LOCI

In this unit, you will explore loci. Loci are collection of points that satisfy given conditions. You will sketch, determine, and describe loci. You will extend your knowledge of loci to solving both linear and nonlinear systems of equations and inequalities.

Loci

locus – A locus is the set of all points that satisfy a given condition or set of conditions.

loci (pronunciation – "low – sigh") – Loci is the plural of locus. Loci are collections of points.

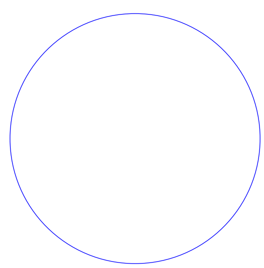
The word locus is derived from the Latin word for "location".

To find a locus, complete the following steps:

1) Draw any given figures.
2) Locate several or all points that satisfy the given condition.
3) Draw a smooth geometric figure.
4) Describe the locus.

Example 1: Sketch the locus that occurs at the end of the minute hand of "Big Ben" as it revolves around a center point as an hour elapses.

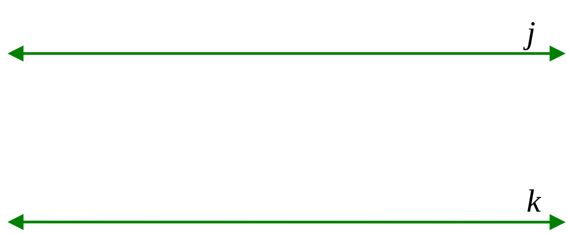
The locus is a circular path of points that are covered as the end of the minute hand travels around the clock.



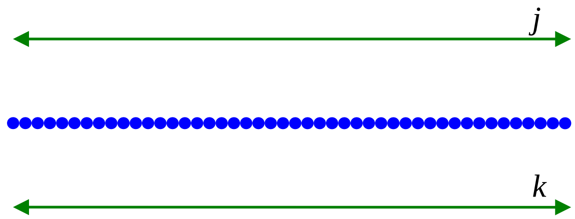
Example 2: Identify the locus described below:

"All points equidistant from lines *j* and *k*."

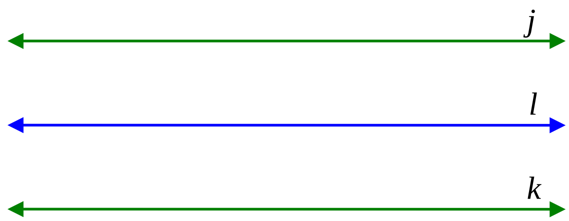
First, make a sketch of the figures referenced in the description.



Next, locate several or all points that satisfy the given condition.



Finally, draw a smooth geometric figure that satisfies ALL points and name it.



Line *l* is the set of points that are equidistant from lines *j* and *k*.

Locus and Systems of Equations

Two equations in two variables are called a system of equations. The solution to a system of equations is the point on a coordinate plane where the two equations intersect.

There are three possible solutions for a system of equations.

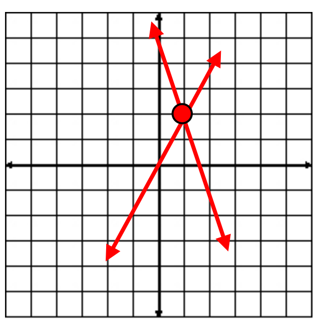
1) The lines may intersect at one point; therefore there would be **one solution**, the ordered pair (x, y).

2) The lines may be parallel and not intersect at all; therefore would be **no solution**.

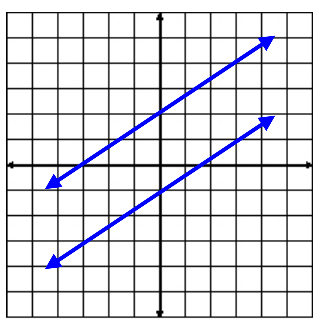
3) The lines may be identical and would lie on top of each other when graphed so the solution would be **many solutions**.

Examples:

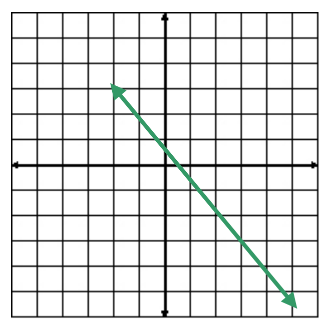
$y = 2x$	$y = \frac{2}{3}x + 2$	$x + y = 2$
$y = -3x + 5$	$y = \frac{2}{3}x - 1$	$y = -x + 2$



One Solution



No Solution



Many Solutions

Solving Systems of Equations by Graphing

To solve a system of equations by graphing, graph each line as follows.

(1) Write the equation of the line in slope-intercept form by solving for y.

(2) Identify the y-intercept and slope.

(3) Plot the y-intercept, and then use the slope ratio $\left(\frac{\text{rise}}{\text{run}}\right)$ to plot additional points.

(4) Draw a straight line through the plotted points.

Example 1: Graph the system of equations to find the locus of points that satisfy the graphs of both equations.

$$\begin{aligned} 3x + y &= 4 \\ 2x - y &= 6 \end{aligned}$$

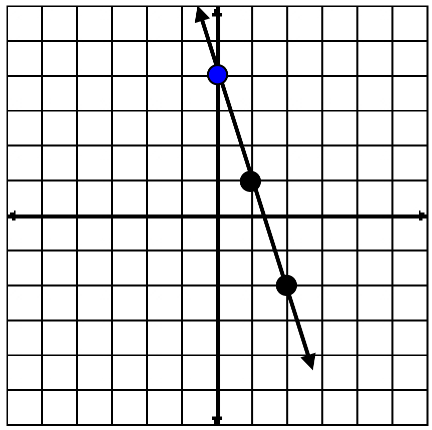
Step 1: Solve each equation for y.

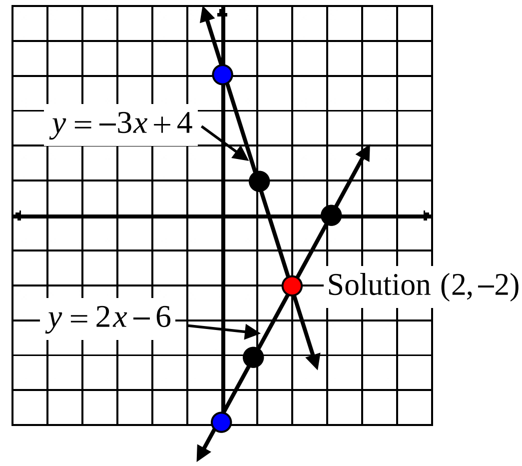
$3x + y = 4$	$2x - y = 6$
$y = -3x + 4$	$-y = -2x + 6$
	$y = 2x - 6$

Step 2: Determine the y-intercept and the slope.

$y = -3x + 4$	$y = 2x - 6$
$y = -3(0) + 4$	$y = 2(0) - 6$
$y = 4$	$y = -6$
$y\text{-int} = (0, 4)$	$y\text{-int} = (0, -6)$
$m = \frac{-3}{1}$	$m = \frac{2}{1}$

Step 3: Plot the y-intercept and graph the first equation. Then, plot the second equation.





The point with coordinates (2, –2) is the locus of points that satisfy the graphs of both equations.

Solving Systems of Equations by Substitution

Another way to solve a system of equations is using substitution. To solve a system of equations in two variables using substitution:

1.) Solve one equation for a variable (it is much easier to solve an equation for a variable that has a coefficient of 1).

2.) Substitute this expression into the other equation and find the value of one variable.

3.) Substitute the value found in the previous step into either of the equations to find the value of the second variable.

Example 2: Find the locus of points that satisfy the equations shown below:

$$x + y = 2 \qquad 3x + y = 8$$

First, solve $x + y = 2$ for *y*. (Note: you could also solve the other equation for *y*.)

$$\begin{aligned} x + y &= 2 \\ -x \quad & \quad -x \\ y &= 2 - x \end{aligned}$$

Next, substitute $2 - x$ into the second equation for *y*.

$3x + y = 8$	
$3x + (2 - x) = 8$	Substitution $(y = 2 - x)$
$3x + 2 - x = 8$	Simplify
$2x + 2 = 8$	Collect Like Terms
$2x = 6$	Subtract 2 from both sides.
$x = 3$	Divide by 2 on both sides.

Finally, substitute the value determined in the previous step into either equation.

$x + y = 2$	
$3 + y = 2$	Substitution $(x = 3)$
$y = -1$	Subtract 3 from both sides.

The point with coordinates (3, –1) is the locus of points that satisfy the graphs of both equations.

Example 3: Find the locus of points that satisfy the equations shown below:

$2x + y = 9$ $\qquad$ $y + 2x = 7$

.

First, solve $2x + y = 9$ for *y*.

$$\begin{aligned} 2x + y &= 9 \\ -2x \quad\; &\;\; -2x \\ y &= 9 - 2x \end{aligned}$$

Next, substitute $9 - 2x$ into the second equation for *y*.

$y + 2x = 7$	
$(9 - 2x) + 2x = 7$	Substitution $(y = 9 - 2x)$
$9 - 2x + 2x = 7$	Simplify
$9 = 7$	Collect Like Terms
$9 \neq 7$	$9 = 7$ is not a true statement.

$\therefore$ No Solution

No locus of points satisfies this system of equations.

Example 4: Find the locus of points that satisfy the equations shown below:

$2x - y = -4 \qquad\qquad y - 2x = 4$

First, solve the second equation for *y*.

$$\begin{aligned} y - 2x &= 4 \\ +2x \quad &\;\; +2x \\ y &= 4 + 2x \end{aligned}$$

Next, substitute $4 + 2x$ into the first equation for *y*.

$2x - y = -4$	
$2x - (4 + 2x) = -4$	Substitution $(y = 4 + 2x)$
$2x - 4 - 2x = -4$	Simplify
$-4 = -4$	Collect Like Terms

Since, $-4 = -4$ is a true statement; but, there is no x value, this indicates the graph of the two lines are the same set of locus of points.

$\therefore$ Many Solutions

Another way to solve a system of equations is by eliminating a variable. This process involves adding or subtracting the equations, depending on whether the terms are opposites (you add) or the same (you subtract).

Example 5: Use elimination to find the locus of points that satisfy the equations shown below:

$3x - 2y = 1 \qquad\qquad -3x + 4y = 7$

First, ask yourself if there are any terms that are the same or opposites. In this case the $3x$ and $-3x$ are opposites.

Next, use the addition property of equality to combine the two equations into one; then, solve for *y*.

$$\begin{array}{r} 3x - 2y = 1 \\ \underline{-3x + 4y = 7} \\ 2y = 8 \\ y = 4 \end{array}$$

Finally, substitute this value into one of the original equation for *y* and solve for *x*.

$3x-2y=1$	
$3x-2(4)=1$	Substitution $(y=4)$
$3x-8=1$	Simplify
$3x=9$	Add 8 to both sides.
$x=3$	Divide by 3.

The point with coordinates (3, 4) is the locus of points that satisfy the graphs of both equations.

Example 6: Use elimination to find the locus of points that satisfy the equations shown below:

$$2x+3y=5 \qquad 2x+y=3$$

First, ask yourself if there are any terms that are the same or opposites. In this case the $2x's$ are the same.

Next, use the subtraction property of equality to combine the two equations into one; then, solve for *y*.

Since the *x*-terms are the same, eliminate them by subtraction.

$$\begin{array}{lcccr} 2x+3y=5 & \rightarrow & 2x+3y=5 & \rightarrow & 2x+3y=5 \\ 2x\ +\ y=3 & \rightarrow & -(2x\ +\ y=3) & \rightarrow & -2x\ -y=-3 \end{array}$$

$$\begin{array}{r} 2x+3y=5 \\ \underline{-2x-\ \ y=-3} \\ 2y=2 \\ y=1 \end{array}$$

Finally, substitute the value determined in the previous step into either equation.

$2x+y=3$	
$2x+1=3$	Substitution $(y=1)$
$2x=2$	Subtract 1 from both sides.
$x=1$	Divide by 2.

The point with coordinates (1, 1) is the locus of points that satisfy the graphs of both equations.

Sometimes it is necessary to multiply one or both equations by a number to produce the same coefficient or opposites. You make the choice. If you are more comfortable adding, then produce opposites; if you are more comfortable subtracting, then make them the same.

Example 7: Use elimination to find the locus of points that satisfy the equations shown below:

$$3x - y = 8 \qquad\qquad x + 2y = -2$$

First, multiply the first equation by 2 to produce opposites for *y*.

$$2(3x - y = 8) \qquad \to \qquad 6x - 2y = 16$$

Next, use the addition property of equality to combine the two equations into one; then, solve for *y*.

$$\begin{array}{r} 6x - 2y = 16 \\ \underline{x + 2y = -2} \\ 7x \qquad = 14 \\ x = 2 \end{array}$$

Finally, substitute the value determined in the previous step into either equation.

$x + 2y = -2$	
$2 + 2y = -2$	Substitution $(x = 2)$
$2y = -4$	Subtract 2 from both sides.
$y = -2$	Divide by 2.

The point with coordinates $(2, -2)$ is the locus of points that satisfy the graphs of both equations.

Example 8: Use elimination to find the locus of points that satisfy the equations shown below:

$$2x - 7y = 20 \qquad\qquad 5x + 8y = -1$$

First, multiply the first equation by 5 and the second equation by 2 to produce the same coefficient for the *x*-term.

$$5(2x-7y=20) \quad \rightarrow \quad 10x-35y=100$$

$$2(5x+8y=-1) \quad \rightarrow \quad 10x+16y=-2$$

Next, use the subtraction property of equality to combine the two equations into one; then, solve for *y*.

$$\begin{array}{rcr} 10x-35y=100 & \rightarrow & 10x-35y=100 \\ -(10x+16y=-2) & \rightarrow & \underline{-10x-16y=+2} \\ & & -51y=102 \\ & & y=-2 \end{array}$$

Finally, substitute the value determined in the previous step into either equation.

$5x+8y=-1$	
$5x+8(-2)=-1$	Substitution $(y=-2)$
$5x-16=-1$	Simplify
$5x=15$	Add 16 to both sides.
$x=3$	Divide by 5.

The point with coordinates $(3,-2)$ is the locus of points that satisfy the graphs of both equations.

Loci and Systems of Inequalities

First we'll review the graphing of linear inequalities, and then examine the intersection of loci in linear systems of inequalities.

A linear inequality is similar to a linear equation except that the equals sign is replaced with an inequality sign, and the solution to the inequality is a region of the coordinate plane.

Linear Equation: $y = \frac{-2}{3}x + 4$. Linear Inequality: $y \geq \frac{4}{5}x - 2$.

A linear inequality can have one of the following signs:

$<$ "less than"

$>$ "greater than"

$\leq$ "less than or equal to"

$\geq$ "greater than or equal to"

To graph a linear inequality:

1) Solve the inequality for y and make sure it is in the slope-intercept form.

2) Plot the y-intercept $(0, b)$.

3) Use the slope ratio $\left(\frac{\text{rise}}{\text{run}}\right)$ to plot more points.

4) Connect the points using a dashed line if the inequality is < or > (◄- - - -►), or a solid line if the inequality is $\leq$ or $\geq$ (◄────►).

5) Shade a region of the graph that satisfies the inequality.

Let's practice a few examples.

Example 1: Graph $x + y < -3$.

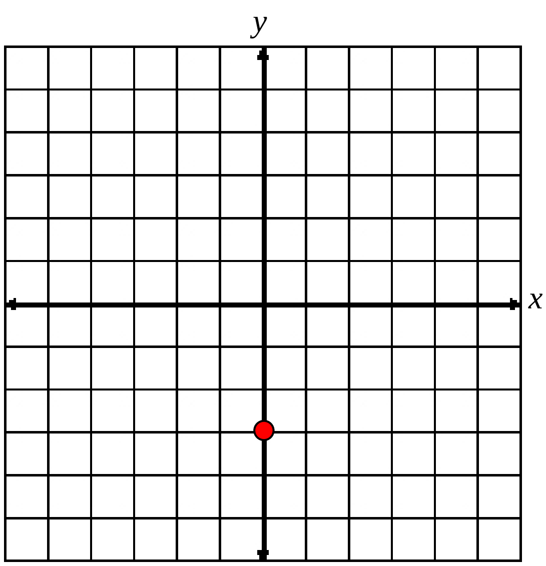
First, solve for *y*.

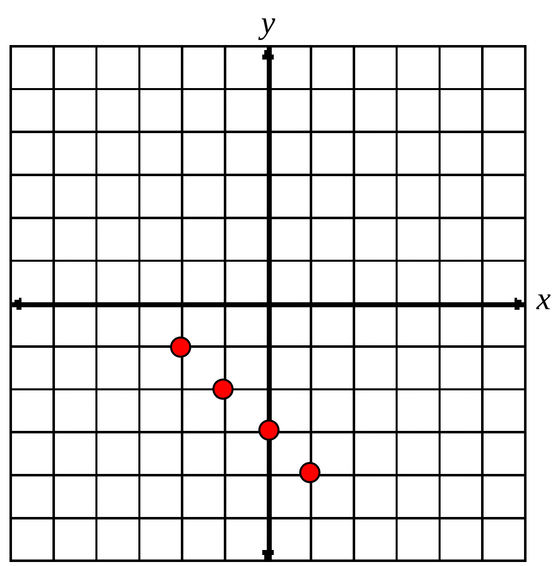
$$\begin{aligned} x + y &< -3 \\ {\color{red}-x} \quad & \quad {\color{red}-x} \\ y &< -x - 3 \end{aligned}$$

Next, identify the slope and *y*-intercept.

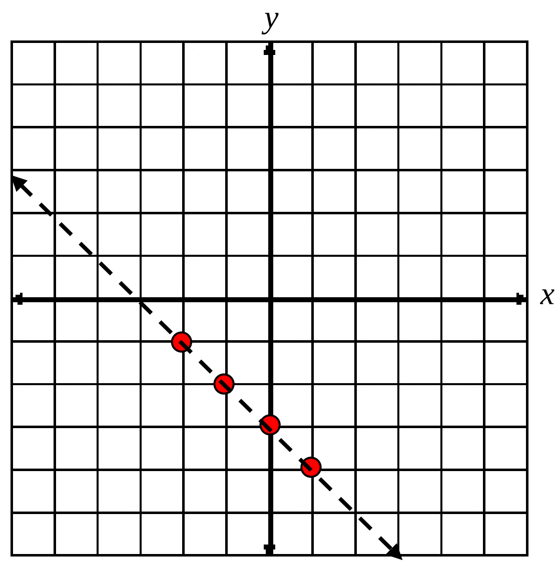
$m = -1$ $\qquad$ $y\text{-int} = (0, -3)$

Next, plot the *y*-intercept $(0, -3)$ and more points based on the slope $(m = \frac{-1}{1})$.





Next, connect the points using a dashed line because the inequality is < “less than”.



Finally, determine the region to shade.

The line just graphed is the boundary of the region.

Where would the *y*-values of the points be less than this boundary?

Will the points be above the dotted line will they be below the dotted line?

*If you are not sure of which region to shade, you can always pick a test point to help you determine the shading. The easiest point to choose is (0, 0). You can use (0, 0) if it is not part of the line that bounds the regions.

Test Rule for Shading Regions of an Inequality

If *x* and *y* is replaced with (0, 0) in the original inequality and the result is a **true** statement, the shade the region **that contains** the point (0, 0). If the result is a **false** statement, the shade the region that **does not** contain the point (0, 0).

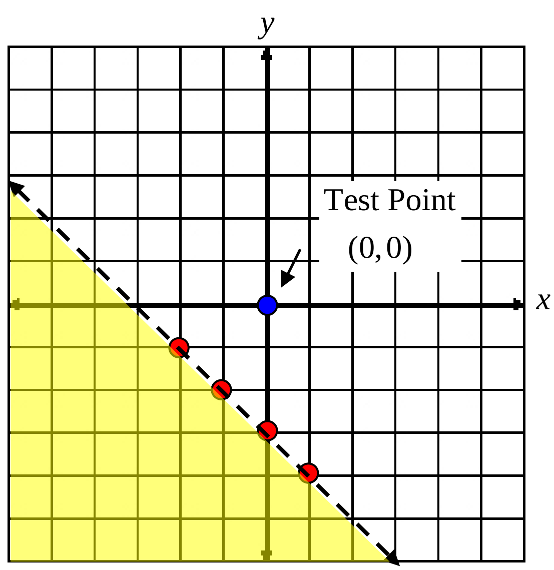
Test $(0,0)$:

$x + y < -3$

$0 + 0 < -3$ Substitution (Test Point: $(0,0)$ $x = 0, y = 0$)

$0 < -3$ Simplify

This is a false statement, so shade the side of the boundary line that does NOT contain (0,0).



Example 2: Graph $-2x + 3y \geq -6$.

First, solve for *y*.

$-2x + 3y \geq -6$

$+2x \qquad\quad +2x$

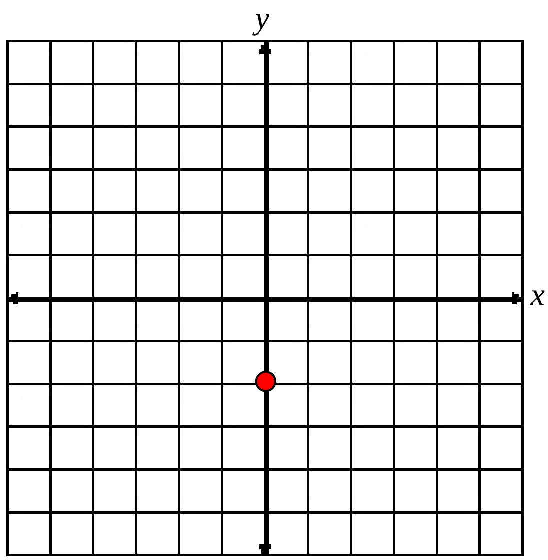
$3y \geq 2x - 6$ Add $2x$ to both sides.

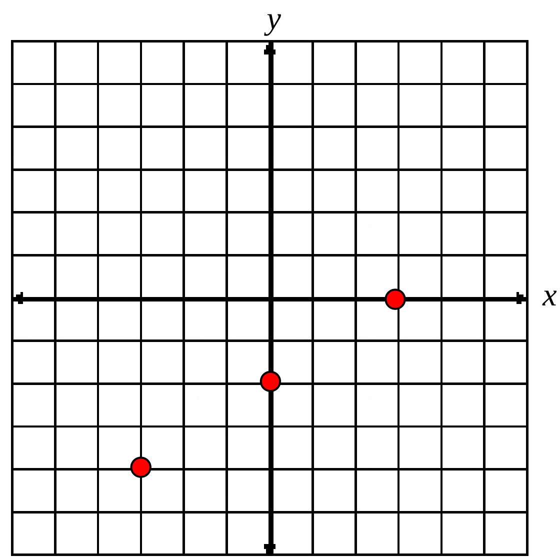
$y \geq \dfrac{2}{3}x - 2$ Divide both sides by 3.

Next, identify the slope and *y*-intercept.

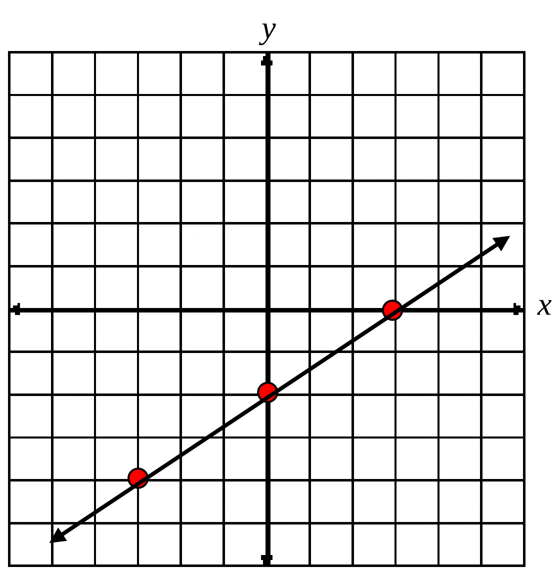
$m = \dfrac{2}{3}$ $\qquad\qquad$ $y\text{-int} = (0, -2)$

Next, plot the *y*-intercept $(0, -2)$ and more points based on the slope $(m = \dfrac{2}{3})$.





Next, connect the points using a solid line because the inequality is $\geq$, "greater than or equal to".



Finally, determine the region to shade.

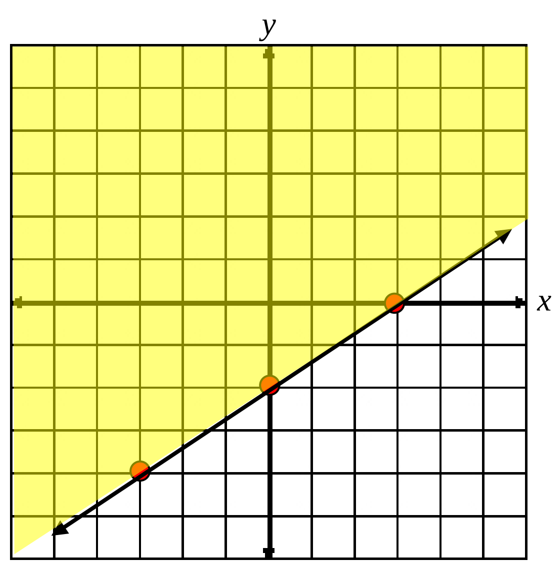
Test $(0,0)$:

$-2x+3y \geq -6$

$-2(0)+3(0) \geq -6$ Substitution (Test Point: $(0,0)$ $x=0, y=0$)

$0 \geq -3$ Simplify

This is a true statement, so shade the side of the boundary line that contains (0,0).



A system of linear inequalities is like a system of equations except that the solution to a system of inequalities is a region on the coordinate plane that represents the intersection of both inequalities.

To solve a system of inequalities:

1) Graph each inequality on a coordinate plane (one at a time).

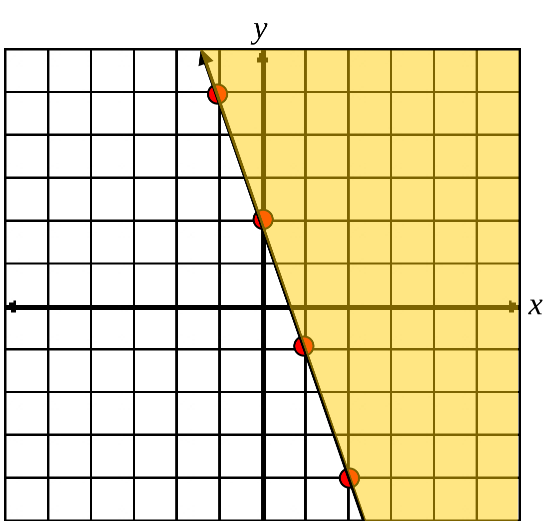
2) Shade the solution to each inequality.

3) Determine where the shading of both inequalities intersects.

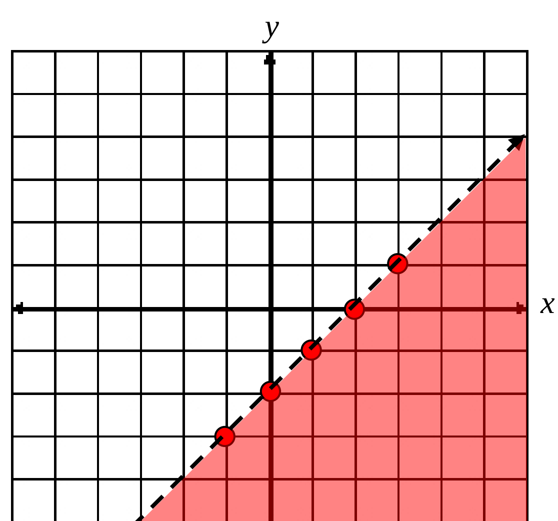
Example 3: Find the locus of points that satisfy the graphs of both inequalities.

$$y \geq -3x + 2$$
$$y < x - 2$$

Graph the boundary line of $y \geq -3x + 2$ and shade the locus of points to the right of the line.

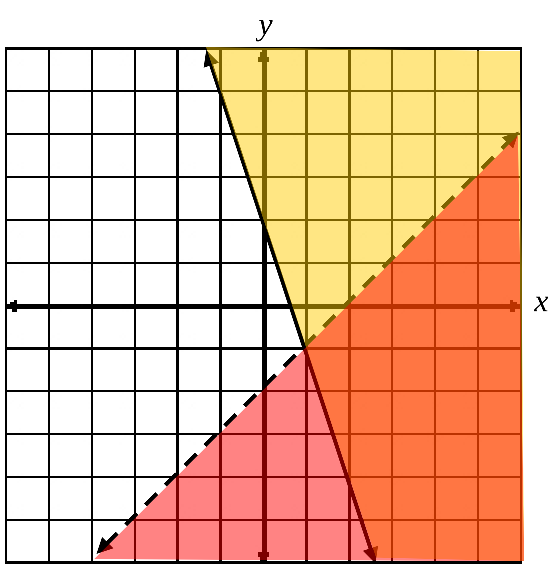


Graph the boundary line of $y < x - 2$ and shade the locus of points below the line.



The two inequalities are shown on separate planes above to help visualize the final solution.

Now, slide the two solutions together to determine the overlapping area.



*Notice that the intersection of the two inequalities is the darker (orange) shading on the right side of the coordinate plane. This region represents all the points below $y < x - 2$ and above $y \geq -3x + 2$, the solution to the system of inequalities.

The locus of points that satisfy the graphs of both given inequalities is shown in the overlapping area of the two inequalities (orange darker shaded area).

Check: To do a "spot check" of the points in the shaded area, test any point located in the shaded region to see if the point tests true in both inequalities.

We'll check $(5, 0)$. $\quad x = 5,\ y = 0$

$$y \geq -3x + 2 \quad \rightarrow \quad 0 \geq -3(5) + 2 \quad \rightarrow \quad 0 \geq -13 \qquad \textit{True}$$

$$y < x - 2 \quad \rightarrow \quad 0 < 5 - 2 \quad \rightarrow \quad 0 < 3 \qquad \textit{True}$$

Intersection of Loci with Nonlinear Equations

You will now investigate the intersection of loci of nonlinear and linear equations.

Example 1: Find the locus of points in a coordinate plane that satisfy both of the graphs of the equations shown below.

$$y = (x+2)^2 - 3 \qquad \text{and} \qquad y = -2x - 4$$

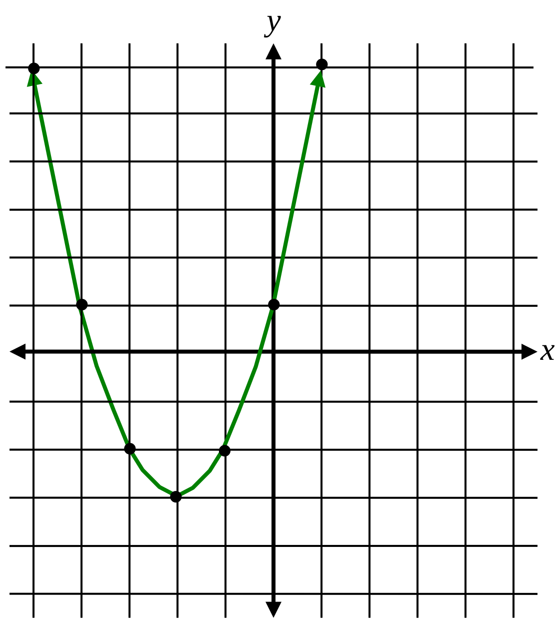
Method 1: Solve by Graphing

Graph both equations and find the locus of points that satisfy both equations.

*Note: A review link to graphing parabolas from Algebra I is provided in the content section of this unit.

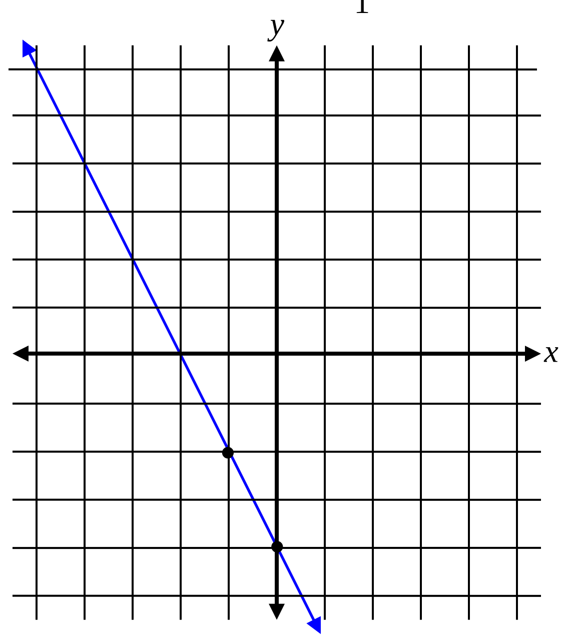
$y = (x+2)^2 - 3$

The graph is a parabola with its vertex at $(-2, -3)$.

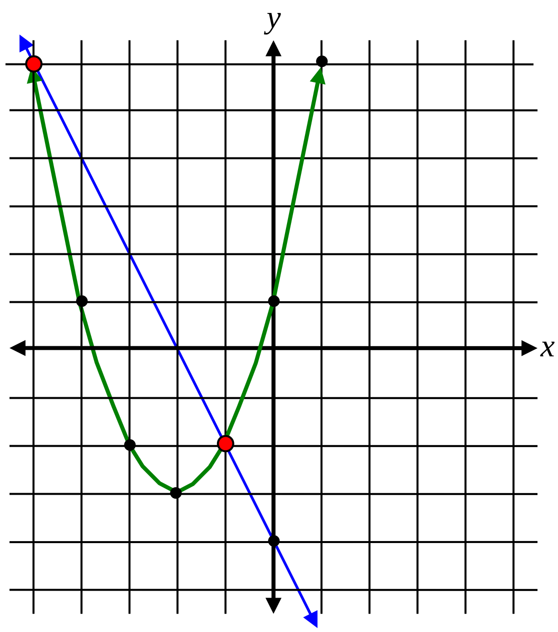


$y = -2x - 4$

The graph is a line with a y-intercept of -4 and a slope of $-\frac{2}{1}$.



The graphs intersect at (–1, –2) and (–5, 6).



The locus of points that satisfy both equations are the points (–1, –2) and (–5, 6).

Method 2: Solve by Using Algebra

First, substitute $-2x-4$ for y in the equation $y=(x+2)^2-3$ and solve for x.

$y=(x+2)^2-3$	Given equation
$-2x-4=(x+2)^2-3$	Substitution $(y=-2x-4)$
$-2x-4=x^2+4x+4-3$	Use "foil" to expand $(x+2)^2$.
$-2x-4=x^2+4x+1$	Simplify
$-4=x^2+6x+1$	Add $2x$ to both sides.
$0=x^2+6x+5$	Add 4 to both sides.
$(x+1)(x+5)=0$	Factor to solve the quadratic equation.
$x=-1$ or $x=-5$	Set each factor equal to 0 and solve.

Next, substitute the x-values determined above to solve for the y-values.

For $x = -1$:

$y = -2x - 4$	Given equation.
$y = -2(-1) - 4$	Substitution $(x = -1)$
$y = -2$	Simplify

One solution (point of intersection) is $(-1, -2)$.

For $x = -5$:

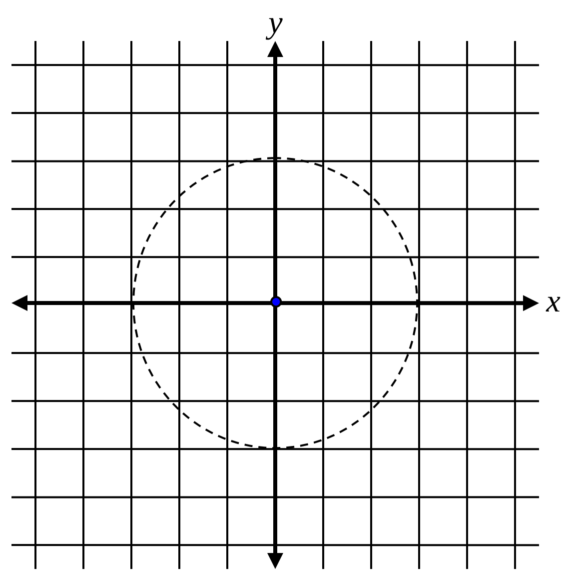
$y = -2x - 4$	Given equation.
$y = -2(-5) - 4$	Substitution $(x = -5)$
$y = 6$	Simplify

A second solution (point of intersection) is $(-5, 6)$.

The locus of points that satisfy both conditions are (–1,–2) and (–5, 6).

Example 2: Find the locus of points in a coordinate plane that satisfy the inequality shown below.

$$x^2 + y^2 > 9$$



First, the boundary of the locus of points for this inequality is a circle with a radius of 3 and its center at the origin. (See the previous unit titled, *Special Segments and Equations of Circles*, for a review of graphing equations of circles.)

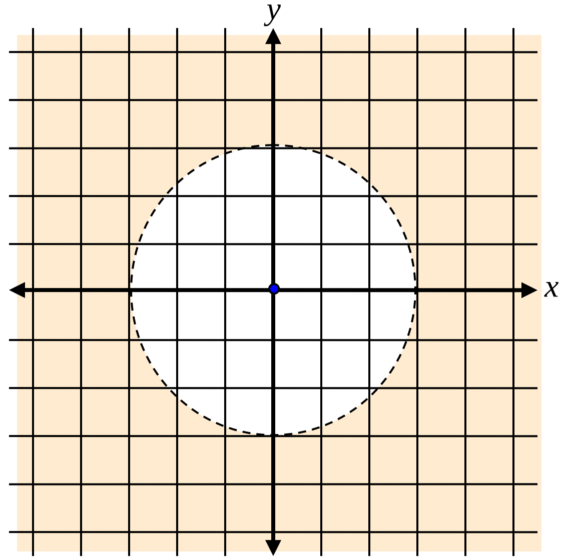
Next, we'll test point (0, 0) to determine whether to shade the interior of the circle or the exterior of the circle.

$0^2 + 0^2 > 9$ Substitute (0, 0) in for x and y.

$0 > 9$ Simplify

This is a false statement, so shade the exterior of the circle, the region bounded by the circle that does NOT contain the point (0, 0).

The graph of the inequality is all points in the coordinate plane that are located on the exterior (outside) of the circle.



The locus of points for the inequality, $x^2 + y^2 > 9,$ are all points in the coordinate plane that are located in the exterior of the equation, $x^2 + y^2 = 9.$

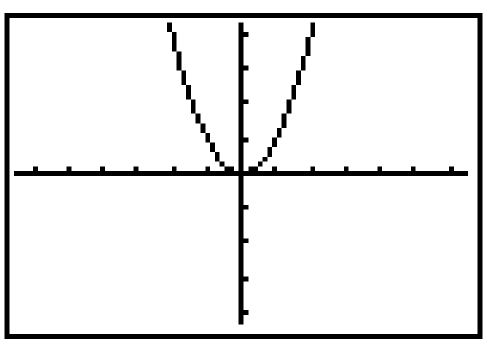
Graphing Quadratic Functions

Quadratic Function

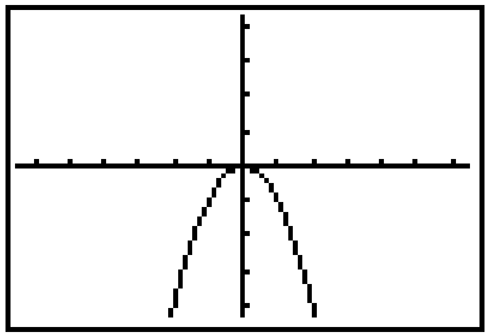
A quadratic function is a function of the form $y = ax^2 + bx + c$ where a, b, and c are real numbers and $a \neq 0$.

The graph of a quadratic function is a curve known as a **parabola** and is shown below. The lowest point on this parabola is the **minimum value** of the function, the point (0, 0) and is called the **vertex** of the parabola. In a parabola there is a vertical line called the axis of symmetry drawn through the vertex, that reflects the parabola across the line of x = h, or in other words, splits the parabola into two equal parts. In the case below, the axis of symmetry would be the y-axis or x = 0.

$y = x^2$



If "*a*" is negative ($y = -x^2$), the parabola opens down and therefore has a maximum value at the vertex (0, 0).



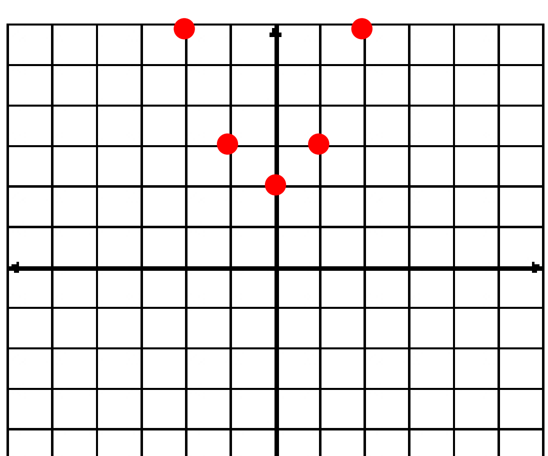
To graph a quadratic equation, you will need to make a table of values. Study the example below.

Example #1: Graph $y = x^2 + 2$

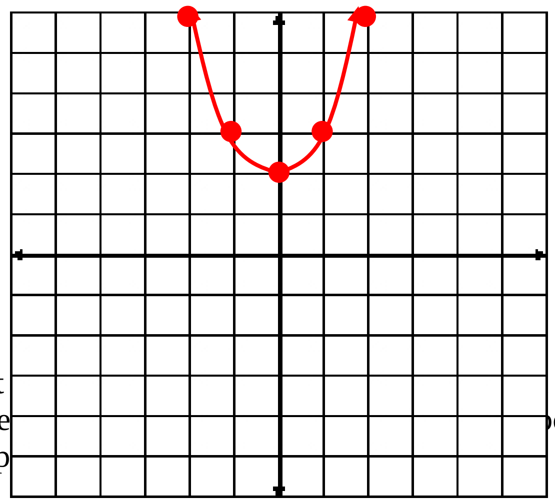
a) Make a table of values using positive and negative *x*-values.

x	$x^2 + 2$	y
–2	$(-2)^2 + 2$	6
–1	$(-1)^2 + 2$	3
0	$(0)^2 + 2$	2
1	$(1)^2 + 2$	3
2	$(2)^2 + 2$	6

b) Graph the points from above on a coordinate plane.

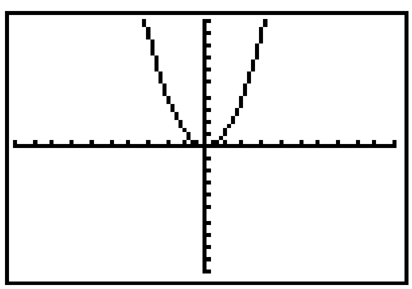


c) Connect the points using a curve (remember that parabolas are curved).

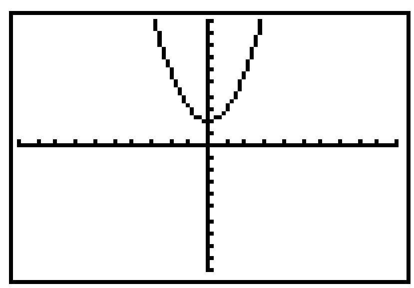


*Notice that … 2) as opposed to (0, 0) as shown in the … both functions and decide why this hap…

a.) $y = x^2$



b.) $y = x^2 + 2$



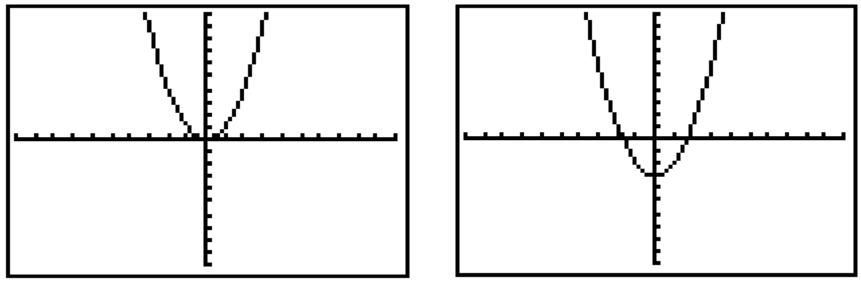
If you compare both equations, you will notice that in the equation in example a, there is no constant term, whereas in the equation in example b, there is a constant of +2. The graph of equation in example b has moved up the *y*-axis 2 units; so you can conclude that a constant moves the graph vertically along the *y*-axis.

Let's use $y = x^2$ as our parent function with a vertex at (0, 0) and compare it with $y = x^2 - 3$. What do you think will happen in this case?

$y = x^2$ $\qquad\qquad$ $y = x^2 - 3$

Table of Values

x	***y***
–2	1
–1	–2
0	–3
1	2



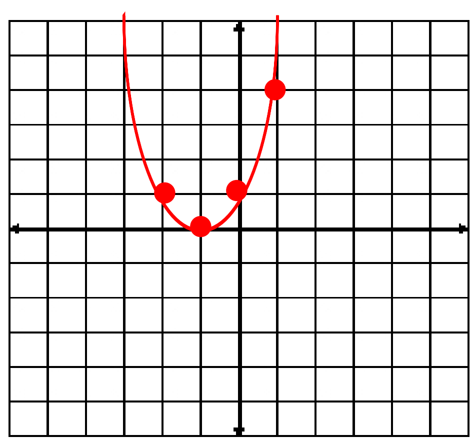
*Due to the pixels in the graphing calculator, it may look like the second equation passes through (2, 0) and (–2, 0). This is not the case and at this point we are only concerned about what the constant does to the graph.

By examining the two equations you can see that by subtracting 3 the graph has moved down 3 units.

Example #2: Graph $y=(x+1)^2$

Make a table of values and graph the points.

x	***y***
–2	1
–1	0
0	1
1	4



Notice that in this example we do not have the same number of points on each side of the vertex. If you want, you can substitute more values for *x* to determine how wide the parabola gets on the left.

Notice that when the constant term is in a quantity with the variable, the vertex moved horizontally along the *x*-axis. In the case above, it moved one unit to the **left**. What do you suppose will happen to the vertex of $y=(x-2)^2$? If you answered that the vertex would move 2 units to the right you are correct. This concept now brings us to the vertex form of a quadratic ($y = -3(x-8)^2 + 10$) which will be explained in another unit.

Vertex Form of a Quadratic

Vertex Form

The vertex form of a quadratic function is $y = a(x-h)^2 + k$, where (h, k) is the vertex and $x = h$ is the axis of symmetry. When "*a*" is positive, the parabola opens up and the vertex is the minimum value. When "*a*" is negative, the parabola opens down and the vertex is the maximum value.

You are now ready to identify the direction of opening, vertex, and axis of symmetry of quadratic functions in vertex form.

Example: Identify the direction of opening, vertex, and axis of symmetry for the following quadratic functions.

a.) $y = 2(x-3)^2 + 5$

direction of opening: up
because "*a*" (2) is +

vertex: (3, 5)

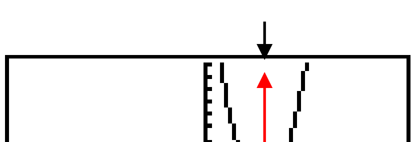
axis of symmetry: $x = 3$

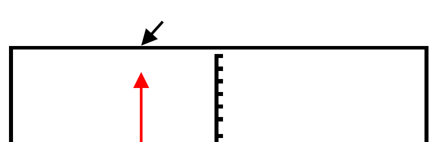
b.) $y = -3(x+4)^2 - 1$

direction of opening: down
because "*a*" (–3) is –.

vertex: (–4, –1)

axis of symmetry: $x = -4$





axis of symmetry
$x = 3$

axis of symmetry
$x = -4$

Graph Paper