INTEGERS: ADDITION AND SUBTRACTION

This unit is about number theory, integers, and the use of signed numbers. The integers are a subgroup of the Real Numbers. Integers have signs, but do not have any decimals or fractions. Integers will be discussed as a subgroup of the set of real numbers. The order of operations will be reviewed as a reminder of the rules that must be followed in mathematics when evaluating expressions. A close look at comparing integers and finding the absolute value of an integer will aid in taking a closer look at the introduction of adding and subtracting integers. In the final part of this unit, the rules that apply to adding and subtracting integers will be applied to adding and subtracting fractions and decimals with signs.

Real Numbers

Order of Operations

Comparing Integers

Absolute Value

Introduction to Adding Integers

Introduction to Subtracting Integers

Adding and Subtracting Integers

Fractions and Decimals with Signs

Real Numbers

Real numbers are the numbers that can be represented on the number line. They include the whole numbers, their opposites, and all the other numbers in between them.

 The real numbers are a union of the rational and irrational numbers.

Rational Numbers - Rational numbers are numbers that can be written as a quotient of two integers.

The natural numbers, whole numbers, and integers are subsets of the rational numbers. Note: A member of any of these sets can be expressed as a quotient of two integers.

> **Natural Numbers**: {1, 2, 3, 4, 5...}

> **Whole Numbers**: {0, 1, 2, 3, 4, 5...}

> **Integers**: {...–5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5...}

Also included in the rational numbers are **fractions** (where the numerator and denominators are integers) and **repeating and terminating decimals**. Fractions may not have zero (0) as the denominator.

Irrational Numbers – Irrational numbers are numbers that **cannot** be expressed as the ratio of two integers. Examples of irrational numbers are (a) square roots of non-perfect squares, (b) pi (π), and (c) decimals that do not develop into a repeating pattern.

> (a) $\sqrt{94}$
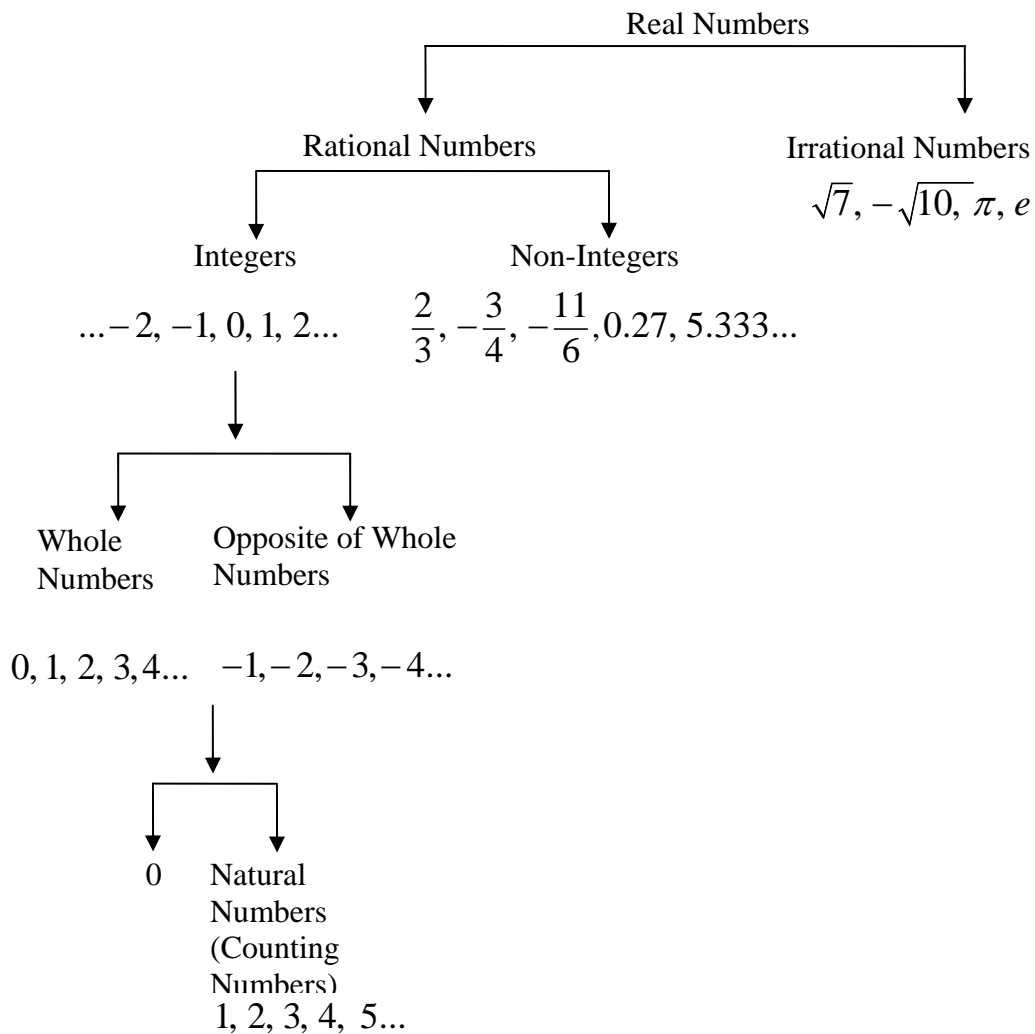> (b) $\pi \approx 3.14159265358979323846264338327950...$
> (c) $9.69535971483265802814888115084530...$

> real numbers = rational numbers + irrational numbers

Study the two sketches below to better understand the subsets of the real numbers.
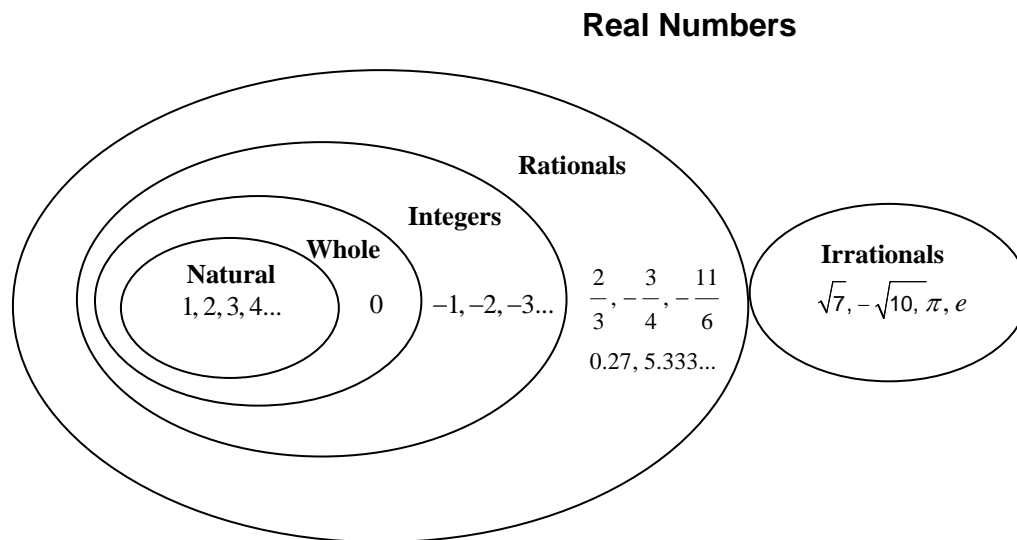
Sketch 1

This sketch shows how each subset can be divided into further subsets. For example, zero is a whole number, but is also an integer and a rational number because it is a member of all those subsets. The only subset of the rational numbers that zero is not a part of is the natural numbers.

Real Numbers

Rational Numbers

Irrational Numbers

$\sqrt{7}, -\sqrt{10}, \pi, e$

Integers

$...-2, -1, 0, 1, 2...$

Non-Integers

$\frac{2}{3}, -\frac{3}{4}, -\frac{11}{6}, 0.27, 5.333...$

Whole Numbers

Opposite of Whole Numbers

$0, 1, 2, 3, 4...$ $-1, -2, -3, -4...$

0   Natural Numbers (Counting Numbers)

$1, 2, 3, 4, 5...$

*The numbers listed in the subsets are just a few examples of the infinite possibilities.

Sketch 2

This sketch shows the connection between the subsets using a Venn diagram. Again, looking at zero, it is in all of the ovals that represent the rational numbers with the exception of the natural numbers.

Real Numbers

Rationals

Integers

Whole

Natural
1, 2, 3, 4...

0

−1, −2, −3...

$\dfrac{2}{3}, -\dfrac{3}{4}, -\dfrac{11}{6}$

0.27, 5.333...

Irrationals

$\sqrt{7}, -\sqrt{10}, \pi, e$

*Notice that the irrationals are in a league of their own.

The **rational numbers** are the types of numbers shown within the "rational circle". The "rational circle" includes all of the numbers within the circles inside it.

Some examples of rational numbers are –11/6, –4, 0, and 4. Rational numbers can be expression as a ratio of two integers.

$$\left(\dfrac{-11}{6}\right), \left(-4=\dfrac{-4}{1}\right), \left(0=\dfrac{0}{1}\right), \text{and } \left(4=\dfrac{4}{1}\right).$$

The **integers** are the types of numbers shown within the "integer circle". The "integer circle" includes all of the numbers within the circles inside it. Some examples of integers are −4, 0, 4 .

The **whole numbers** are the types of numbers shown within the "whole number circle". The "whole number circle" includes all of the numbers within the circle inside it. Some examples of whole numbers are 0 and 4.

The **natural numbers** are the types of numbers shown within the "natural number circle". Some examples of whole numbers are 3 and 4.

The **irrational numbers** are in a "league of their own". Some examples of rational numbers are $\sqrt{26}$, $-\sqrt{43}$, π, and e.

The **real numbers** are the types of numbers shown within all of the circles. The real numbers are the union of the rational and the irrational numbers.

Order of Operations

In order to find the numerical value (**evaluate**) of any combination of numbers and operations (**expression**) correctly, mathematicians have established the order of operations which tells us which operations to do first in any mathematical problem.

P (parentheses)

E (exponents – powers)

M (multiply)⎫
              ⎬ **work left to right**
D (divide) ⎭

A (add) ⎫
           ⎬ **work left to right**
S (subtract) ⎭

This "saying" may be used to help remember the order of operations.

Please
Excuse
My **D**ear
Aunt **S**ally

*Note: Multiplication and division are at the same level, meaning multiplication does NOT take priority over division. Work these two operations as they occur, left to right. The same is true about addition and subtraction. Work the two operations as they occur, left to right.

Thus, if "multiplication and division" or "addition and subtraction" are the only two operations in the expression, work the problem from left to right!

    Example 1: Evaluate $6 \times 4 + 2$.

       $6 \times 4 + 2$ Multiply 6×4

        $24 + 2$ Add $24 + 2$

          26

Example 2: Evaluate $4(6+3)-5\cdot2$.

$$4(6+3)-5\cdot2 \qquad \text{Parentheses } (6+3)$$
$$4(9)-5\cdot2 \qquad \text{Multiply } 4(9)$$
$$36-5\cdot2 \qquad \text{Multiply } 5\cdot2$$
$$36-10 \qquad \text{Subtract}$$
$$26$$

Example 3: Evaluate $5[(3+12)-2(4)]$.

$$5[(3+12)-2(4)] \qquad \text{Work within brackets [ ]}$$
$$5[(3+12)-2(4)] \qquad \text{Parenthesis } (3+12)$$
$$5[15-2(4)] \qquad \text{Multiply } 2(4)$$
$$5[15-8] \qquad \text{Subtract } 15-8$$
$$5(7) \qquad \text{Multiply } 5(7)$$
$$35$$

Example 4: Evaluate $4[3(3+2)^2]$.

$$4[3(3+2)^2] \qquad \text{Work within brackets [ ]}$$
$$4[3(3+2)^2] \qquad \text{Parenthesis } (3+2)$$
$$4[3(5)^2] \qquad \text{Powers } (5)^2$$
$$4[3(25)] \qquad \text{Multiply } 3(25)$$
$$4(75) \qquad \text{Multiply } 4(75)$$
$$300$$

Example 5: Evaluate $4[3(3+2)^2]$.

$4[3(3+2)^2]$	Work within brackets [ ]
$4[3(3+2)^2]$	Parenthesis $(3+2)$
$4[3(5)^2]$	Powers $(5)^2$
$4[3(25)]$	Multiply $3(25)$
$4(75)$	Multiply $4(75)$
300	

Example 6: Evaluate $4\cdot5-18\div6+2\cdot3$.

$4\cdot5-18\div6+2\cdot3$	Multiply and divide left to right
$20-3+6$	Add and subtract left to right
$17+6$	Add
23	

You will use the **order of operations** throughout many mathematics courses that you continue to study.

Comparing Integers

The set of whole numbers consists of 0, 1, 2, 3 … and can be represented on a number line. We can match each whole number with another number that is the same distance from 0 but on the opposite side of 0.
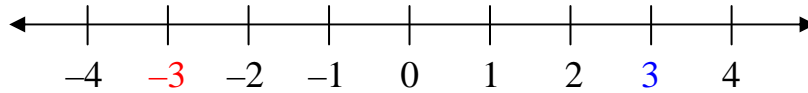
If you take a look at the number line above, 3 or (positive 3) and –3 (negative 3) are on opposite sides of 0 but the same distance from 0. These numbers are called opposites and make up the set of **integers.** Integers are the set of positive and negative whole numbers.

> *Examples*: Name the integer that is suggested by each situation.

a) The temperature is 5° below 0.

–5   Below 0 suggests a **negative** integer.

b) Emily's lemonade stand made a $24 profit.

24   A profit suggests a **positive** integer.

To compare integers, we will use the symbol "<" which means *less than* or the symbol ">" which means *greater than*. If you remember from the previous unit, these symbols are called **inequality** signs. An inequality can either be true or false. For example, the sentence 12 > 8 is true and the sentence 6 > 9 is false.

On a number line the numbers increase as you move from left to right. For any two numbers, the number that is farther to the right is the larger number and the number farther to the left is the lesser number. Let's take a look at comparing some integers using the number line below.
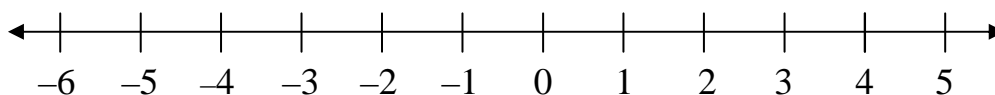
Write a true sentence using $<$ or $>$ in place of $\square$.

a) $3 \, \square \, 9 \, \Rightarrow \quad 3 < 9$

Since 3 is to the left of 9 on the number line, 3 is less than 9.

b) $-5 \, \square \, 11 \Rightarrow \quad -5 < 11$

Since -5 is to the left of 11 on the number line, -5 is less than 11.

c) $-3 \, \square \, -6 \Rightarrow \quad -3 > -6$

Since -3 is to the right of -6 on the number line, -3 is greater than -6.

Hint: An easy way to remember the direction of the inequality sign is that it will always point to the smaller number.

If you study the number line above, you will notice that the integers 5 and -5 are the same distance from 0. This brings us to another algebraic concept called **absolute value**. The absolute value of a number is the distance it is from 0, which means that the absolute value will always be **positive**. Absolute value is symbolized by using two straight bars around a number.

Example 1: $|24|$ means the absolute value of 24, which is 24 because 24 is 24 units away from 0.

Example 2: $|-57|$ means the absolute value of -57, which is 57 because -57 is 57 units away from 0.

Absolute Value

Absolute value of an integer is the distance the integer is from zero. The distance is measure in positive units.

> Absolute value is represented by two vertical bars around the number.
> $| \quad |$

Example 1: The absolute value of –5 is 5 because –5 is five units from 0.

$$|-5| = 5$$

5 units

{…–8, –7, –6, –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, 6, 7, 8...}

Example 2: The absolute value of 5 is 5 because 5 is 5 units from 0.

$$|5| = 5$$

5 units

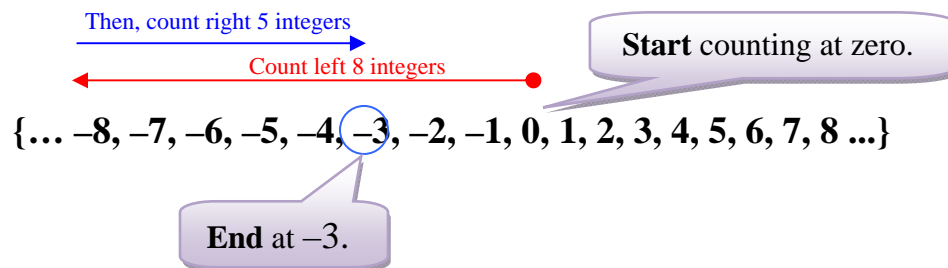{…–8, –7, –6, –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, 6, 7, 8...}

Introduction to Adding Integers

$$\{\dots -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8 \dots\}$$

Integers are used to show positive and negative quantities. When adding integers, start at 0 and count left for negative values or count right for positive values.

Example 1: Solve: $-8 + 5$

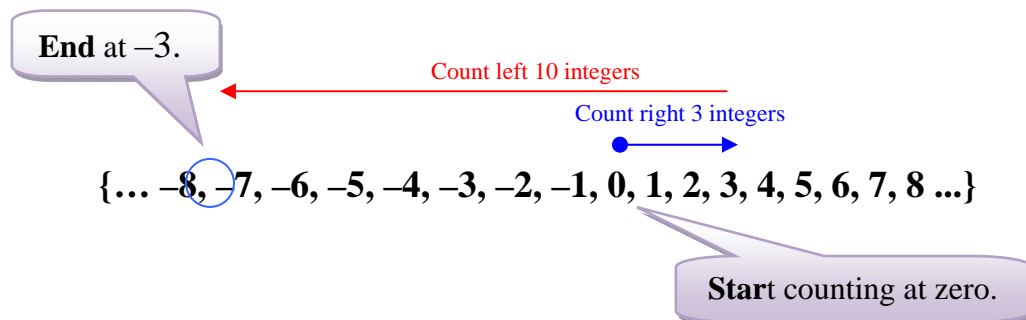Start at 0 and count left for negative 8, and then count right for positive 5 from negative 8. The final answer is –3.

Then, count right 5 integers

Count left 8 integers

Start counting at zero.

$$\{\dots -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8 \dots\}$$

End at –3.

Solution: $-8 + 5 = -3$

Example 2: Solve: $3 + (-10)$

Start at 0 and count right for positive 3, and then count left for negative 10 from positive 3. The final answer is –7.

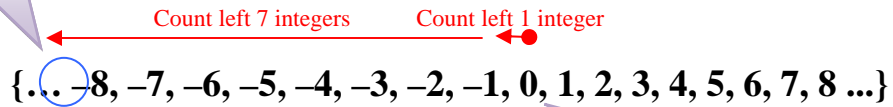**End** at –3.

Count left 10 integers

Count right 3 integers

$$\{\dots -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8 \dots\}$$

Start counting at zero.

Solution: $3 + (-10) = -7$

Example 3: Solve: −1 + (−7)

Start at 0 and count left for negative 1, and then count left for negative 7. The final answer is –8.

End at –8.

Count left 7 integers   Count left 1 integer

{... –8, –7, –6, –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, 6, 7, 8 ...}

Solution: −1 + (−7) = −8

Start counting at zero.

Rules for Adding Integers

Rule 1: When adding integers with the **same sign**, **add** and express the answer with the **same sign**.

$$6 + 4 = 10 \qquad\qquad −1 + (−7) = −8$$

Rule 2: When adding integers with **different signs**, **subtract** and express the answer with the **sign of the integer farthest from zero**.

$$−8 + 5 = (−3) \qquad\qquad 3 + (−10) = −7$$

# Introduction to Subtracting Integers

$$\{... -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8 ...\}$$

Integers are used to show positive and negative quantities. For subtracting integers, let's look at the *difference* between two integers.

Example 1: Solve: $7 - (-4)$

Locate the 7 and the –4. What is the *difference* between the two numbers? Counting right from –4 to 7, the difference is 11.

Start at –4   Count right 11. (11)   **End** at 7

$$\{... -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8 ...\}$$
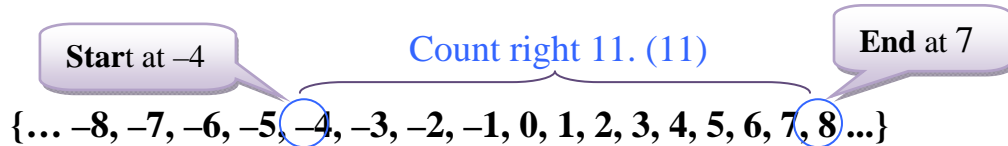
Solution: $7 - (-4) = 11$

Example 2: Solve: $-8 - 5$
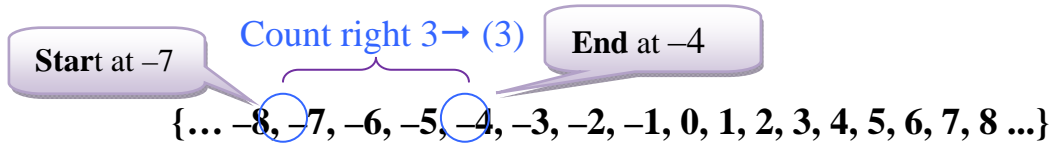
Locate the –8 and the 5. What is the *difference* between the two numbers? Counting left from 5 to –8, the difference is –13.

End at –8   Count left 13. (–13)   **Start** at 5

$$\{... -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8 ...\}$$
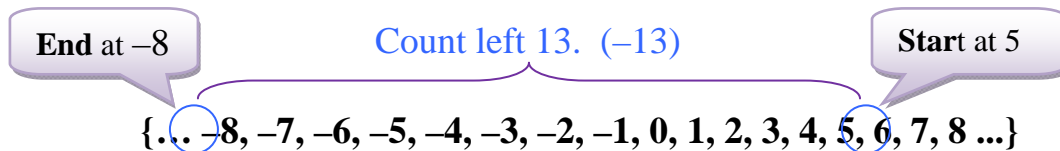
Solution: $-8 - 5 = -13$

Example 3: Solve: –4 – (–7)

Locate the –4 and the –7. What is the *difference* between the two numbers? Counting right from –7 to –4, the difference is 3.

Start at –7   Count right 3→ (3)   **End** at –4

{... –8, –7, –6, –5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5, 6, 7, 8 ...}

Solution: –4 – (–7) = 3

Counting out the answer on a number line is a good way to understand the types of answers that can be attained in subtraction of integers; but, let's take a look at a method that will simplify the work.

A method for **subtracting integers** is simply:

"add the opposite".

The Problem	Switch to addition. Change the second number to its opposite.	Follow the rules for **adding integers**.	State the final answer.
$7 - (-4) =$	$7 + 4 =$	$7 + 4 = 11$	$7 - (-4) = 11$
$-8 - 5 =$	$-8 + (-5) =$	$-8 + (-5) = -13$	$-8 - 5 = -13$
$-4 - (-7) =$	$-4 + 7 =$	$-4 + 7 = 3$	$-4 - (-7) = 3$

Adding and Subtracting Integers

When adding and subtracting integers, it will be very helpful to understand the following rules:

Adding Integers

Same sign: If the numbers have the same sign, add the numbers and keep the sign given.

Example 1: $(-5) + (-9) = -14$ Both integers are negative, so the answer is negative.

Example 2: $6 + 15 = 21$ Both integers are positive, so the answer is positive.

Different signs: If the numbers have different signs, subtract the absolute value of the numbers and use the sign of the larger absolute value.

Example 3: $(-9) + 3 = -6$ Subtract 9 and 3 to get 6, and then determine that –9 has the larger absolute value; so, you will use the negative sign.

Example 4: $17 + (-4) = 13$ Subtract 17 and 4 to get 13, and then determine that 17 has the larger absolute value; so, you will use the positive sign, or in this case no sign at all represents positive.

Subtracting Integers **Subtracting**

Change the problem to addition and add the opposite of the second number. Then go to addition rules.

Example 5:　15 – (–8) =

　　　　　　　15 + (–8) =　　　change the problem to addition

　　　　　　　15 + (+8) =　　　change (–8) to a (+8)

　　　　　　　23　　　　　　　Add the numbers because they now have the same sign.

Example 6:　18 – 26 =

　　　　　　　18 + (26) =　　　change the problem to addition

　　　　　　　18 + (–26) =　　　change 26 to (–26)

　　　　　　　–8　　　　　　　Subtract the numbers because they now have different signs and use the sign of the larger absolute value.

Fractions and Decimals with Signs

Fractions and decimals may have signs also. The rules for determining the signs of the answers to computations with integers also apply to computations with fractions and decimals. Let's take a look at a few examples and see how to apply the rules.

Example 1: Find the sum of –2 3/4 + –5 7/8.

Addition Rule: When the signs are the same (both negative), ADD and use the same sign.

$$-2\frac{3}{4}=2\frac{6}{8}$$
$$+ -5\frac{7}{8}=5\frac{7}{8}$$
$$-7\frac{13}{8}=-8\frac{5}{8}$$

The sum of –2 3/4 + –5 7/8 is –8 5/8.

Example 2: Find the sum of 5 2/3 + –7 1/9.

> *Addition Rule*: When the signs are different, SUBTRACT and
> take the sign of the number farther from zero on the number line.

*Note: Since –7 1/9 is farther from zero than 5 2/3 on the number line, the
sign of the answer is **negative**. Also, write the –7 1/9 above the 5 2/3 to
subtract.

$$
\begin{array}{r}
\left(\frac{9}{9}+\frac{1}{9}\right) \\
-7\frac{1}{9}=\cancel{7}^{6}\frac{1}{9}=6\frac{10}{9} \\
+\quad 5\frac{2}{3}=\ 5\frac{6}{9}=5\frac{6}{9} \\
\hline
-1\frac{4}{9}
\end{array}
$$

The sum of 5 2/3 + –7 1/9 is –1 4/9.

Example 3: Find the difference of 2/3 – 7/8.

$$\frac{2}{3} = \frac{16}{24}$$
$$-\frac{7}{8} = \frac{21}{24}$$

Subtraction Rule: Rewrite the problem to "add the opposite".

2/3 – 7/8 = 2/3 + –7/8   *The opposite of 7/8 is –7/8.

Since the signs are different, "subtract" and take the sign of the number farther from zero on the number line.

$\downarrow$

(2/3 + –**7/8**)

*To subtract, write –7/8 as the top number since 21/24 is larger than 16/24.

$$-\frac{7}{8} = \frac{21}{24}$$ *Subtract 21 – 16.

$$+\ \frac{2}{3} = \frac{16}{24}$$ *Take the sign of –7/8.

$$-\frac{5}{24}$$

The difference of 2/3 – 7/8 is –5/24.

Example 4: Find the difference of –2 3/10 – (–5 3/4).

> Subtraction Rule: Rewrite the problem to "add the opposite".

$\qquad$ –2 3/10 + (+5 3/4) $\qquad$ *The opposite of –5 3/4 is + 5/3/4.

*To subtract, write 5 3/4 as the top number since 5 is larger than 2.

$$5\frac{3}{4} = 5\frac{15}{20}$$
$$+- 2\frac{3}{10} = 2\frac{6}{20}$$
$$\overline{ 3\frac{9}{20}}$$

*Subtract 15 – 6 (no borrowing is needed).

*Take the **sign of 5 3/4**.

The difference of –2 3/10 – (–5 3/4) is 3 9/20.