INTRODUCTION TO RATIONAL NUMBERS

This unit covers many facets of foundation topics that will be useful when working with rational numbers. Prime factors of a number are used to determine the Greatest Common Factor (GCF) and the Least Common Multiple (LCM). Finding equivalent fractions, simplifying fractions, and examining mixed numbers are basic concepts that are used in preparation for comparing and calculating with fractions.

Rational Numbers

Squared and Cubed Numbers

Prime factorization

GCF

LCM

Introduction to Fractions

Comparing Fractions

Mixed Numbers

Rational Numbers

Rational numbers are numbers that can be written as a quotient of two integers.

Integers are the whole numbers and their opposites: $\{...-3,-2,-1,0,1,2,3...\}$.

In general, a rational number is $\dfrac{a}{b}$ where a can be any integer and b can be any integer except zero.

*Zero is not acceptable in the denominator because division by zero is undefined.

$\dfrac{n}{0}$ is undefined because there is no number times zero that gives the number. Think: $0\overline{)n}^{\;?}$ $(? \times 0 = n)$ No integer will make this statement true.

Three-fifths, $\dfrac{3}{5}$, is a rational number because it can be written in fraction form.

$\dfrac{3}{5}$ means 3 parts out of 5

$\dfrac{3}{5}$ means $3 \div 5$

$\dfrac{3}{5}$ means 3 to 5

In a whole pie that is divided into five equal parts, $\dfrac{3}{5}$ means 3 of the five equal parts.
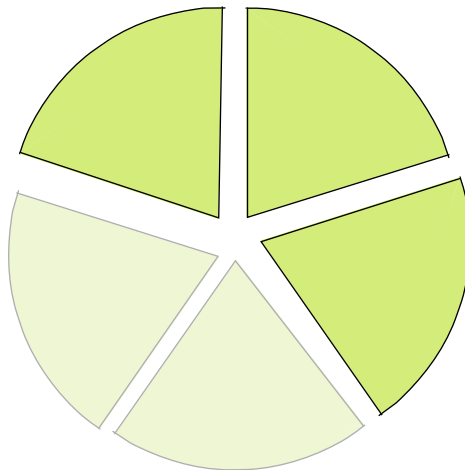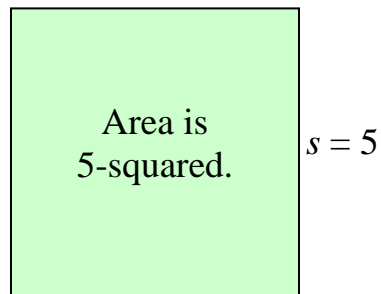
Squared and Cubed Numbers

Squared Numbers

Five to the second power, 5^2, equals 5×5 and can be read "5-squared".

Think of the area of a square with a side that measures 5 units.

$$A = s^2$$
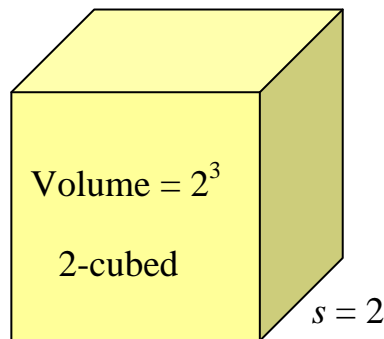$$A = s \times s$$
$$A = 5 \times 5$$
$$A = 5^2 = 5\text{-squared} = 25$$

Area is 5-squared.

$s = 5$

Five to the second power, 5^2, equals 5×5 and can be read "5-squared".

Cubed Numbers

Two to the third power, 2^3, equals 2×2×2 and can be read "2-cubed".

Think of the volume of a cube with a side that measures 2 units.

Volume = 2^3

2-cubed

$s = 2$

$$V = l \times w \times h$$
$$V = 2 \times 2 \times 2$$
$$V = 2^3 = 2\text{-cubed} = 8$$

Two to the third power, 2^3, equals 2×2×2 and can be read "2-cubed".

Three to the sixth power, 3^6, equals $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 729$.

Prime Factorization
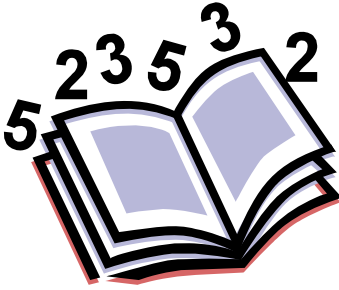
A **prime** number has *exactly* two factors, which are one (1) and the number itself.

Five is a prime number because its only factors are 5 and 1. $5 \times 1 = 5$

The factors of 5 can be listed as $\{1, 5\}$.

A **composite** number has *more than* two factors.

Eight is a composite number because it has four factors.

$8 \times 1 = 8$ and $2 \times 4 = 8$

The factors of 8 can be listed as $\{1, 2, 4, \text{and } 8\}$.

*Note: When listing factors in this notation, put the factors in order from least to greatest.

*** One is *neither* prime nor composite. ***

One has only one factor, and that is itself, 1; thus, it does not fit the definition of a prime nor a composite number.

To find the prime factors of a composite number, write the number as a product of prime numbers.

Here are the prime numbers that exist that are less than 20:

Primes: $\{2, 3, 5, 7, 11, 13, 17, 19\ldots\}$

Example 1: Find the prime factorization of 40.

Think of two factors of 40, and then continue on until all factors are prime factors.

$$40 = 2 \times 20$$

$$\downarrow \searrow$$

$$40 = 2 \times 2 \times 10 \qquad\qquad (20 = 2 \times 10)$$

$$\downarrow \searrow$$

$$40 = 2 \times 2 \times 2 \times 5 \qquad\qquad (10 = 2 \times 5)$$

✓ Check: Is $2 \times 2 \times 2 \times 5$ equal to 40? Yes!

The prime factorization of 40 is $2 \times 2 \times 2 \times 5$ which can be expressed as $2^3 \times 5$.

A **factor tree** is an organized structure used to find the prime factorization of a number.

Example 2: Find the prime factorization of 32 using a factor tree.

$$32$$

$$\swarrow \searrow$$

$$4 \quad \times \quad 8$$

$$\swarrow \searrow \quad \swarrow \searrow$$

$$2 \times 2 \times 2 \times 4$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \swarrow \searrow$$

$$2 \times 2 \times 2 \times 2 \times 2$$

All prime numbers … finished!

✓ Check: Is $2 \times 2 \times 2 \times 2 \times 2$ equal to 32? Yes!

The prime factorization of 32 is $2 \times 2 \times 2 \times 2 \times 2$ which can be expressed as 2^5.
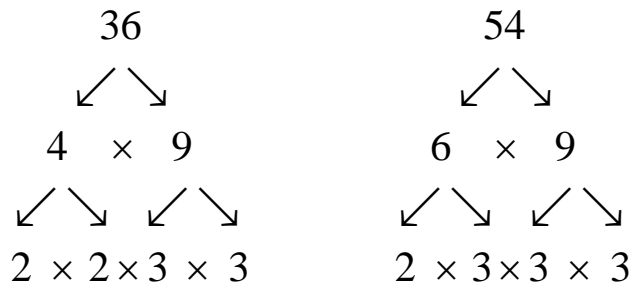
Greatest Common Factor

Finding GCF Using Prime Factorization

To find the **Greatest Common Factor (GCF)** of two numbers, apply prime factorization by finding *all* the common factors in each prime factorization, and then multiplying them.

Example 1: Find the GCF of 36 and 54.

Find the prime factorization of both numbers.

$$36$$

$$4 \times 9$$

$$2 \times 2 \times 3 \times 3$$

$$54$$

$$6 \times 9$$

$$2 \times 3 \times 3 \times 3$$

Find the common factors. The GCF is the product of *all* the common prime factors; that is, the factors appearing in both factorizations.

$$36 = 2 \times 2 \times 3 \times 3$$
$$54 = 2 \times 3 \times 3 \times 3$$

$$36 = 2 \times 2 \times 3 \times 3$$
$$54 = 2 \times 3 \times 3 \times 3$$

There is a two (2), a three (3), and a second three (3) that is found in both prime factorizations.

The GCF is $2 \times 3 \times 3 = 18$.

Finding GCF By Listing Factors

The GCF may also be determined by listing factors.

To find GCF by listing factors, think of all the numbers that multiply to make each of the given numbers, write those numbers in order, and then find the greatest factor that occurs in both sets.

Example 2: Find the GCF of 18 and 24.

List all of the factors of 18.	List all of the factors of 24.
1 × 18 2 × 9 3 × 6	1 × 24 2 × 12 3 × 8 4 × 6
Write the factors in order from least to greatest. {1, 2, 3, 6, 9, 18}	Write the factors in order from least to greatest. {1, 2, 3, 4, 6, 8, 12, 24}

Now look for the largest factor that is the same in both lists.

$${1, 2, 3, 6, 9, 18}$$
$${1, 2, 3, 4, 6, 8, 12, 24}$$

There are three common factors, 2, 3, and 6, but the largest of them is 6.

The GCF is 6.

Using GCF to Simplify Fractions

The GCF may be used to reduce fractions.

Example 3: Reduce $\dfrac{18}{24}$ to simplest form.

In the previous example, we determined that the GCF of 18 and 24 is 6. When reducing a fraction, divide both the numerator (top) and the denominator (bottom) of the fraction by the GCF (6).

$$\frac{18 \div 6}{24 \div 6} = \frac{3}{4}$$

The fraction 18/24 reduces to 3/4.

Least Common Multiple

Finding LCM Using Prime Factorization

To find the **Least Common Multiple (LCM)** of two numbers, apply prime factorization to find all the prime factors in each prime factorization, and then multiply the *highest occurrence of each **different*** factor.

Example 1: Find the LCM of 18 and 24 using prime factorization.

Step 1: Find the prime factorization of both numbers.

$$18$$ $$\quad\quad\quad\quad 24$$

$$2 \times 9 \quad\quad\quad\quad 6 \times 4$$

$$2 \times 3 \times 3 \quad\quad\quad 2 \times 3 \times 2 \times 2$$
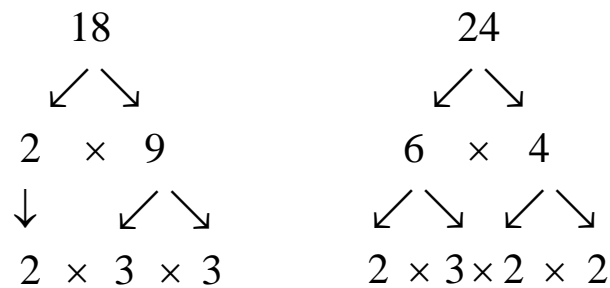
Step 2: Write each factorization using exponents.

$$18 = 2 \times 3^2 \quad\quad\quad\quad 24 = 2^3 \times 3$$

Step 3: Find the LCM; that is, find the product of the highest occurrence of each prime factor.

Look for the highest exponent of each factor.

$$18 = 2 \times 3^2 \quad\quad\quad\quad 24 = 2^3 \times 3$$

The highest frequency of the factor of two occurs in the 24….2^3.

The highest frequency of the factor of three occurs in the 18….3^2.

The LCM is $3^2 \times 2^3 = 9 \times 8 = 72$.

Finding LCM By Listing Multiples

The LCM may also be determined by listing multiples.

To find the least common multiple (LCM) by listing multiples:

1. List the multiples of both numbers.
2. Find the first multiple (the least) that is common to both sets of multiples.

Example 2: Find LCM for 8 and 16.

Step 1: List the multiples of each number.

Multiples of 16:	16 = {16, 32, 48, 64...}
Multiples of 8:	8 = {8, 16, 24, 32, 40, 48 ...}

Step 2: Find the least multiple that is common to both sets.

Multiples of 16:	16 = {*16*, *32*, *48*, 64...}
Multiples of 8:	8 = {8, *16*, 24, *32*, 40, *48* ...}

*Notice that 32 and 48 are also common multiples, but they are not the **least** common multiple.

The LCM is 16.

Example 3: Find LCM for 8 and 6.

Step 1: List the multiples of each number.

Multiples of 8:	8 = {8, 16, 24, 32, 40, 48 ...}
Multiples of 6:	6 = {6, 12, 18, 24, 30, 36, 42, 48…}

Step 2: Find the least multiple that is common to both sets.

Multiples of 8:	8 = {8, 16, 24, 32, 40, *48* ...}
Multiples of 6:	6 = {6, 12, 18, 24, 30, 36, 42, *48*…}

*Notice that 48 is also a common multiple, but it is not the **least** common multiple.

The LCM is 24.

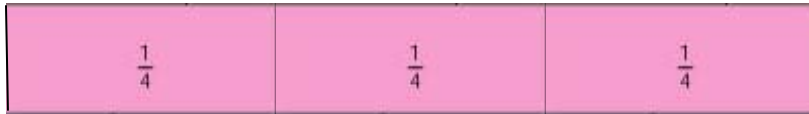## Introduction to Fractions

Equivalent Fractions

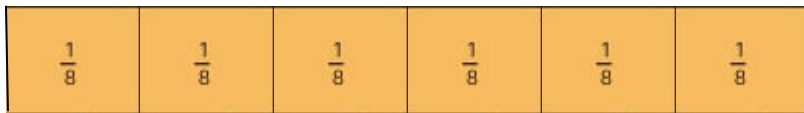Fractions that represent the same amount are called **equivalent** fractions. Look at the fraction strips below that represent the quantity $\frac{3}{4}$.

Three-fourths can also be expressed in eighths.

It takes two $\frac{1}{8}$'s to make one $\frac{1}{4}$. Therefore, three $\frac{1}{4}$'s = six $\frac{1}{8}$'s.

$$\frac{3}{4} = \frac{6}{8}$$

Equivalent fractions can be found by multiplying the numerator (top) and the denominator (bottom) of a fraction by the same number.

$$\frac{3 \times 2}{4 \times 2} = \frac{6}{8}$$

*Notice that this process is actually multiplying the original fraction by 2/2 which equals 1; thus, the value of 3/4 remains the same, it just has a different appearance (6/8).

$$\frac{3}{4} \times 1 \rightarrow \frac{3}{4} \times \frac{2}{2} = \frac{6}{8}$$

Example 1: Find three equivalent fractions for $\frac{2}{3}$.

To find the three new equivalent fractions, multiply by any fraction that equals one. We'll use 2/2, 3/3, and 4/4.

$$\frac{2}{3} \times \frac{2}{2} = \frac{4}{6} \qquad\qquad \frac{2}{3} \times \frac{3}{3} = \frac{6}{9} \qquad\qquad \frac{2}{3} \times \frac{4}{4} = \frac{8}{12}$$

Therefore, three equivalent fractions of 2/3 are 4/6, 6/9, and 8/12.

*There are infinitely many more fractions equivalent to 2/3.

Simplifying Fractions

Simplifying a fraction (**reducing**) is finding an equivalent fraction that is in simplest form. Simplest form means that the only number that will divide into both the numerator and denominator is *one*.

Example 2: Find the simplified form of $\frac{8}{12}$.

The quickest way to reduce the fraction is to find the GCF and use that number to divide into both the numerator (top) and the denominator (bottom) of the fraction.

In this problem, the GCF is 4.

$$\frac{8}{12} \div \frac{4}{4} = \frac{2}{3}$$

The simplified form of 8/12 is 2/3.

*Notice that the reduced form could have been found by dividing by 2/2, and then by 2/2 again.

$$\frac{8}{12} \div \frac{2}{2} = \frac{4}{6} \qquad \rightarrow \qquad \frac{4}{6} \div \frac{2}{2} = \frac{2}{3}$$

Example 3: Find the simplified form of $\dfrac{72}{144}$.

In this problem, the GCF is 72.

$$\frac{72}{144} \div \frac{72}{72} = \frac{1}{2}$$

The simplified form of 72/144 is 1/2.

*Notice that the reduced form could have been found by performing many smaller division steps.

$$\frac{72}{144} \div \frac{2}{2} = \frac{36}{72} \qquad \rightarrow \qquad \frac{36}{72} \div \frac{2}{2} = \frac{18}{36}$$

$$\frac{18}{36} \div \frac{9}{9} = \frac{2}{4} \qquad \rightarrow \qquad \frac{2}{4} \div \frac{2}{2} = \frac{1}{2}$$

Example 4: Find the simplified form of $\dfrac{51}{68}$.

In this problem, the GCF is 17.

$$\frac{51}{68} \div \frac{17}{17} = \frac{3}{4}$$

The simplified form of 51/68 is 3/4.

51

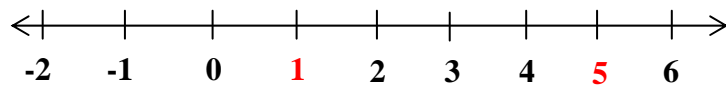3 × 17

68

2 × 34
↓
2 × 2 × 17

Comparing Fractions

To compare fractions, make sure the denominators are the same, and then compare the numerators.

Comparing Fractions with the Same Denominator

Example 1: Compare $\dfrac{5}{6}$ and $\dfrac{1}{6}$.

Compare the **numerators**: $5 > 1$

Since $5 > 1$, it follows that $\dfrac{5}{6} > \dfrac{1}{6}$.

Comparing Fractions with the Different Denominators

To compare fractions with *different denominators*, find equivalent fractions that have the same denominator.

Express each fraction with the **least common denominator (LCD)**. The LCD is the *least common multiple (LCM)* of the denominators.

To compare fractions with different denominators:

1. Find the least common denominator (LCD).
2. Find equivalent fractions by expressing each fraction with like denominators.
3. Compare the numerators.

Example 1: Compare $\dfrac{3}{4}$ and $\dfrac{4}{5}$.

Step 1: Find the least common denominator of 4 and 5.

$$4 = 2 \times 2 = 2^2$$
$$5 = 5$$

$$\text{LCD} = 2^2 \times 5 = 4 \times 5 = 20$$

Step 2: Set up equivalent fractions with the LCD.

x 5   **x 5**   **x 4**   **x 4**

$$\frac{3}{4} = \frac{\square}{20} \qquad \frac{3}{4} = \frac{15}{20} \qquad\qquad \frac{4}{5} = \frac{\square}{20} \qquad \frac{4}{5} = \frac{16}{20}$$

x 5   **x 5**   **x 4**   **x 4**

*Note: To find the five, think $20 \div 4 = ?$ To find the four, think $20 \div 5 = ?$

Step 3: Compare the numerators of the fractions expressed in the same denominator.

$$\frac{15}{20} \text{ is less than } \frac{16}{20}$$

Since $15 < 16$, it follows that: $\dfrac{15}{20} < \dfrac{16}{20}$ and $\dfrac{3}{4} < \dfrac{4}{5}$.

Mixed Numbers

Mixed numbers are numbers that have a whole number and a fraction.

Examples of Mixed Numbers: $2\dfrac{2}{3}$ $7\dfrac{5}{8}$ $29\dfrac{3}{4}$

Improper fractions are fractions where the numerator is larger than the denominator.

Examples of Improper Fractions: $\dfrac{8}{3}$ $\dfrac{61}{8}$ $\dfrac{119}{4}$

When working with fractions, it is necessary to know how to convert mixed numbers to improper fractions and vice versa.

Changing Mixed Numbers to Improper Fractions

Example 1: Express $1\dfrac{5}{12}$ as an improper fraction.

$$1\dfrac{5}{12} \quad = \quad 1+\dfrac{5}{12} \quad = \quad \dfrac{12}{12}+\dfrac{5}{12} \quad = \quad \dfrac{12+5}{12} \quad = \quad \dfrac{17}{12}$$

*A quick way to find the improper fraction is to multiply the denominator by the whole number, and add on the numerator. Then, place that number over the denominator.

$$1\dfrac{5}{12} \quad = \quad \dfrac{12 \times 1 + 5}{12} \quad = \quad \dfrac{12+5}{12} \quad = \quad \dfrac{17}{12}$$

The improper fraction for 1 5/12 is 17/12.

Example 2: Express $2\dfrac{4}{9}$ as an improper fraction.

$$2\dfrac{4}{9} \quad = \quad \dfrac{9 \times 2 + 4}{9} \quad = \quad \dfrac{18+4}{9} \quad = \quad \dfrac{22}{9}$$

The improper fraction for 2 4/9 is 22/9.

Changing Improper Fractions to Mixed Numbers

Example 3: Express $\dfrac{13}{10}$ as a mixed fraction.

Think of $\dfrac{13}{10}$ as $\dfrac{10}{10} + \dfrac{3}{10}$, then as $1 + \dfrac{3}{10}$ because $\dfrac{10}{10} = 1$, then as $1\dfrac{3}{10}$.

In this problem, the whole number is 1 $(\dfrac{10}{10})$ and the remaining part is $\dfrac{3}{10}$.

*A quick way to find the mixed number is to divide the numerator by the denominator and express the remainder as a fraction.

$$\frac{13}{10} \;=\; 10\overline{)13}^{\,1} \;=\; 1\frac{3}{10}$$
$$\underline{10}$$
$$3$$

The mixed number for 13/10 is 1 3/10.

Example 4: Express $\dfrac{27}{15}$ as a mixed fraction.

$$\frac{27}{15} \;=\; 15\overline{)27}^{\,1} \;=\; 1\frac{12}{15}$$
$$\underline{15}$$
$$12$$

The fraction must be simplified.

$$1\frac{12}{15} = 1\frac{12 \div 3}{15 \div 3} = 1\frac{4}{5}$$

The mixed number for 27/15 is 1 4/5.