# POLYGONS, TESSELLATIONS, AND AREA

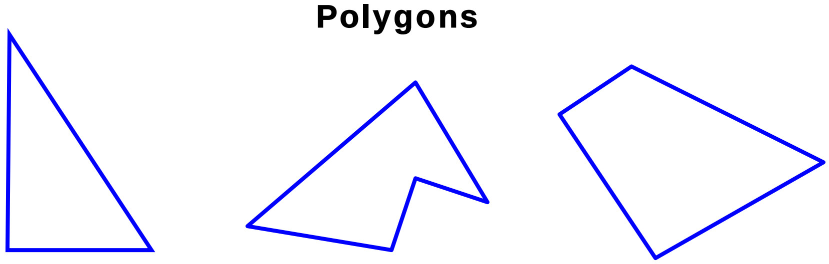
In this unit, you will learn what a polygon is and isn't. You will investigate shapes that tessellate a plane and shapes that do not. You will also compute area and find dimensions of triangles and various types of quadrilaterals.

## Polygons

**polygon** - A polygon is a closed figure that is made up of line segments that lie in the same plane. Each side of a polygon intersects with two other sides at its endpoints.

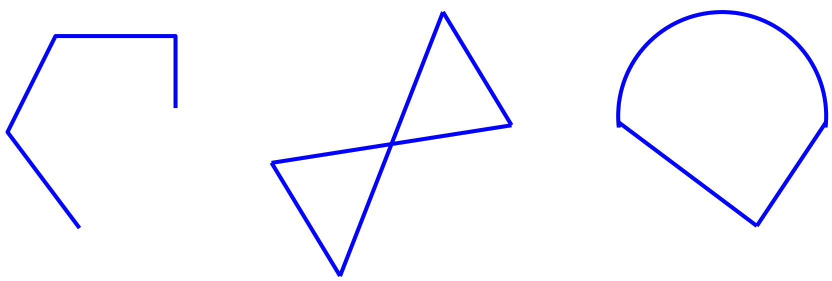
Here are some examples of polygons:

**Polygons**



Here are some examples of figures that are NOT classified as polygons. The reason the figure is not a polygon is shown below it.

**Not Polygons**

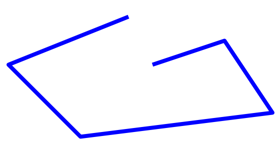


The path is open.

Two sides of the figure intersect at a point other than the endpoints.

One side of the figure is curved.

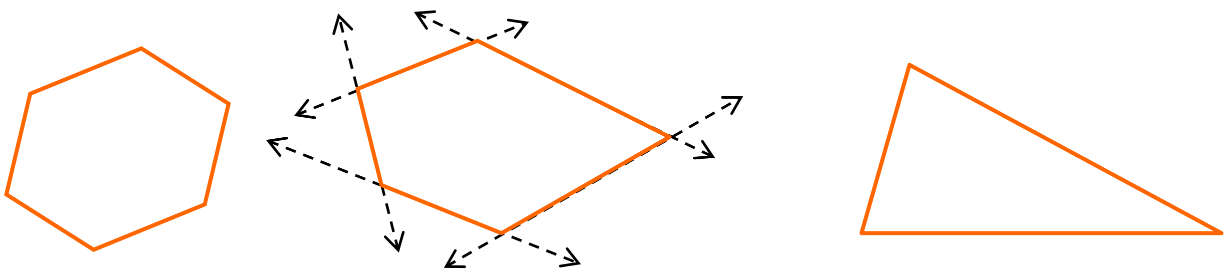
*Example 1*: Is the shape a polygon? Explain why or why not.



The shape is not a polygon since the path is open.

**convex polygon** – A convex polygon is a polygon in which **none** of its sides lie in the interior of the polygon.

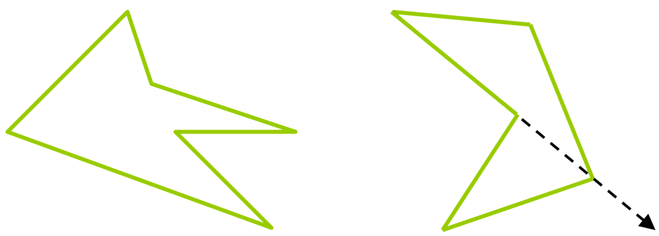
**convex polygons**



When the sides are extended as lines, none of the lines fall in the interior of the polygons.

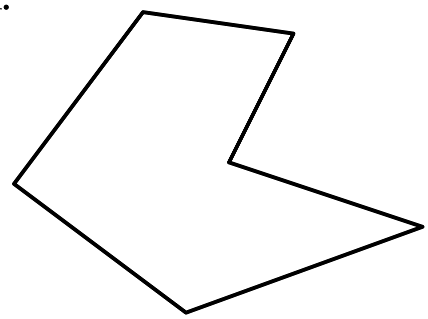
**concave** polygon – A concave polygon is a polygon which has sides that, when extended, lie within  the interior of the polygon.

**concave polygons**

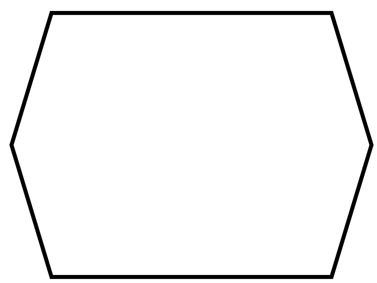


*Example 2*:  Which polygon is convex?  Please explain.

A**.**



B.



Choice B is the convex polygon.  A convex polygon is a polygon in which none of its sides lie in the interior of the polygon when they are extended.  Choice A is a concave polygon.

In general, a polygon with $n$ sides is called an *n-gon*.   Several common polygons have been given names based on the number of sides.

| Number of Sides | Polygon |
|---|---|
| 3 | triangle |
| 4 | quadrilateral |
| 5 | pentagon |
| 6 | hexagon |
| 7 | heptagon |
| 8 | octagon |
| 9 | nonagon |
| 10 | decagon |
| 12 | dodecagon |
| $n$ | $n$-gon |

**regular polygon –** A regular polygon is a convex polygon with **all sides and angles congruent**.

**Theorem 28-A**

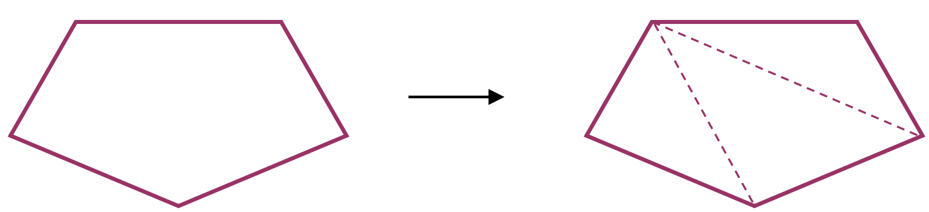
**In a convex polygon with $n$ sides, the sum of its interior angles equals 180($n$ - 2) degrees.**

$$S = 180(n - 2)$$

$S$ represents the sum of all interior angles.
$n$ represents the number of sides in a polygon.

*Example 3*:  Draw a pentagon and draw all possible non-overlapping diagonals from one vertex, and then answer the following questions.

(a) How many triangles are formed?



Three triangles are formed.

(b) Apply the Triangle Sum Theorem to determine the total number of degrees for the five angles in a pentagon.

Three triangles are formed by the non-overlapping diagonals. The sum of the angles in three triangles formed is equal to the sum of all five angles in the pentagon. Thus, the following statements can be made:

| | |
|---|---|
| $S = 180 + 180 + 180$ | Results of dividing the pentagon into triangles. |
| $S = 180(3)$ | Meaning of Multiplication |
| $S = 540$ | Simplify |

The sum of the five angles in a pentagon is 540 degrees.

(c) Now apply Theorem 28-A to solve the same problem.

| | |
|---|---|
| $S = 180(n-2)$ | Theorem 28-A |
| $S = 180(5-2)$ | Substitution (A pentagon has 5 sides.) |
| $S = 180(3)$ | Simplify |
| $S = 540$ | Simplify |

By theorem 28-A, we have determined that the sum of the five angles in a pentagon is 540 degrees.

The formula stated in Theorem 28-A simplifies the method to determine the sum of the angles in any convex polygon.

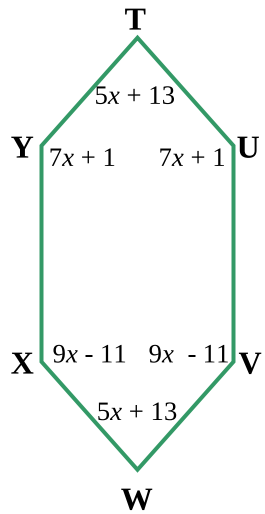
*Example 4*: What is the sum of the angles in a dodecagon?

Since there are twelve sides in this polygon, we will use Theorem 28-A to shorten our work.

| | |
|---|---|
| $S = 180(n-2)$ | Theorem 28-A |
| $S = 180(12-2)$ | Substitution (A dodecagon has 12 sides.) |
| $S = 180(10)$ | Simplify |
| $S = 1800$ | Simplify |

The sum of the twelve angles in a dodecagon is 1800 degrees.

*Example 5*: Find the measure of each angle in hexagon TUVWXY. Each angle is represented by the expression displayed in the diagram.




*Step 1:* Find the measure of the sum of the angles in a hexagon.

| | |
|---|---|
| $S = 180(n-2)$ | Theorem 28-A |
| $S = 180(6-2)$ | Substitution (A hexagon has 6 sides.) |
| $S = 180(4)$ | Simplify |
| $S = 720$ | Simplify |

*Step 2:* Solve for *x*.

| | |
|---|---|
| $2(5x+13)+2(7x+1)+2(9x-11)=720$ | Write an equation. |
| $10x+26+14x+2+18x-22=720$ | Distributive Property |
| $42x+6=720$ | Collect Like Terms |
| $42x=714$ | Subtraction |
| $x=17$ | Division |

*Step 3:* Find the measures of each of the angles.

**Angles *T* and *W***

| | |
|---|---|
| $5x+13$ | Expression for $\angle T$ and $\angle W$. |
| $5(17)+13$ | Substitution |
| $98$ | Simplify |

Angles *T* and *W* measure 98 degrees.

**Angles *Y* and *U***

| | |
|---|---|
| $7x+1$ | Expression for $\angle Y$ and $\angle U$. |
| $7(17)+1$ | Substitution |
| $120$ | Simplify |

Angles *Y* and *U* measure 120 degrees.

**Angles *X* and *V***

| | |
|---|---|
| $9x-11$ | Expression for $\angle X$ and $\angle V$. |
| $9(17)-11$ | Substitution |
| $142$ | Simplify |

Angles *X* and *V* measure 142 degrees.

*Check:* $2(98)+2(120)+2(142)=720$
The six angles total 720 degrees.

*Example 6*: What is the size of an interior angle of a regular dodecagon?

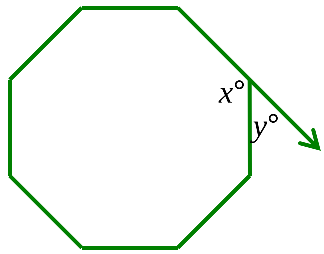
| | |
|---|---|
| $S=180(n-2)$ | Theorem 28-A |
| $S=180(12-2)$ | Substitution (A dodecagon has 12 sides.) |
| $S=180(10)$ | Simplify |
| $S=1800$ | Simplify |

$\frac{1800}{12}$ Divide by 12 for the twelve congruent angles.

Each angle equals 150 degrees.

The size of an interior angle in a regular dodecagon is 150 degrees.

*Example 7*:  For the regular octagon shown below, answer each question.




(a) What is the size of one interior angle ($x$)?

| | |
|---|---|
| $S = 180(n-2)$ | Theorem 28-A |
| $S = 180(8-2)$ | Substitution (An octagon has 8 sides.) |
| $S = 180(6)$ | Simplify |
| $S = 1080$ | Simplify |

| | |
|---|---|
| $\frac{1080}{8}$ | Divide by 8 for the eight congruent interior angles. |

One interior angle measures 135 degrees; $x = 135$ degrees.

(b) What is the size of one exterior angle ($x$)?

| | |
|---|---|
| $x + y = 180$ | $x$ and $y$ are linear angles. |
| $135 + y = 180$ | Substitution ($x = 135$ from part a) |
| $y = 45$ | Subtraction |

One exterior angles measures 45 degrees; $y = 45$ degrees.

(c) What is the sum of all the exterior angles?

| | |
|---|---|
| 45(8) | There are 8 exterior angles in an octagon. |
| 360 | Simplify |

The sum of the exterior angles of a regular octagon is 360 degrees.

Examine the exterior angles of other regular polygons. You will find that the sum of the exterior angles of any regular polygon total 360 degrees.

**Theorem 28-B**

**In a convex polygon, the sum of the exterior angles is 360 degrees.**

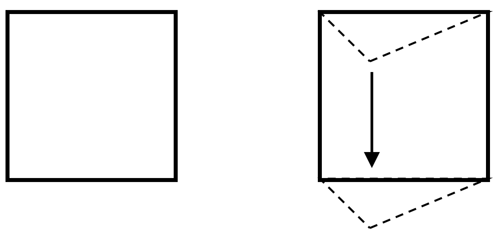
## Tessellations

**tessellation** – A tessellation is a complete pattern of repeating shapes or figures that cover a plane leaving no spaces or gaps.

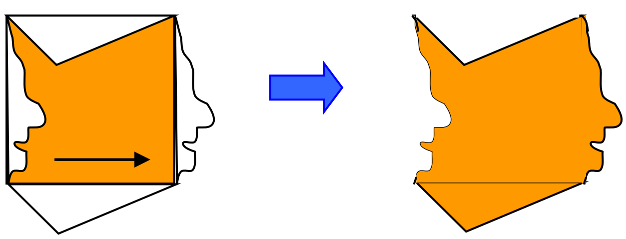
A tessellation may be created using slides, flips, and turns. Interesting tessellations may be formed beginning with a square or equilateral triangle.

*Example 1*: Create a figure from a square that will tessellate a plane.

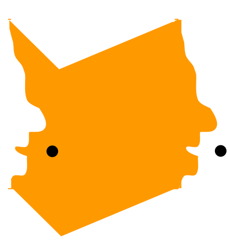
1) Start by drawing a square. Cut a triangle from its side and slide the triangle to the opposite side. (Use tape to attach the triangle.)



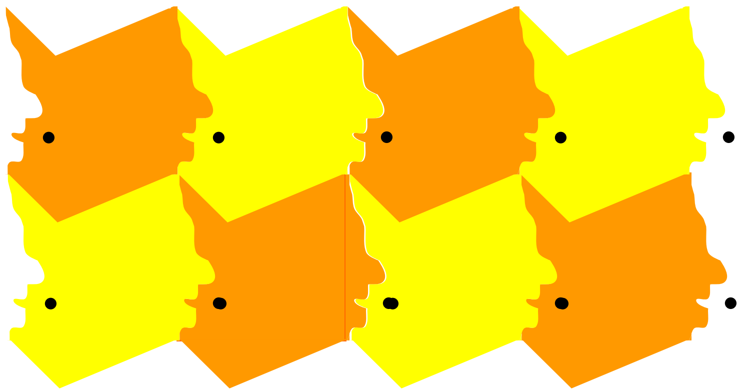
2) Sketch a face on the left side, cut it out and slide it to the opposite side.



3) Add additional features to make the figure more interesting.



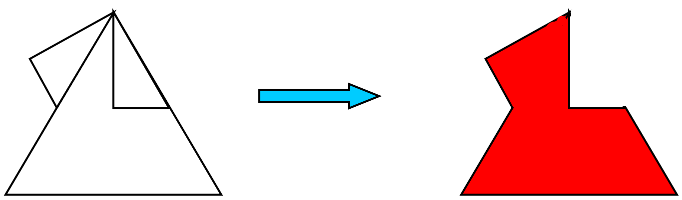
4) Copy the shape across and down, fitting the shapes together like a puzzle.



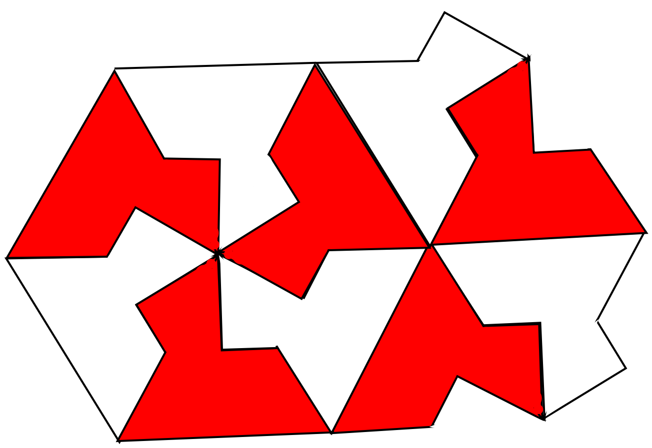
A design is created that tessellates a plane.

*Example 2*: Create a figure from an equilateral triangle that will tessellate a plane.

1) Draw an equilateral triangle and cut out a shape from one vertex and then rotate it around the vertex to another side.

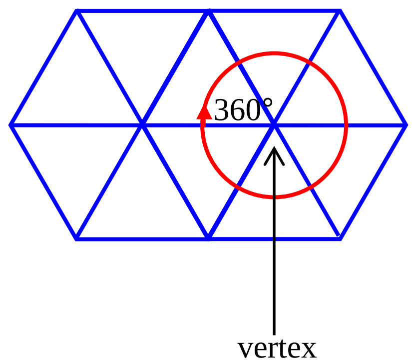


2) Tessellate the plane.



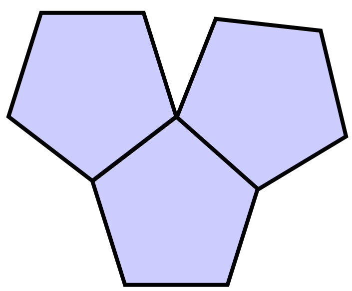
You can tessellate a plane with regular polygons when they form a 360 angle at their touching vertices.

*Example 3*: Will an equilateral triangle tessellate a plane?




Yes, the vertices meet to form a 360 degree angle. Recall that the three angles in an equilateral triangle are congruent with each one measuring 60 degrees.

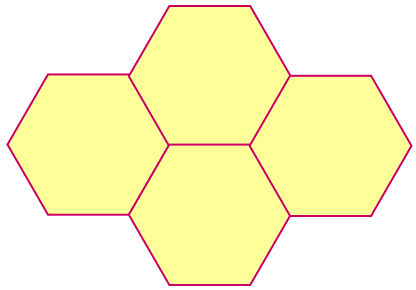
*Example 4*: Will a regular pentagon tessellate a plane?



Recall that one angle in a regular pentagon measures 108 degrees. Three angles of three pentagons meeting at a vertex will total 324 degrees and the sum of four angles would be greater than 360 degrees. Since the three angles do not total 360 degrees, there is a gap; and therefore, regular pentagons alone will not tessellate a plane.

**regular tessellation –** A regular tessellation uses one type of regular polygon to tessellate a plane. Not all regular polygons tessellate a plane. In the examples above, we have seen that equilateral triangles tessellate a plane, but pentagons do not tessellate a plane.

*Example 5*: Give an example of a regular polygon that may be used to create a regular tessellation.



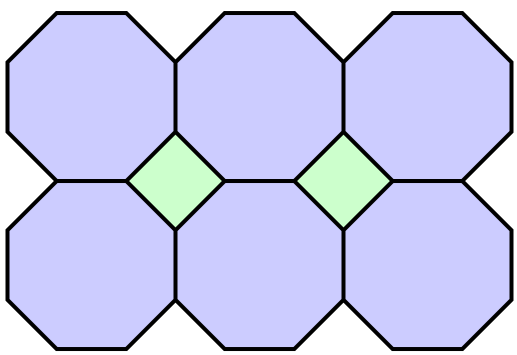
A regular hexagon tessellates the plane. At any one vertex of the tessellation, three angles of a hexagon, each measuring 120 degrees, meet.

$$120(3) = 360$$

Therefore, hexagons tessellate a plane and are an example of a regular tessellation.

**semi-regular tessellation** – A semi-regular tessellation is a tessellation that contains two or more regular polygons that tessellate the plane.

*Example 6*: What combination of regular shapes tessellates a plane?



A regular octagon by itself will not tessellate a plane; however, combine it with a square that is turned to give the appearance of a diamond shape and the two together tessellate the plane. Let's examine why this is true.

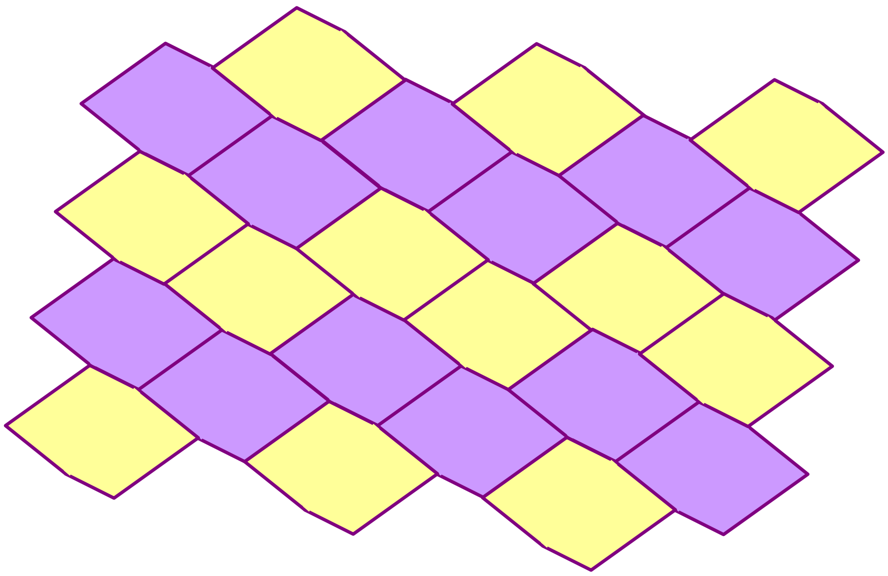
At any one vertex of the figure, two angles from the two octagons and an angle from the square meet. Each of the angles in the regular octagon measure 135 degrees. The angle from the square measures 90 degrees.

$$135 + 135 + 90 = 360$$

The combination of regular octagons and squares will tessellate the plane when they are arranged such that three angles meeting at one vertex total 360 degrees. The tessellation is an example of a semi-regular tessellation.

**uniform tessellation** – A uniform tessellation is a tessellation that has the same combination of shapes and angles at each vertex.

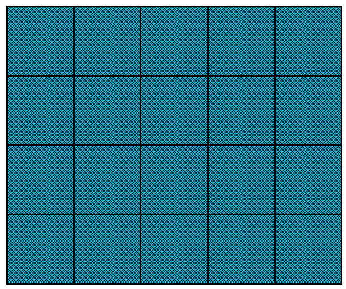
The figure below is an example of a uniform tessellation.



## Area of Parallelograms

The **area of a rectangle** is the product of its base and height. Area is a measurement of coverage and is measured in **square units**.

**Height = 4 units**



**Base = 5 units**

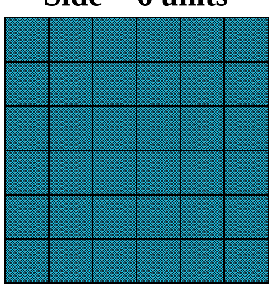
$$A = b \times h$$
$$A = 5(4)$$
$$A = 20 \text{ square units}$$

The **area of a rectangle** is the product of its base and height.

***A = bh***

The **area of a square** is the product of its base and height. Since squares have sides of equal length, the area of a square is the product of its length (side) and its width (side). Area is a measurement of coverage and is measured in **square units**.

**Side = 6 units**



| | |
|---|---|
| $A = bh$ | A square is a rectangle. |
| $A = s \times s$ | A square's base and height are the same (*s*). |
| $A = s^2$ | Simplify |
| $A = 6^2$ | |
| $A = 36$ | Simplify |

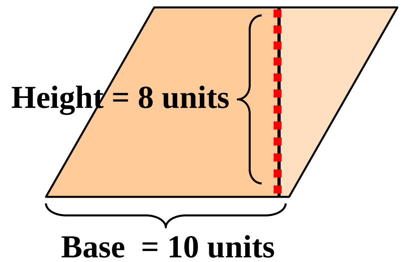
The area of a square that measures 6 units on one side equals 36 square units.

The **area of a square** is the square of the length of one side.

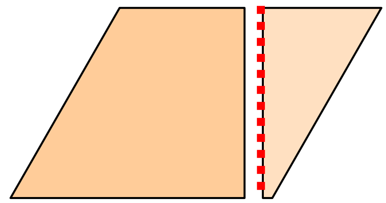
$$\boldsymbol{A = s^2}$$

The **area of a parallelogram** can be rearranged into the shape of a rectangle if the parallelogram is cut along an altitude (perpendicular height) from the top to its base.

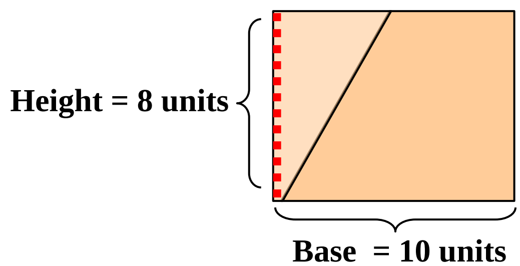
Given: Parallelogram with a base of 10 units and a height of 8 units.




Cut the parallelogram along the height.



Rearrange the area into the shape of a rectangle by sliding the piece that is right of the altitude to the left, aligning it with the left side of the parallelogram.

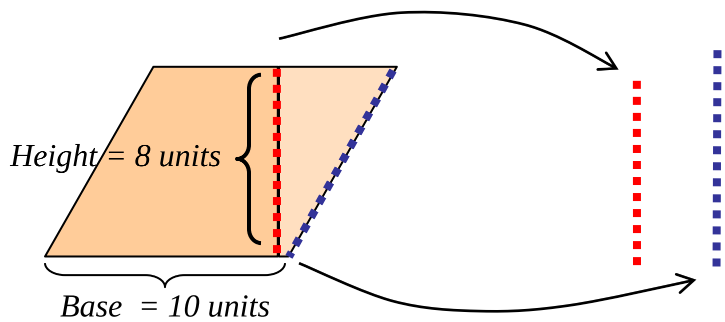



The amount of coverage, the height, and the base all remain the same even after the rearrangement of the pieces of the parallelogram. Thus, the formula for finding the area of a parallelogram is the same as finding the area of a rectangle. Just remember to use the height of the parallelogram as the height, not the length of the slanted sides.

The **area of a parallelogram** is the product of its base and height.

$$A = bh$$

Note:  The height of a parallelogram is shorter than the length of the slanted side.  Be sure to measure the height of a parallelogram, not its slanted side, when determining is area.




*Example 1*:  Find the area of a parallelogram with a base of 10 units and a height of 8 units.

| | |
|---|---|
| $A = b \times h$ | Formula for area of parallelogram |
| $A = 10(8)$ | Substitution |
| $A = 80$ square units | Multiply |

*Example 2*: Find the area of parallelogram ABCD.

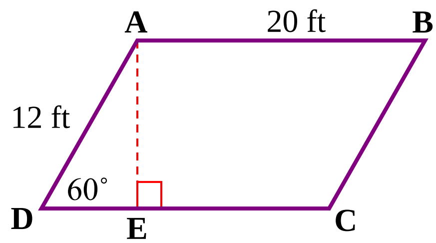
Given:

Parallegram $ABCD$

$\angle ADE = 60°$

$AD = 12$ ft

$AB = 20$ ft




We know that the formula for finding the area of a parallelogram is $A = bh$.

We are given the base (20 ft), but must calculate the height.

Triangle $AED$ is a 30-60-90 degree right triangle. Recall that the shorter leg of this type of triangle is equal to $\frac{1}{2}$ of the hypotenuse and the longer leg equals the the length of the short leg times $\sqrt{3}$..

| | |
|---|---|
| $AD$ (Hypotenuse) $= 12$ | Given |
| $DE$ (Shorter Leg) $= 6$ | $\frac{1}{2}$ of the hypotenuse |
| $AE$ (Longer Leg) $= 6\sqrt{3}$ | shorter leg $\times$ $\sqrt{3}$ |

Thus, the height of the parallelogram is $6\sqrt{3}$.

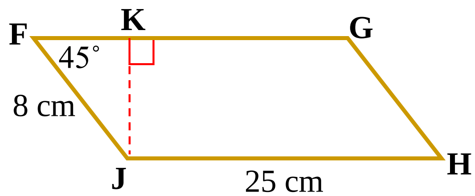
Now we have all that we need to find the area of the parallelogram.

| | |
|---|---|
| $A = b\ h$ | Formula for area of a parallelogram. |
| $A = 20 \times 6\sqrt{3}$ | Substitution |
| $A = 120\sqrt{3}$ | Simplify by multiplying the whole numbers outside the radical. |

The area of the parallelogram is $120\sqrt{3}$ square feet.

*We can express the answer as a radical or evaluate it as a rounded decimal (207.8 sq ft).

*Example 3*: Find the area of parallelogram FGHJ.




We know that the formula for finding the area of a parallelogram is $A = bh$.

We are given the base (25 cm), but must calculate the height.

Triangle *FKJ* is a 45-45-90 degree right triangle. Recall that the legs of this type of triangle are equal in length and that the hypotuse equals the length of one leg times $\sqrt{2}$.

| | |
|---|---|
| $FJ$ (Hypotenuse) $= 8$ | Given |
| Let $x =$ length of $KJ$. | |
| $8 = x\sqrt{2}$ | The hypotenuse $=$ length of one leg times $\sqrt{2}$. |
| $\frac{8}{\sqrt{2}} = x$ | Division Property |
| $\frac{8 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = x$ | Multiply by $\frac{\sqrt{2}}{\sqrt{2}}$. |
| $\frac{8 \cdot \sqrt{2}}{2} = x$ | $\sqrt{2} \cdot \sqrt{2} = 2$ |
| $4 \cdot \sqrt{2} = x$ | Simplify |
| $\therefore KJ = 4\sqrt{2}$ | |

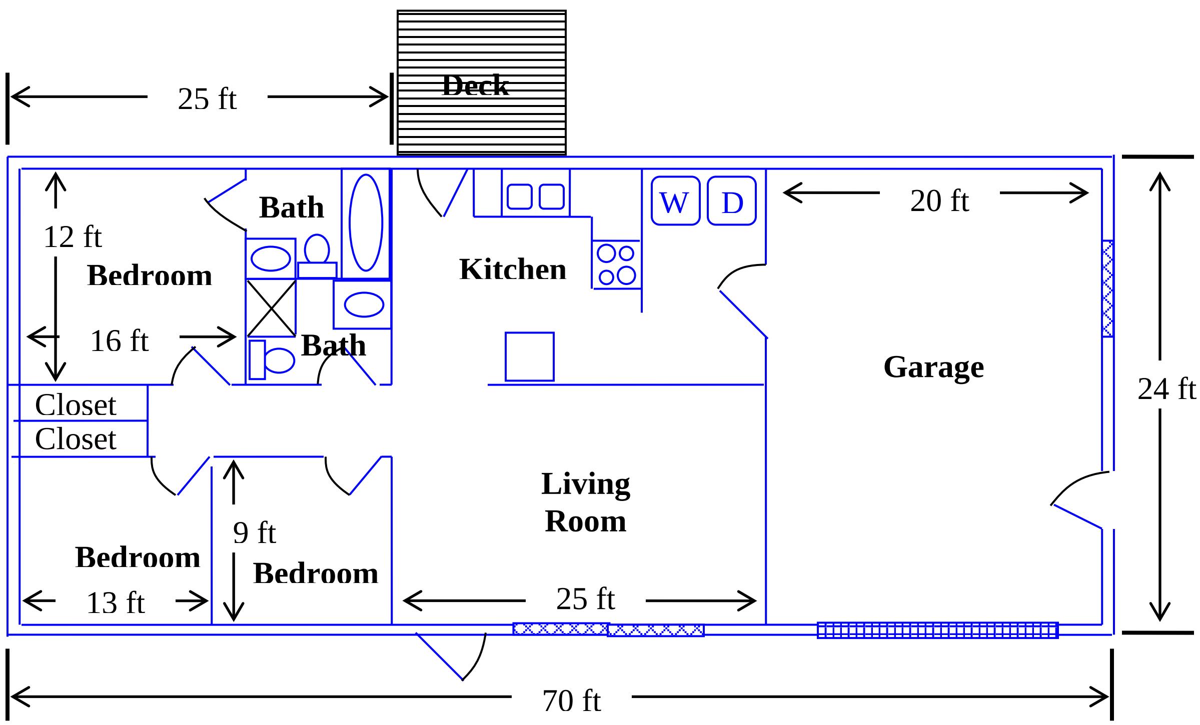
Thus, the height of the parallelogram is $4\sqrt{2}$.

Now we have all that we need to find the area of the parallelogram.

| | |
|---|---|
| $A = b\ h$ | Formula for area of a parallelogram. |
| $A = 25 \times 4\sqrt{2}$ | Substitution |
| $A = 100\sqrt{2}$ | Simplify by multiplying the whole numbers outside the radical. |

The area of the parallelogram is $100\sqrt{2}$ square centimeters.
*We can express the answer as a radical or evaluate it as a rounded decimal (141.4 sq cm).

*Example 4*: Find the amount of tiling that would be needed to cover the two bath areas for the given floor plan.




Study the floor plan and notice that the width of the bedroom and the two baths is 25 feet. Also notice that the width of the bedroom is 16 feet.

Therefore, the width of the two baths:

$$25 - 16 = 9\,\text{ft}$$

Examining the floor plan closely, notice that the length of the two baths is given as 12 feet.

Now we have all that we need to determine the area of the two bathrooms in square feet.

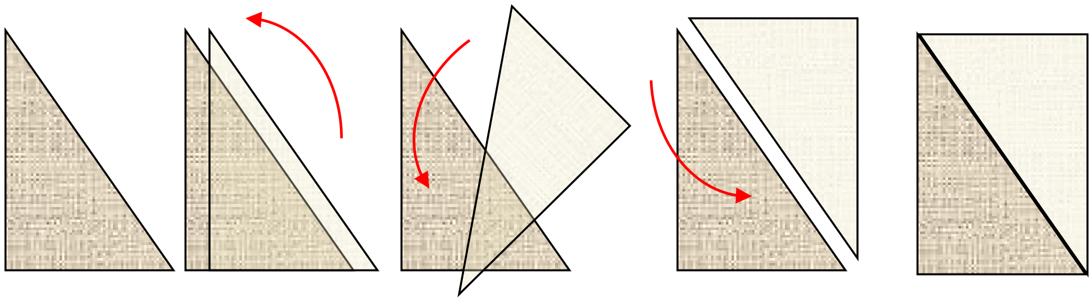
| | |
|---|---|
| $A = b\,h$ | Formula for area of a rectangle. |
| $A = 9(12)$ | Substitute |
| A = 108 | Simplify |

The area of the two baths is 108 square feet.

## Area of Other Polygons

### Triangles

A **triangle's area** is equal to half the area of a rectangle with the same base and height.



*Example 1*: Find the area of a right triangle with a base of 5 units and a height of 6 units.

**Height = 6 units**

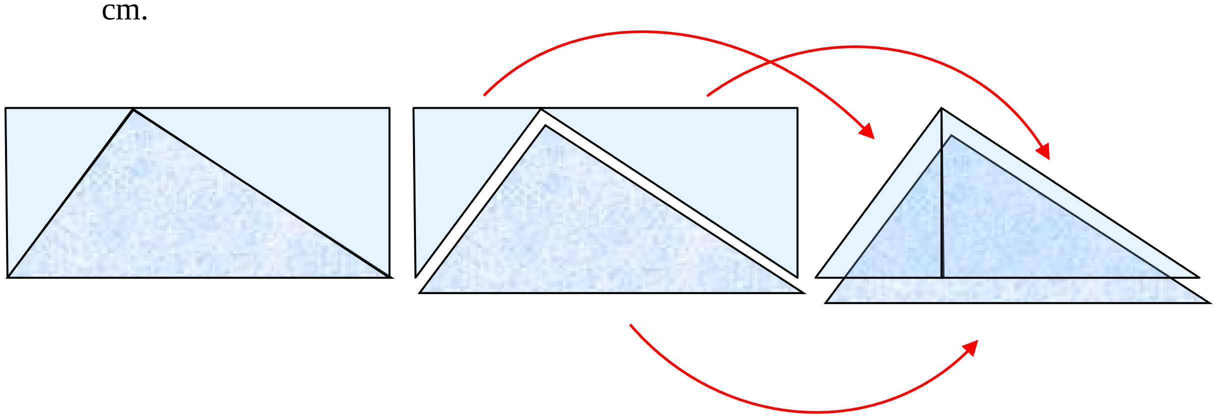
**Base = 5 units**

$$A = \frac{1}{2}bh$$

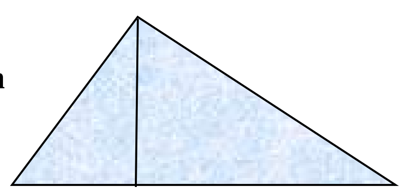
$$A = \frac{1}{2}(5)(6)$$

$$A = 15 \text{ square units}$$

*Example 2*: Find the area of a triangle with a base of 8.9 cm and a height of 4.3 cm.



**Height = 4.3 cm**



**Base = 8.9 cm**

$$A = \frac{1}{2}bh$$

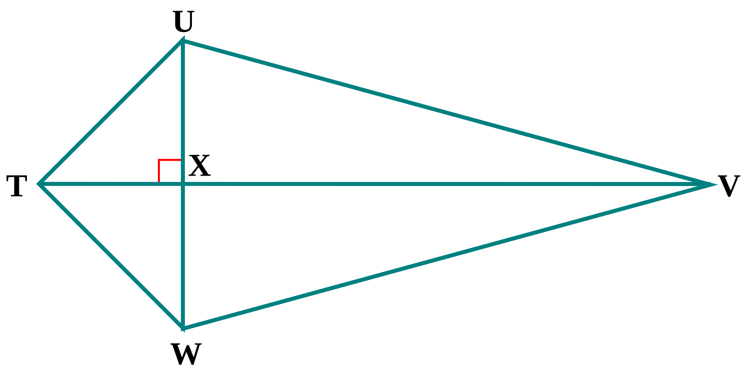
$$A = \frac{1}{2}(4.3)(8.9)$$

$$A = 19.135 \text{ sq cm}$$

The **area of a triangle** is the product of one-half its base(b) and height(h).

$$\boldsymbol{A = \frac{1}{2}bh}$$

*Example 3*:  Find the area of kite TUVW if TV = 25 cm and UX = 10 cm.




First we will examine the properties and similarities of triangles TUV and TWV to see if those observations will help us in determining the area of the kite.

| | |
|---|---|
| $\overline{TU} \cong \overline{TW}$ | Definition of kite. |
| $\overline{UV} \cong \overline{WV}$ | Definition of kite. |
| $\overline{TV} \cong \overline{TV}$ | Reflexive Property |
| $\triangle TUV \cong \triangle TWV$ | SSS |

$\therefore$ We can find the area of $\triangle TUV$, and then double it to find the area of kite $TUVW$.

Area of $\triangle TUV$

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}(25)(10)$$

$$A = 125$$

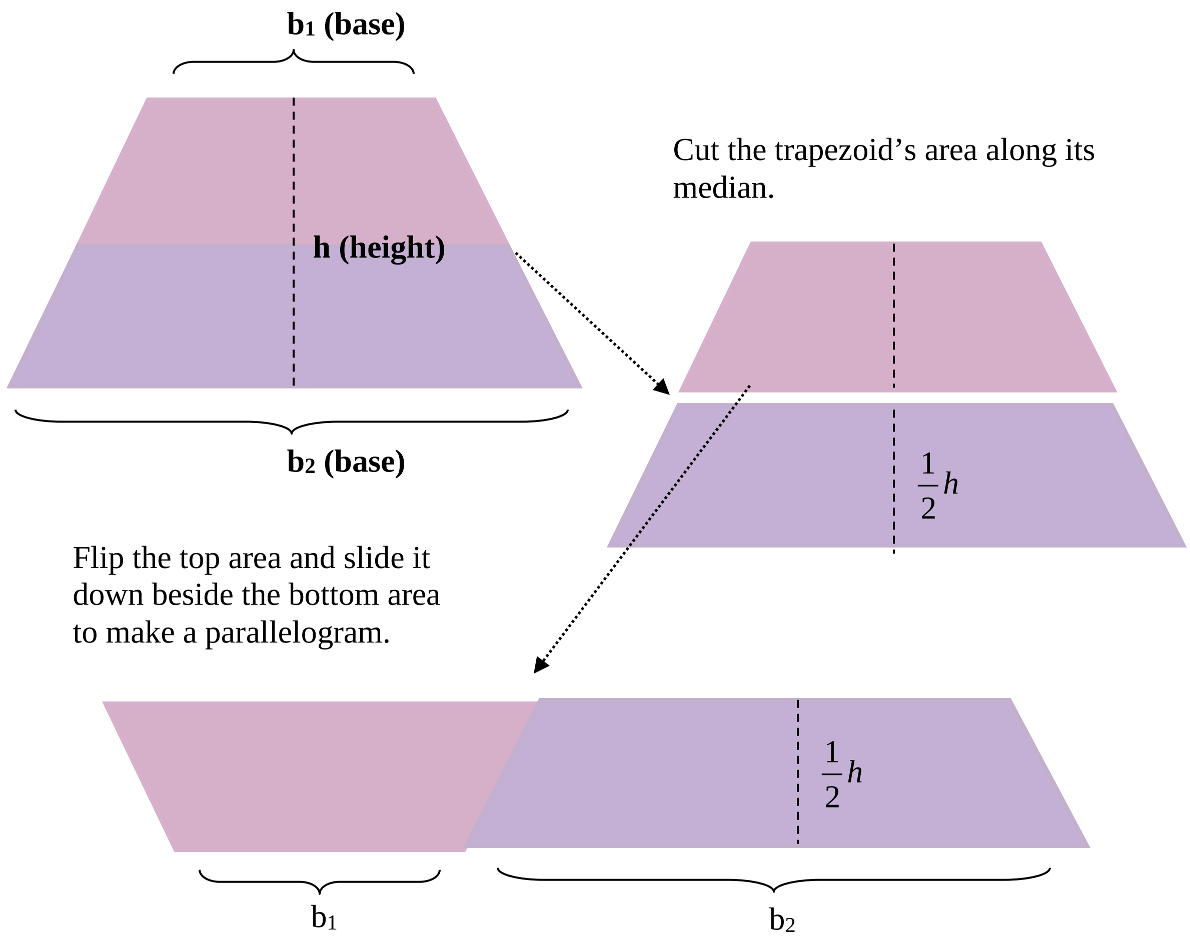
Area of $\triangle TUV$ + Area of $\triangle TWV$ = Area of kite $TUVW$

$$125 + 125 = 250$$

The area of kite $TUVW$ is 250 square centimeters.

## Area of Trapezoid

**trapezoid –** A trapezoid is a quadrilateral with exactly one pair of parallel sides.



Use the formula for the area of a parallelogram to derive the area of a trapezoid.

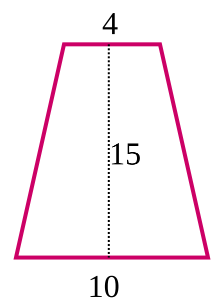
| | |
|---|---|
| $A = bh$ | Formula for area of parallelogram. |
| $A = (b_1 + b_2)(\frac{1}{2}h)$ | Substitution |
| $A = (\frac{1}{2}h)(b_1 + b_2)$ | Commutative Property |
| $A = \frac{1}{2}h(b_1 + b_2)$ | Simplify |

The **area of a trapezoid** is the product of one-half its height($h$) times the sum of its bases($b_1$ and $b_2$).

$$\boldsymbol{A = \frac{1}{2}h(b_1 + b_2)}$$

*Example 4*: Find the area of a trapezoid whose length of the two parallel sides is 4 feet and 10 feet and the height is 15 feet.

| | |
|---|---|
| $A = \frac{1}{2}h(b_1 + b_2)$ | Formula for area of a trapezoid. |
| $A = \frac{1}{2}(15)(4+10)$ | Substitute |
| $A = \frac{1}{2}(15)(14)$ | Simplify |
| $A = \frac{1}{2}(210)$ | Simplify |
| $A = 105$ | Simplify |




The area of the trapezoid is 105 square feet.

*Example 5*: Find the height of a trapezoid that has an area of 108 square millimeters and bases measure 14 and 10 millimeters.

| | |
|---|---|
| $A = \frac{1}{2}h(b_1 + b_2)$ | Formula for area of a trapezoid |
| $108 = \frac{1}{2}(h)(14+10)$ | Substitute |
| $108 = \frac{1}{2}(h)(24)$ | Simplify |
| $108 = \frac{1}{2}(24)(h)$ | Commutative Property |
| $108 = 12h$ | Simplify |
| $h = 9$ | Divide |

The height of the trapezoid is 9 millimeters.

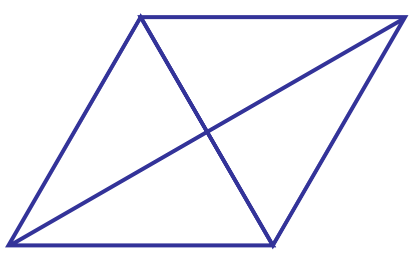
## Area of Rhombus

The area of a rhombus is determined by finding the product of its diagonals, and then taking half of that product.

The **area of a rhombus** may be expressed as half the product of its diagonals.

$$\boldsymbol{A = \frac{1}{2}d_1\, d_2}$$

*Example 6*: Find the length of the shorter diagonal of a rhombus if the longer diagonal measures 27 feet and the area of the rhombus is 324 square feet.



Given:

Area $= 324$ sq ft

$d_1 = 27$ ft

| | |
|---|---|
| $A = \frac{1}{2}d_1 d_2$ | Formula for area of a rhombus. |
| $324 = \frac{1}{2}(27)d_2$ | Substitution |
| $324 = \frac{27}{2}d_2$ | Simplify |
| $324\left(\frac{2}{27}\right) = \frac{27}{2}\left(\frac{2}{27}\right)d_2$ | Multiplication Property |
| $d_2 = 24$ | Simplify |

*Example 7*: For the previous problem, have we gathered enough information to determine the length of one side of the rhombus?

We now know the following:

$A = 324$ ft $\qquad d_1 = 27$ ft $\qquad d_2 = 24$ ft

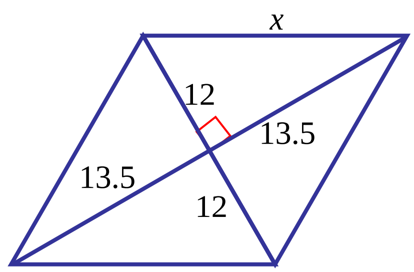
Recall the following properties of a rhombus:

(a) The diagonals of a rhombus bisect each other.

(b) The diagonals of a rhombus are perpendicular to each other.

Therefore: $\qquad \frac{1}{2}d_1 = 13.5$ ft $\qquad \frac{1}{2}d_2 = 12$ ft

Now we will label our information on a rhombus to determine if we have enough information to find the length of one side of the rhombus.




Recall that the sides of a rhombus are all equal.

From the information displayed in our figure, we can see that applying the Pythagorean Theorem is a solution for determining the length of one side of the rhombus.

| | |
|---|---|
| $a^2 + b^2 = c^2$ | Pythagorean Theorem |
| $(13.5)^2 + (12)^2 = s^2$ | Substitute |
| $182.25 + 144 = s^2$ | Simplify |
| $326.25 = s^2$ | Simplify |
| $s \approx 18.1$ | Take the square root of both sides. |

The length of each of the sides of the rhombus is about 18.1 feet.

## Summary of Area Formulas

| Summary of Area Formulas | |
|---|---|
| **Square** | $A = s^2$ |
| **Rectangle** | $A = bh$ |
| **Parallelogram** | $A = bh$ |
| **Triangle** | $A = \frac{1}{2}bh$ |
| **Trapezoid** | $A = \frac{1}{2}h(b_1 + b_2)$ |
| **Rhombus** | $A = \frac{1}{2}d_1 d_2$ |