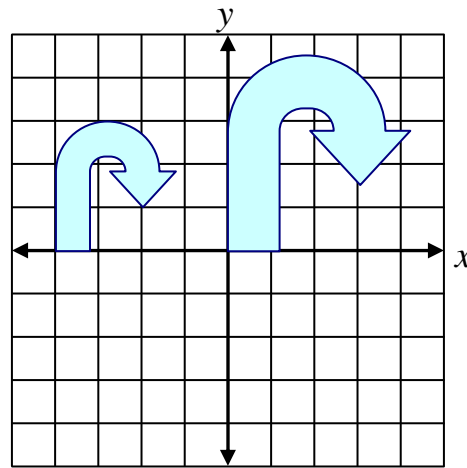Four types of transformations are *translations, reflections, rotations, and dilations.*
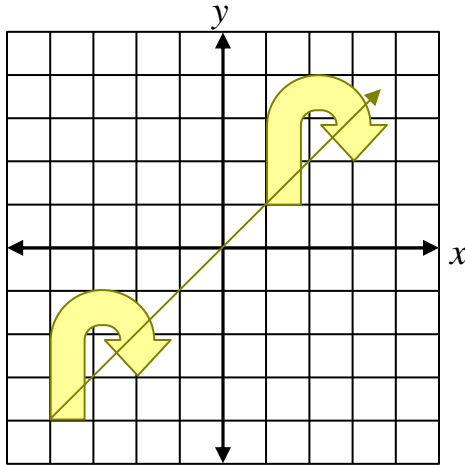
Translation - sliding a figure along a straight line without turning to another location.

Reflection - flipping a figure over a line of reference creating a mirror image.

Rotation - turning a figure around a fixed point called the center of rotation.

The **angle of rotation** is an angle formed by rays drawn from the center of rotation through corresponding points on an original figure and its image.

Dilation - enlarging or reducing a figure.

Translations

A **translation** is a transformation in which each point of the figure moves the same distance in the same direction. A figure and its image are congruent.
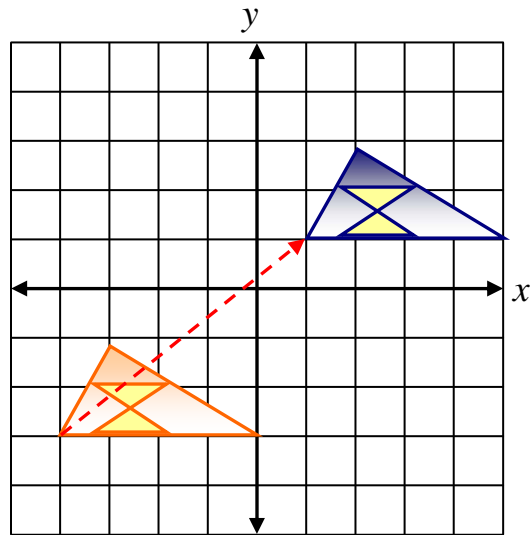
A translation slides a figure along a line without turning.
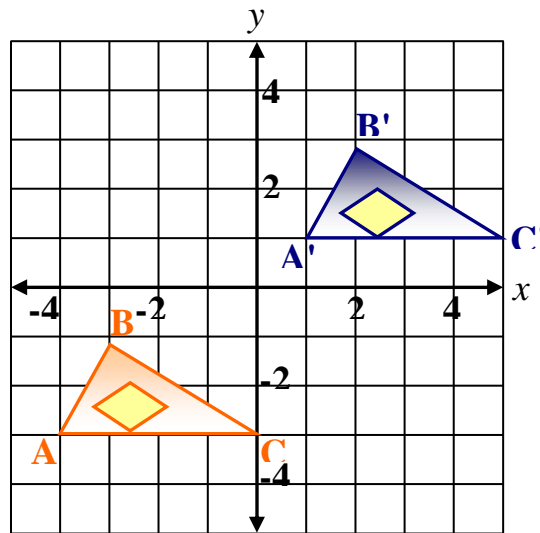
Example 1: Describe the translation in words.

Figure ABC is translated to figure A'B'C'. (A'B'C' is read "A-primed, B-primed, C-primed".) The translation is 5 units to the right and 4 units up.

To check, start at point **A** and count **five** units to the **right**, and then count **four** units **up**. You will end up at point **A'**. Do the same for the other two pairs of corresponding points.

Notation in the Coordinate Plane: The translation of each point (x, y) of a shape can be represented using coordinate notation.

Translation: $(x, y) \rightarrow (x + a, y + b)$

a = how many units a point moves horizontally

b = how many units a point moves vertically
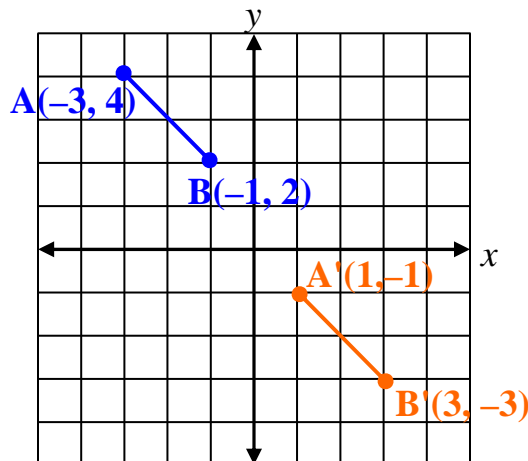
If $a > 0$ (positive), the point moves to the right.

If $a < 0$ (negative), the point moves to the left.

If $b > 0$ (positive), the point moves up.

If $b < 0$ (negative), the point moves down.

Example 2: Given segment AB with vertices A(–3, 4) and B(–1, 2), find the coordinates of the image after the translation $(x, y) \rightarrow (x+4, y-5)$ and draw the image.

Translation :

$(x, y) \quad \rightarrow \quad (x+a, y+b)$

$(x, y) \quad \rightarrow \quad (x+4, y-5)$

$A(-3,4) \rightarrow A'(-3+4, 4-5) = A'(1,-1)$
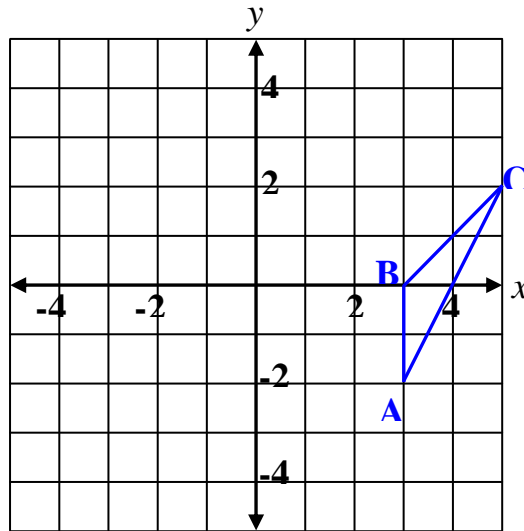
$B(-1,2) \rightarrow B'(-1+4, 2-5) = B'(3,-3)$

The endpoints of the translated segment are $A'(1,-1)$ and $B'(3,-3)$.

Example 3: Draw triangle ABC with vertices of A(3, –2), B(3, 0), and C(5, 2). Then find the coordinates of the vertices of the image after the given translation. Draw the image.

Translation : $(x, y) \rightarrow (x - 7, y + 2)$

Step 1: Graph triangle ABC with vertices of A(3, –2), B(3, 0), and C(5, 2).

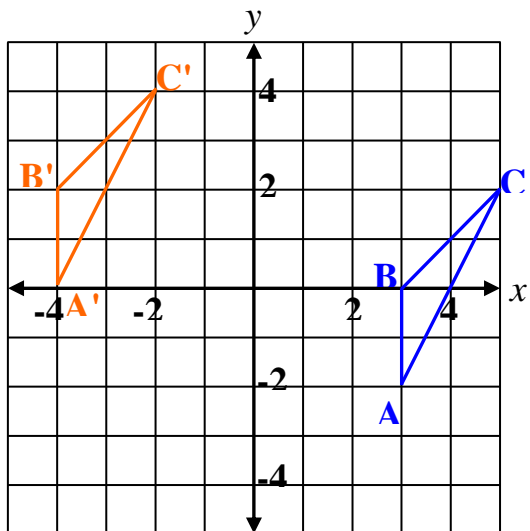*Step 2*: Find the coordinates of the vertices of the image.

Translation : $(x, y) \rightarrow (x - 7, y + 2)$

Subtract 7 from each *x*-coordinate.   Add 2 to each *y*-coordinate.

| Original		Image
$P(x, y)$	$\rightarrow$	$P'(x - 7, y + 2)$
$A(3, -2)$	$\rightarrow$	$A'(-4, 0)$
$B(3, 0)$	$\rightarrow$	$B'(-4, 2)$
$C(5, 2)$	$\rightarrow$	$C'(-2, 4)$

Step 3: Draw the image, A'B'C'.

Each point on triangle ABC moves 7 units to the left and 2 units up to translate to triangle $A'B'C'$.

The vertices of the translated triangle are $A'(-4, 0), B'(-4, 2),$ and $C'(-2, 4).$
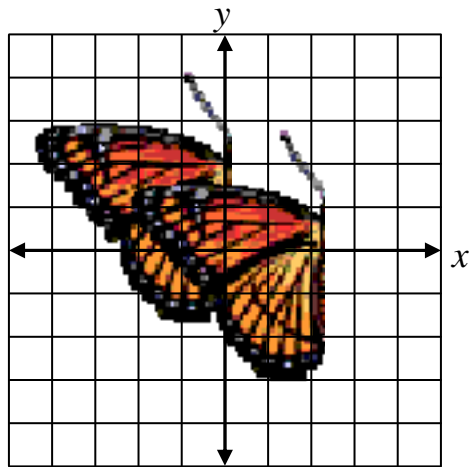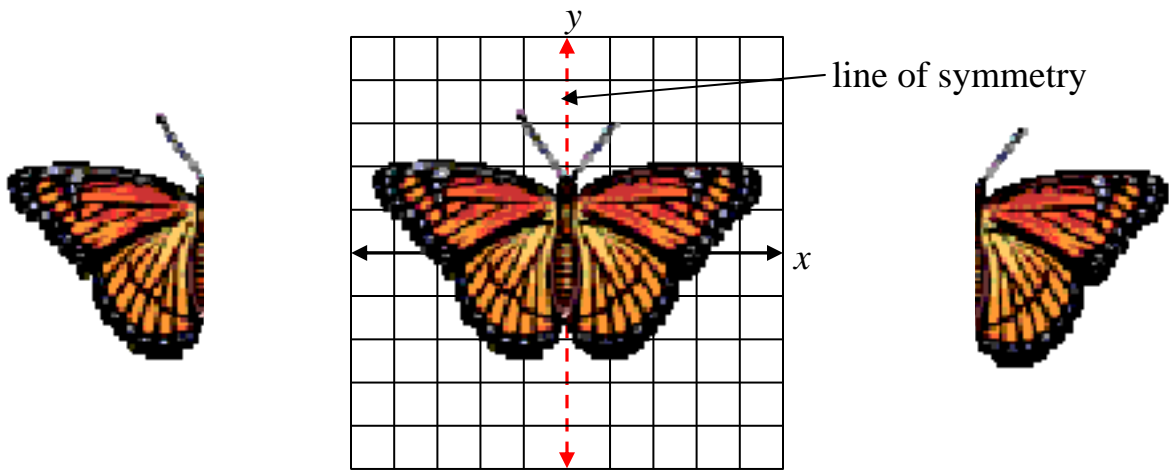
Reflections and Symmetry

A **reflection** is a transformation of a shape in which the shape is reflected or flipped across a line. The line is called the **line of reflection**.

line of reflection
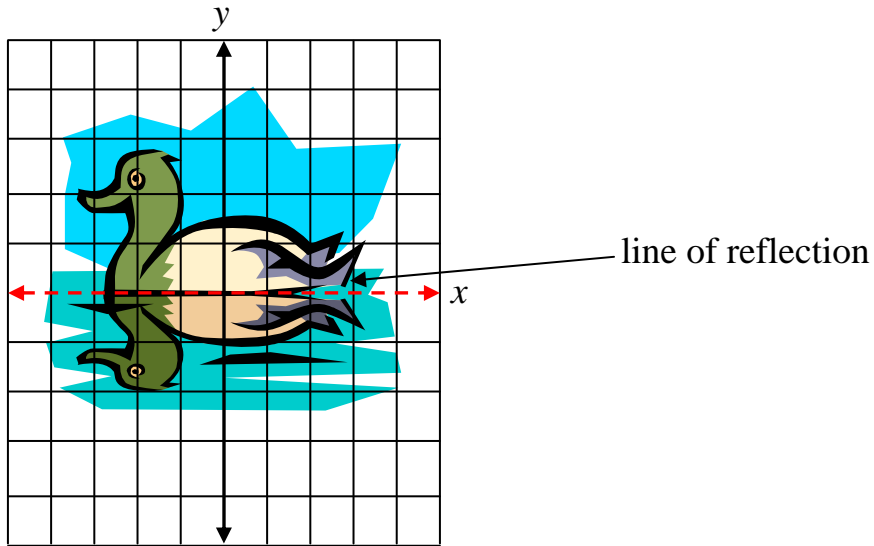
The duck is reflected in a pool of water creating a mirror image.

A figure has *line symmetry* if it can be folded over a line so that one half of the figure matches the other half. The line is called the **line of symmetry**.

The **line of symmetry** divides the figure into two parts that are reflections of each other. The two sides are mirror images of each other. The structure of a butterfly is a natural example where line symmetry enhances its beauty.
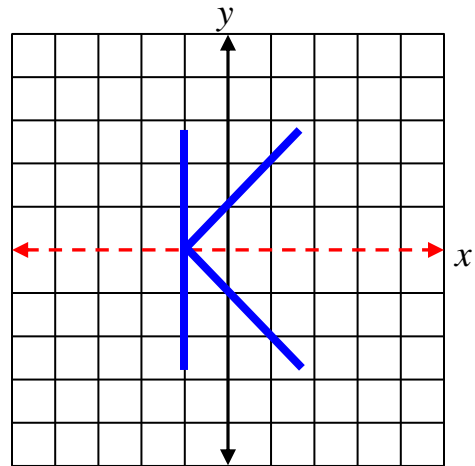
line of symmetry

The left side of the butterfly is symmetrical to the right side. Symmetrical objects have parts that are congruent. The wings are congruent.
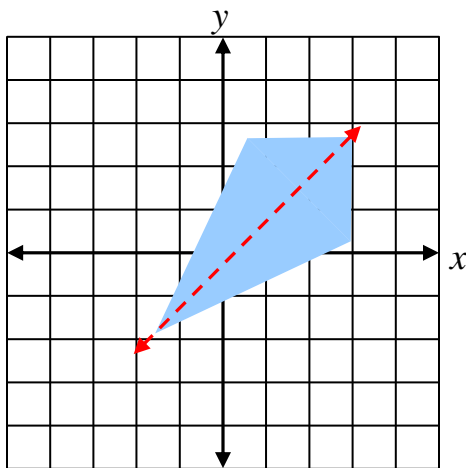
Samples of Line Symmetry
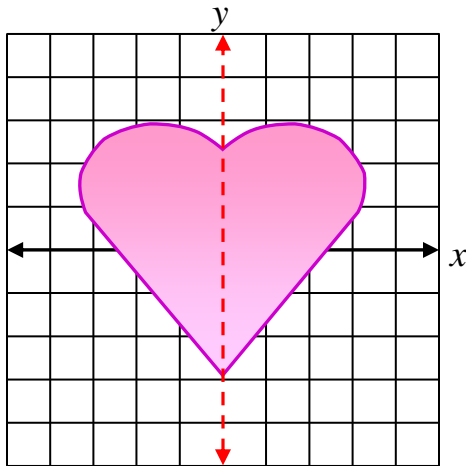
Vertical Line Symmetry

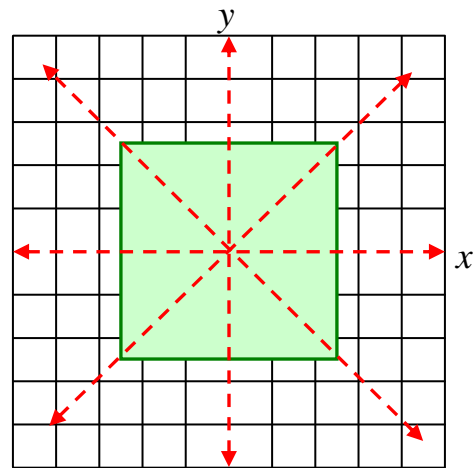The left side of the heart is a reflection of the right side.

Horizontal Line Symmetry

The top part of the K is a reflection of the bottom part.

Diagonal Line Symmetry

The diagonal line cuts the kite into two symmetrical parts.


Four Lines of Symmetry

In this figure, there are four lines of symmetry; that is, four lines that cut the square into two symmetrical parts.

Notation in the Coordinate Plane: Reflections may be described using coordinate notation.

Reflection over the *x*-axis

A reflection over the *x*-axis occurs when the *y*-coordinate of an ordered pair is multiplied by –1. The line of symmetry is the *x*-axis.

Example 1: Reflect segment AB with endpoints A(–3, 4) and B(3, 1) over the *x*-axis.
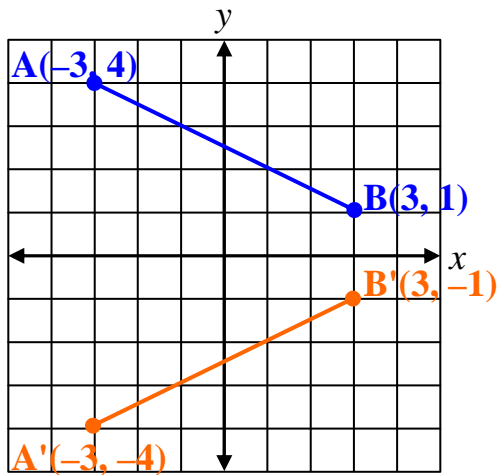
Multiply the y-coordinate by -1.

$$P(x,\ y) \quad \rightarrow \quad P'(x,-y)$$
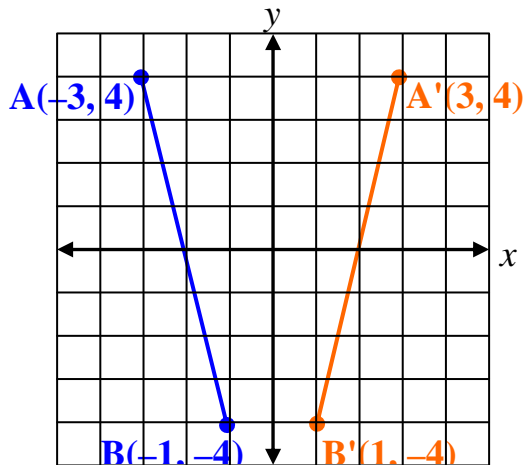
$$A(-3,4) \quad \rightarrow \quad A'(-3,-4)$$

$$B(3,1) \quad \rightarrow \quad B'(3,-1)$$

The endpoints of the reflected segment are $A'(-3,-4)$ and $B'(3,-1)$.

Reflection over the *y*-axis

A reflection over the *y*-axis occurs when the *x*-coordinate of an ordered pair is multiplied by –1. The line of symmetry is the *y*-axis.

Example 2: Reflect segment AB with endpoints A(–3, 4) and B(–1,–4) over the *y*-axis.

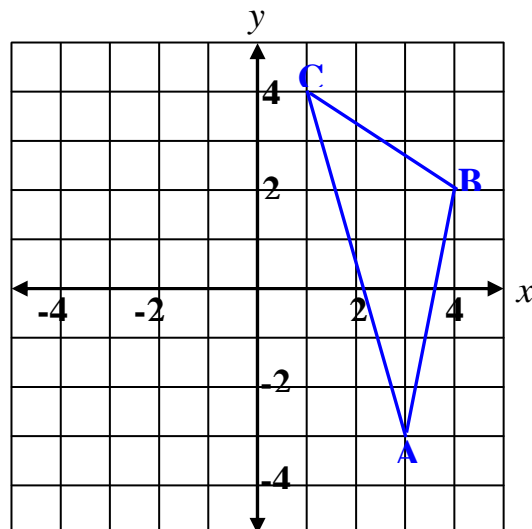Multiply the x - coordinate by -1.

$$P(x, y) \qquad \rightarrow \qquad P'(-x, y)$$
$$A(-3, 4) \qquad \rightarrow \qquad A'(3, 4)$$
$$B(-1, -4) \qquad \rightarrow \qquad B'(1, -4)$$

The endpoints of the reflected segment are $A'(3,4)$ and $B'(1,-4)$.

Example 3: Draw triangle *ABC* with vertices $A(3,-3)$, $B(4,2)$, and $C(1,4)$. Then find the coordinates of the vertices of the image after a reflection over the *y*-axis.
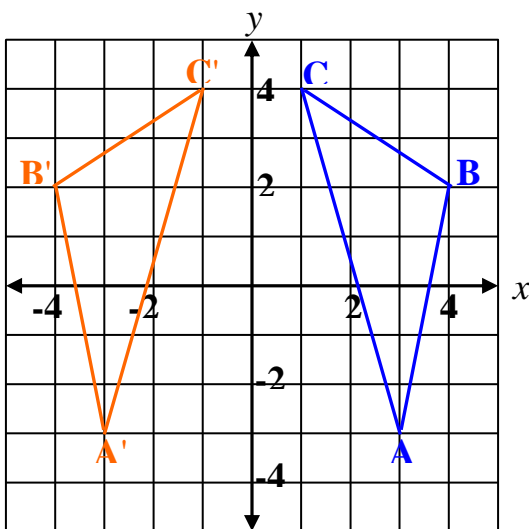
Step 1: Graph triangle ABC.

Step 2: Since the reflection is to made over the *y*-axis, multiply each
x-coordinate by –1 to find the coordinates of the vertices of the image.

| Original		Image
$P(x, y)$	$\rightarrow$	$P'(-x , y)$
$A(3, -3)$	$\rightarrow$	$A'(-3, -3)$
$B(4, 2)$	$\rightarrow$	$B'(-4, 2)$
$C(1, 4)$	$\rightarrow$	$C'(-1, 4)$

Step 3: Draw the image $A'B'C'$.

Triangle $A'B'C'$ is a reflection of triangle ABC. The *y*-axis is the
line of symmetry.

Rotations

A **rotation** is a transformation in which a figure is turned about a fixed point called the **center of rotation**.

The **angle of rotation** is an angle formed by rays drawn from the center of rotation through corresponding points on an original figure and its image. The direction of rotation can be *clockwise* or *counterclockwise*. The figure and its image are congruent.
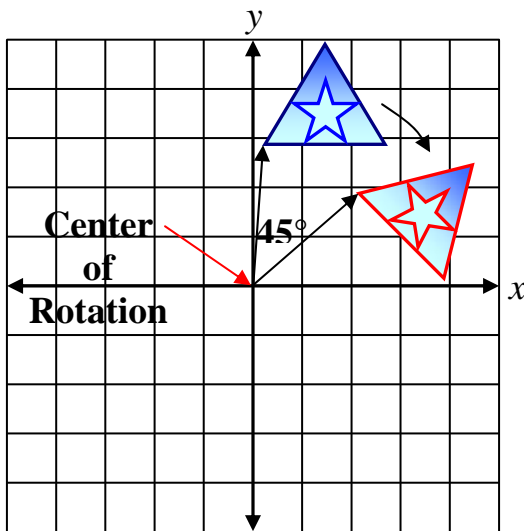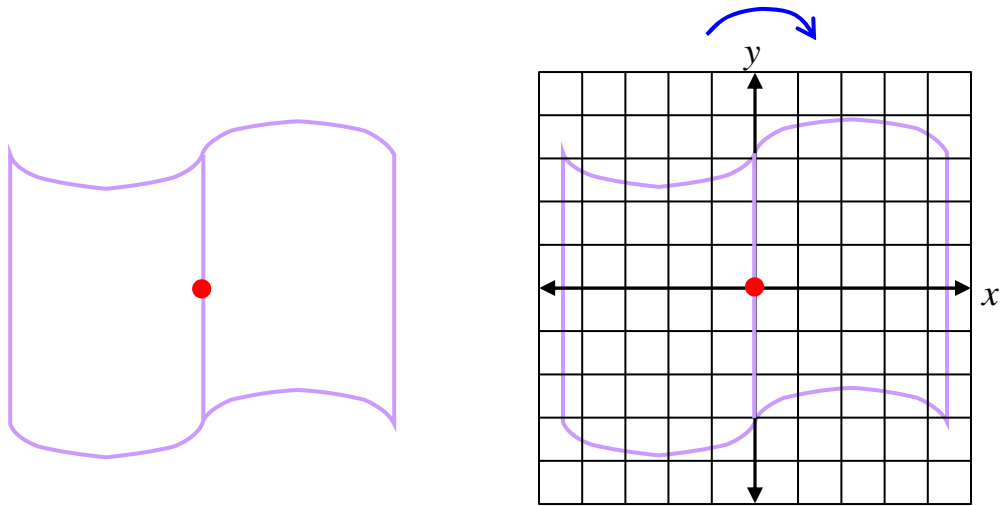

45° clockwise rotation
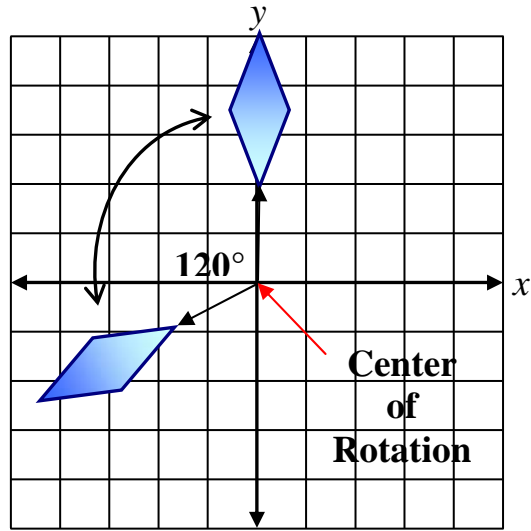
120° counterclockwise rotation

A figure has **rotational symmetry** if the figure can be rotated around some point so that it coincides with itself. The point is the center of rotation, and the amount of rotation must be less than one full turn; that is, less than 360°.

If you turn this figure around the center point, it coincides with itself every rotation of 45°, so it has rotational symmetry.
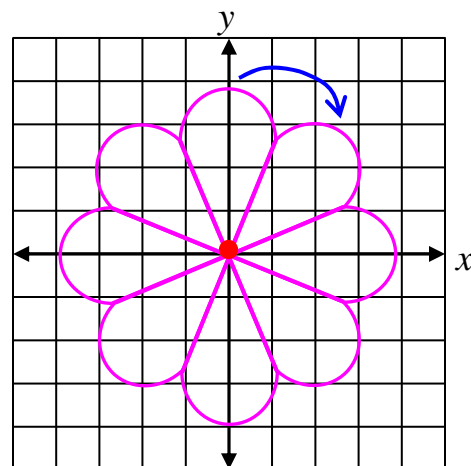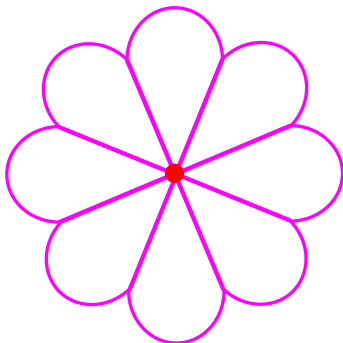
If you turn this figure around the given point of rotation, it coincides with itself every rotation of 180°, so it has rotational symmetry.

Notation in the Coordinate Plane: Rotations of each point (*x, y*) of a figure can be represented using coordinate notation.

Ninety-degree Clockwise Rotation

A 90-degree clockwise rotation occurs about the origin if the coordinates of the ordered pair are switched and the new *y*-coordinate is multiplied by –1.

Example 1: Rotate segment AB with endpoints A(1, 2) and
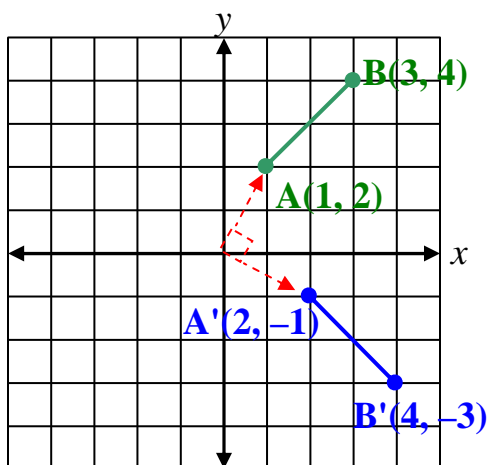B(3,4) about the origin with a 90-degree clockwise rotation.

To rotate segment AB ninety degrees, switch the coordinates, and then
multiply the new *y*-coordinate by –1.

$$P(x, y) \rightarrow P'(y, -x)$$
$$(1,2) \rightarrow (2,-1)$$
$$(3,4) \rightarrow (4,-3)$$

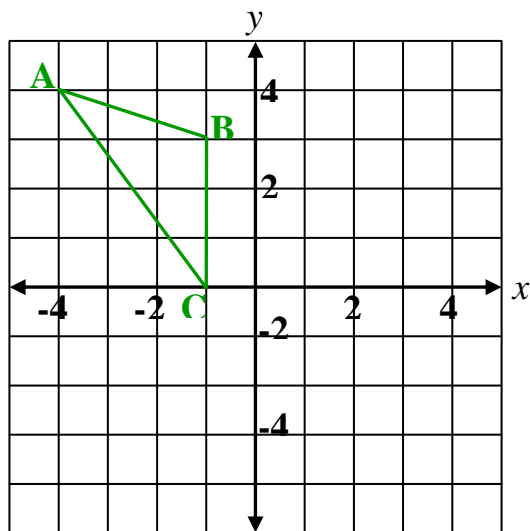Graph the original segment (AB) and its image $(A'B')$.

The endpoints of the rotated segment are $A'(2,-1)$ and $B'(4,-3)$.

Example 2: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0). Then find the coordinates of the vertices of the image after a **90°** clockwise rotation around the origin. Draw the image, A'B'C'.
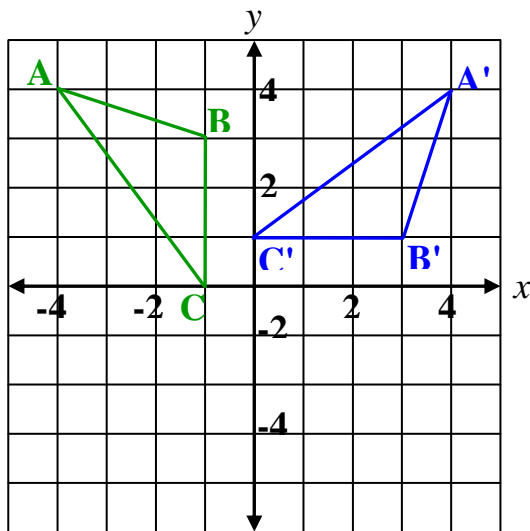
Step 1: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0).

Step 2: Find the coordinates of the vertices of the image. Switch the coordinates and multiply the new *y*-coordinate by –1.

Original		Image
$P(x, y)$	$\rightarrow$	$P'(y, -x)$
$A(-4, 4)$	$\rightarrow$	$A'(4, 4)$
$B(-1, 3)$	$\rightarrow$	$B'(3, 1)$
$C(-1, 0)$	$\rightarrow$	$C'(0, 1)$

Step 3: Draw the image, A'B'C'.

Triangle $A'B'C'$ is a 90-degree clockwise rotation of triangle ABC.

Ninety-degree Counterclockwise Rotation

A 90-degree counterclockwise rotation occurs about the origin if the coordinates of the ordered pair are switched and the new *x*-coordinate is multiplied by –1.

Example 3: Rotate segment AB with endpoints A(1, 2) and B(3,4) about the origin with a 90-degree counterclockwise rotation.

To rotate segment AB ninety degrees, switch the coordinates, and then multiply the new *x*-coordinate by –1.

$$P(x, y) \quad \rightarrow \quad P'(-y, x)$$
$$(1, 2) \quad \rightarrow \quad (-2, 1)$$
$$(3, 4) \quad \rightarrow \quad (-4, 3)$$
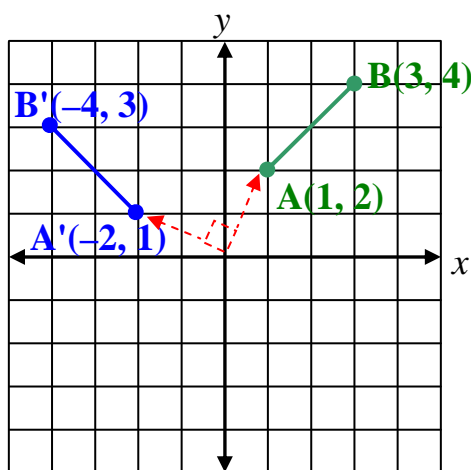
The endpoints of the rotated segment are $A'(-4, 3)$ and $B'(-2, 1)$.

Dilations

A **dilation** is a transformation in which the size is changed but not the shape. A dilation can be an enlargement or a reduction of a figure. The dilation of a figure is similar to the original image.

A dilation has a fixed point that is the **center of dilation**. The figure stretches or shrinks with respect to the **center of dilation**. To find the center of dilation, draw a line that connects each pair of corresponding vertices.

The **scale factor** describes how much a figure is enlarged or reduced. It is the ratio of the side length of the image to the corresponding side length of the original figure. A scale factor can be expressed as a decimal, a fraction, or a percent.
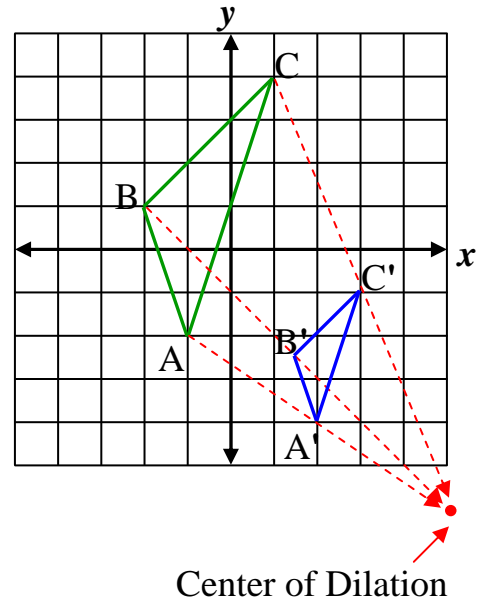
A **dilation** produces an image similar to the original figure.

Multiplying by a scale factor > 1 ⟶ Enlarges a figure

Multiplying by a scale factor < 1 ⟶ Reduces a figure

Center of Dilation
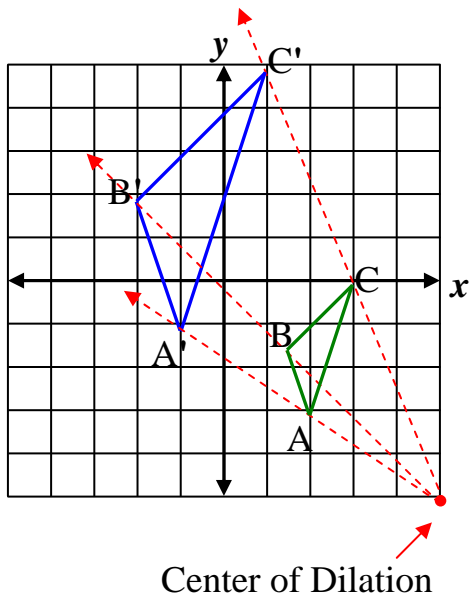
Triangle A'B'C' is a dilation of triangle ABC and has a 100% increase in size.
The *scale factor* is 2.

Center of Dilation

Triangle A'B'C' is a dilation of triangle ABC and has a 100% decrease in size.
The *scale factor* is 0.5.

Example 1: An artist uses a computer program and a coordinate grid to enlarge a design. Find the scale factor of the design.

Step 1: Find the width of the original design.

The width of the original design is the difference of the original design's x- coordinates.

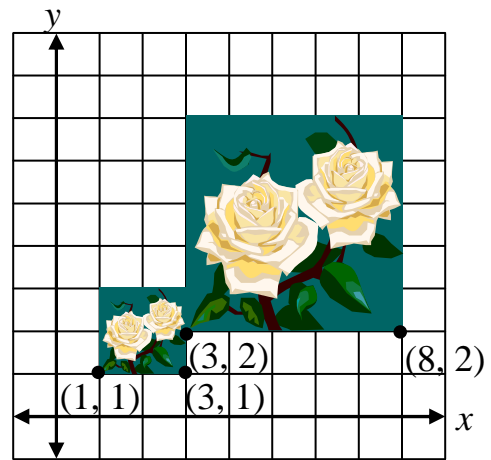$(1,1)$ $(3,1)$ $\rightarrow 3 - 1 = 2$ units

Step 2: Find the width of the image.

The width of the enlargement is the difference of the enlargement's x- coordinates.

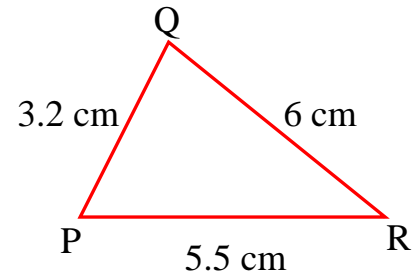$(3,2)$ $(8,2)$ $\rightarrow 8 - 3 = 5$ units

Step 3: Find the scale factor.

$$\text{Scale Factor} = \frac{\text{image}}{\text{original design}} = \frac{5}{2} = 2.5$$
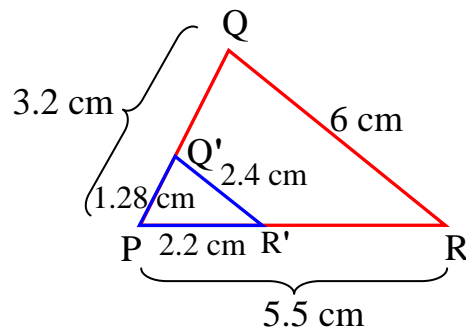
The scale factor is 2.5.

Example 2: Dilate triangle PQR by a scale factor of 0.4. Let vertex P be the center of dilation. Segment PR has a length of 5.5 centimeters, segment PQ has a length of 3.2 centimeters and segment QR has a length of 6 centimeters. P and P' are the same point.

Step 1: Multiply each side by 0.4.

Original	**Image**
(Length)	(0.4 × Length)
PR = 5.5 cm	PR' = 2.2 cm
PQ = 3.2 cm	PQ' = 1.28 cm
QR = 6 cm	Q'R' = 2.4 cm

Step 2: Draw the image, PQ'R'.

Triangle PQ'R' is a dilation of triangle PQR by a scale factor of 0.4 with point P as the center of dilation.

Problem Solving – Working Backwards

When using the strategy of working backwards, start with the end results and reverse the steps needed to get those results.

Let's take a look at an example where this stragegy works nicely.

Example: Bill purchased a tractor that was to be delivered to his farm. The price included a 7% sales tax on the original cost of the tractor and $750 for delivery charges. The total price was $17,870. What was the original price of the tractor.

What is given?
- Total price: $17,870
- Delivery charges: $750
- Sales Tax: 7%

What is asked?
- What was the original price of the tractor?

Since the final price is known, use the method of "working backwards", step by step, to find the original price.

First Step: Undo the delivery charges which was added on last.

Subtract the delivery charges from the total amount.

$$\begin{array}{r} 17{,}870 \\ -750 \\ \hline 17{,}120 \end{array}$$

Thus, the cost before the delivery charges was added was $17,120.

Next Step: Now undo the sales tax that was added on to the original price.

The price of $17,120, which includes the sales tax, was increased by 7% from the original price.

Therefore, $17,120 is 107% (100% + 7%) of the original price.
Note: 100% represents the original price and 7% represents the tax.

Let n represent the original price and write a statement to represent the problem.

17,120 is 107% of n

Then, use a proportion solve the percent problem.

$$\% = \frac{is}{of} \quad \rightarrow \quad \frac{107}{100} = \frac{17120}{n}$$

Cross Multiply
$$107n = 100(17,120)$$
$$107n = 1,712,000$$
$$\frac{107n}{107} = \frac{1,712,000}{107}$$
$$n = 16,000$$

Thus, the original price of the tractor was $16,000.

The chart summarizes the steps for working backwards. Start at the **bottom** of the chart ($16,000) and work up (forward) to check the work.

Final Price
$17,870
(delivery charge added on)

$17,120
(7% sales tax added on)

$16,000
Original Price
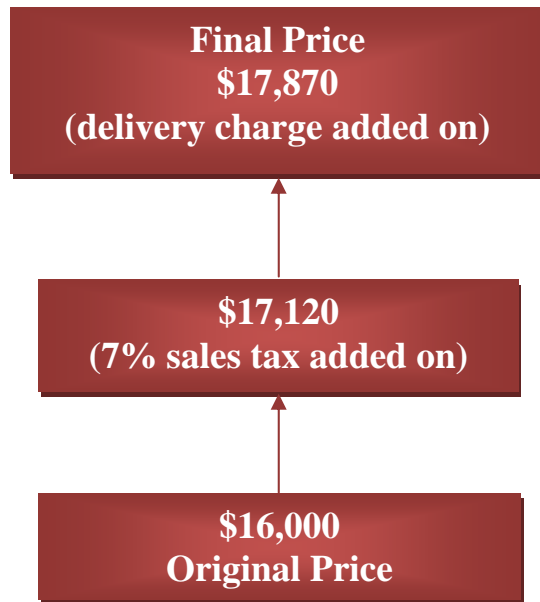
Working backwards can be a very useful problem-solving strategy.