INSCRIBED ANGLES, TANGENTS, AND SECANTS

In this unit, you will learn about inscribed angles, tangents, and secants. You will explore the relationship between inscribed angles and their intercepted arcs. You will investigate polygons inscribed in circles and polygons circumscribed about circles. You will learn about a point of tangency and examine lines that are tangent to circles and how they relate to radii. You will learn about secants and their connection with arcs, arc measure, and tangents.

Inscribed Angles

Tangents

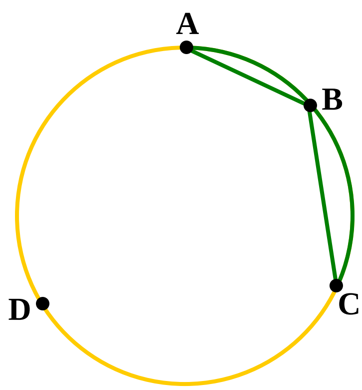
Secants

Construction of an Inscribed Hexagon

Inscribed Angles

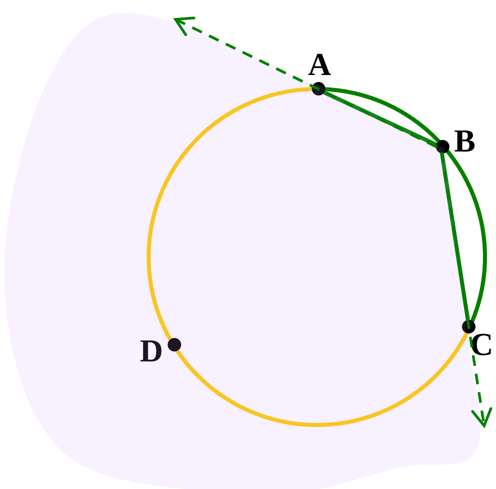
inscribed angle – An inscribed angle in a circle is an angle that has its vertex located on the circle and its rays are chords.

intercepted arc – An intercepted arc is an arc that lies in the interior of an inscribed angle and is formed by the intersection of the rays of an inscribed angle with the circle.



$\angle ABC$ is an inscribed angle.

$\overset{\frown}{ADC}$ is an intercepted arc of $\angle ABC$.



*Note: $\overset{\frown}{ADC}$ lies in the interior of $\angle ABC$.

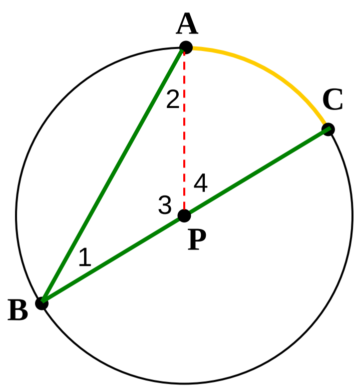
Theorem 25-A

If an angle is inscribed in a circle, then the measure of the angle is one-half the measure of the intercepted arc.

There are three cases to this proof.

Let's take a look at the case where the center of the circle lies on one of the rays of the inscribed angle.

Case 1: The center of the circle lies on one of the rays of the inscribed angle. (In the figure below, center point P lies on inscribed $\angle ABC$.)



Given: $\odot$ P, Inscribed $\angle ABC$

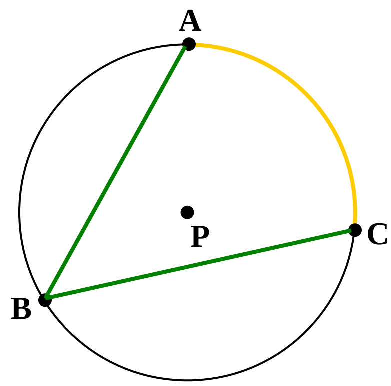
Prove: $m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$

*Numbers have been used to make easy reference to the angles.

Statement	**Reason**
Draw radius *PA*.	Radius *PA* is added as an auxilary line.
$m\angle 4 = m\angle 1 + m\angle 2$	Exterior Angle Theorem
$\overline{PB} \cong \overline{PA}$	Radii of the same circle are congruent.
$m\angle 2 = m\angle 1$	Angles that are opposite congruent sides in a triangle are congruent.
$m\angle 4 = m\angle 1 + m\angle 1$	Substitution
$m\angle 4 = 2(m\angle 1)$	Simplify
$\frac{1}{2}(m\angle 4) = m\angle 1$	Division
$m\angle 1 = \frac{1}{2}(m\angle 4)$	Symmetric Property of Equality
$m\angle 4 = m\overset{\frown}{AC}$	Definition of Arc Measure
$m\angle 1 = \frac{1}{2} m\overset{\frown}{AC}$	Substitution
$\therefore m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$	

In Case 2, the center of the circle lies within the inscribed angle.

Case 2: The center of the circle lies in the interior of the inscribed angle. (In the figure below, center point P lies in the interior of inscribed $\angle ABC$.)



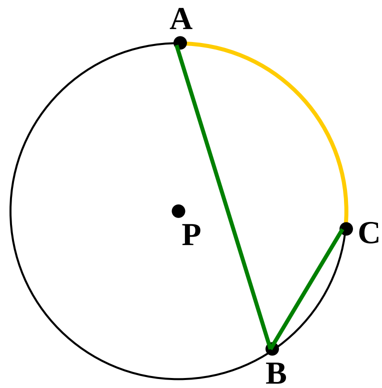
A similar proof could be developed as illustrated for Case 1.

Example 1: If the measurement of $\angle ABC = 56°$, what is the measure of $\overset{\frown}{AC}$?

$m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$	Theorem 25-A
Let $x = m\overset{\frown}{AC}$.	
$56° = \frac{1}{2}x$	Substitution
$112° = x$	Multiplication Property
$m\overset{\frown}{AC} = 112°$	

In Case 3 the center of the circle lies outside of the inscribed angle.

Case 3: The center of the circle lies in the exterior of the inscribed angle. (In the figure below, center point P lies in the exterior of inscribed $\angle ABC$.)



A similar proof could be developed as illustrated for Case 1.

Example 2: If the measurement of $\overset{\frown}{AC} = 139°$, what is the measure of $\angle ABC$?

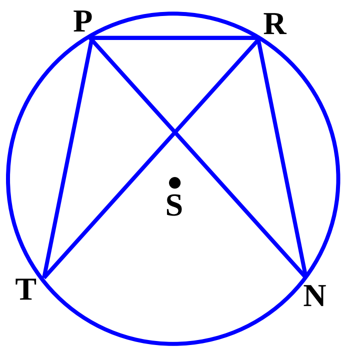
$m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$	Theorem 25-A
$m\angle ABC = \frac{1}{2}(139)$	Substitution
$m\angle ABC = 69.5°$	Simplify

Theorem 25-A holds true for all three of these cases; that is, the measurement of an inscribed angle is 1/2 the measure of its intercepted arc.

Theorem 25-B

If two inscribed angles intercept the same arc, then the angles are congruent.

Example 3: In the figure below, what angle is congruent to $\angle PTR$?



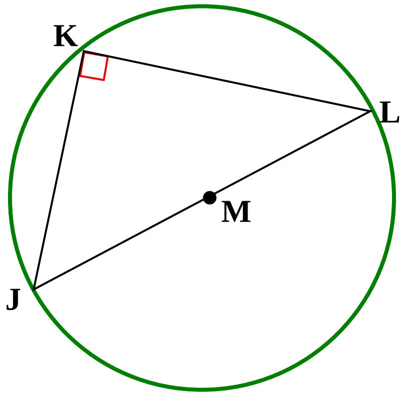
$\angle PTR$ intercepts $\overset{\frown}{PR}$.

$\angle RNP$ intercepts $\overset{\frown}{PR}$.

$\therefore\ \angle RNP \cong \angle PTR$ — Theorem 25-B The inscribed angles are congruent because they intercept the same arc.

Theorem 25-C

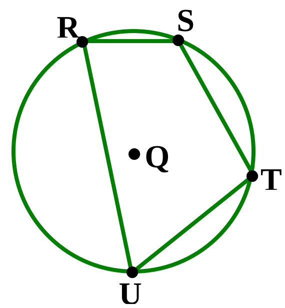
An angle that is inscribed in a circle is a right angle if and only if its intercepted arc is a semicircle.



$\overset{\frown}{JKL}$ is a semicircle.	Given
$\angle JKL$ is an inscribed angle.	Given
$m\overset{\frown}{JKL} = 180°$	Definition of arc measure
$m\angle JKL = 90°$	Theorem 25-A

inscribed polygon – An inscribed polygon within a circle is a polygon whose vertices lie on the circle.

Theorem 25-D	**If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.**



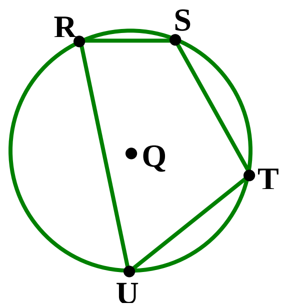
Example 4: Take any pair of opposite angles in the quadrilateral RSTU and explain Theorem 25-D.

We will organize the answer into statements and reasons.

Statement	Reason
$\angle RST$ and $\angle RUT$ are opposite angles in quadrilateral $RSTU$.	Given
$\angle RST$ is an inscribed angle of $\overset{\frown}{RUT}$.	Definition of Inscribed Angle
$m\angle RST = \frac{1}{2} m\overset{\frown}{RUT}$	Theorem 25-A
$\angle RUT$ is an inscribed angle of $\overset{\frown}{RST}$.	Definition of Inscribed Angle
$m\angle RUT = \frac{1}{2} m\overset{\frown}{RST}$	Theorem 25-A
$m\angle RST + m\angle RUT = \frac{1}{2} m\overset{\frown}{RUT} + \frac{1}{2} m\overset{\frown}{RST}$	Addition Property
$m\angle RST + m\angle RUT = \frac{1}{2}(m\overset{\frown}{RUT} + m\overset{\frown}{RST})$	Distributive Property
$m\overset{\frown}{RUT} + m\overset{\frown}{RST} = 360°$	Definition of Arc Measure
$m\angle RST + m\angle RUT = \frac{1}{2}(360°)$	Substitution
$m\angle RST + m\angle RUT = 180°$	Simplify
$m\angle RST$ and $m\angle RUT$ are supplementary angles.	Defintion of Supplementary Angles

Therefore, opposite angles $\angle RST$ and $\angle RUT$ of quadrilateral $RSTU$ are supplementary angles.

Example 5: For inscribed quadrilateral $RSTU$, if $\angle RST$ measures $125°$ and $\angle STU$ measures $95°$, then find the measures of $\angle TUR$ and $\angle URS$.



$\angle RST$ and $\angle TUR$ are opposite angles and are supplementary by Theorem 25-D.

$m\angle RST + m\angle TUR = 180°$	Definition of Supplementary Angles
$125° + m\angle TUR = 180°$	Substitution
$m\angle TUR = 55°$	Subtraction

$\angle STU$ and $\angle URS$ are opposite angles and are supplementary by Theorem 25-D.

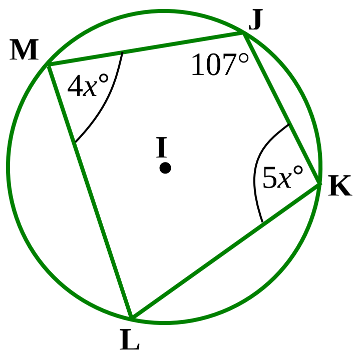
$m\angle STU + m\angle URS = 180°$	Definition of Supplementary Angles
$95° + m\angle URS = 180°$	Substitution
$m\angle URS = 85°$	Subtraction

To check the answers:

$m\angle RST + m\angle STU + m\angle TUR + m\angle URS = 360°$	The four angles of a quadrilateral total 360°.
$125° + 95° + 55° + 85° = 360°$	Substitution
$360° = 360°$	Simplify

Angles TUR and URS measure 55 degrees and 85 degrees, respectively.

Example 6: For inscribed quadrilateral *JKLM*, find the size of angles *K*, *L*, and *M*.



$m\angle M + m\angle K = 180$	Theorem 25-D and Definition of Supplementary Angles
$4x + 5x = 180$	Substitution
$9x = 180$	Simplify
$x = 20$	Division Property
$\angle M = 4x = 4(20) = 80°$	Substitution
$\angle K = 5x = 5(20) = 100°$	Substitution
$\angle J + \angle L = 180°$	Theorem 25-D and Definition of Supplementary Angles
$107° + \angle L = 180°$	Substitution
$\angle L = 73°$	Subtraction

To check the answers:

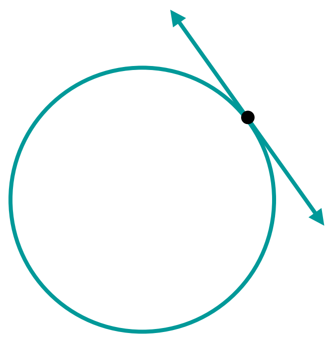
$m\angle J + m\angle K + m\angle L + m\angle M = 360°$	The four angles of a quadrilateral total 360°.
$107° + 100° + 73° + 80° = 360°$	Substitution
$360° = 360°$	Simplify

Angles M, K, and L measure 80, 100, and 73 degrees, respectively.

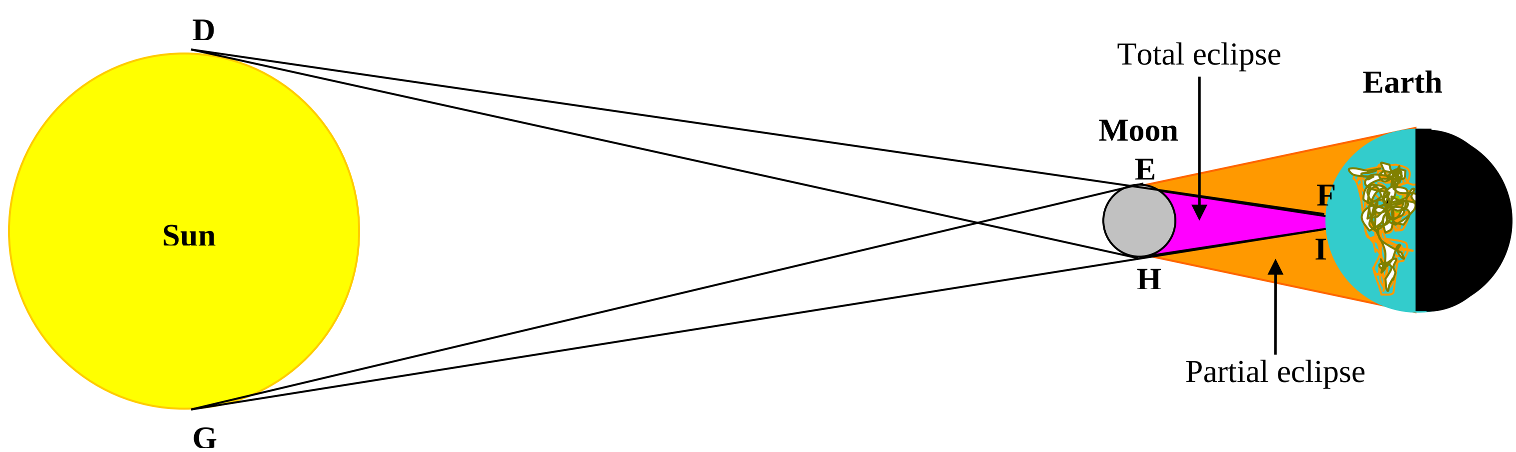
Tangents

tangent line – A tangent line to a circle is a line that intersects the circle at *exactly* one point. (It appears to brush the edge of a circle.)

point of tangency – A point of tangency is the point where a tangent line intersects with a circle.



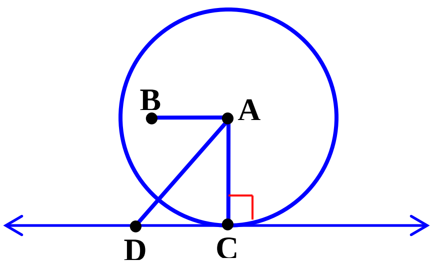
Example 1: The drawing below shows how the sun, moon, and earth are aligned for a solar eclipse. Identify the tangents lines which partition an area on the earth that experiences a total solar eclipse.



The pink area, the area between $\overline{EF}$ and $\overline{HI}$, is the area that experiences the total solar eclipse. One tangent line, $\overline{DE}$, that creates this area, is drawn from the upper most point on the sun, point D, through the uppermost point on the moon, point E, to the earth, point F. The second tangent line, $\overline{GH}$, that creates the area that experiences the total solar eclipse, is drawn from the lowest point on the sun, point G, through the lowest point on the moon, Point H, to the earth, point I.

Theorem 25-E

If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency.



Study the diagram to the left. $\overline{AB}$ is shorter than $\overline{AC}$. $\overline{AD}$ is longer than $\overline{AC}$. Any segment other than $\overline{AC}$ drawn to tangent $\overleftrightarrow{DC}$ will be longer than $\overline{AC}$. The shortest segment from a point to a line is a perpendicular segment. Thus, radius $\overline{AC} \perp \overleftrightarrow{DC}$.

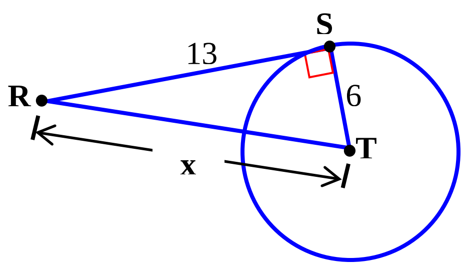
Example 2: Find x for the given information.

Given:

$\odot T$

Tangent RS = 13

Radius ST = 6



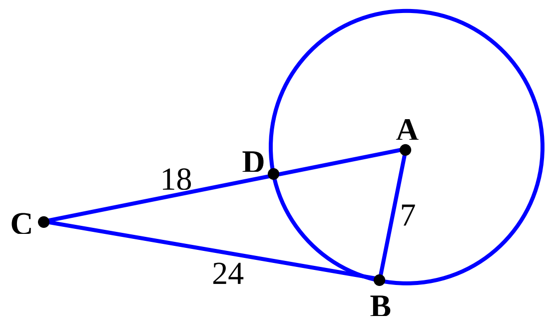
By theorem 25-E, $\overline{ST} \perp \overline{RS}$; thus, $\triangle RST$ is a right triangle.

$(13)^2 + (6)^2 = x^2$	Pythagorean Theorem
$169 + 36 = x^2$	Simplify
$205 = x^2$	Simplfiy
$x = \sqrt{205}$	Take the square root of both sides of the equation.
$x \approx 14.3$	Simplify

Theorem 25-F (Converse of 25-E)

If a radius is perpendicular to a line at the point at which the line intersects the circle, then the line is a tangent.

Example 3: Determine if $\overline{BC}$ is a tangent line to circle A.



Step 1: First determine the length of *AC*.

$AD = 7$	$\overline{AD}$ and $\overline{AB}$ are radii; therefore they are congruent.
$CD + DA = AC$	Segment Addition
$18 + 7 = AC$	Substitution
$AC = 25$	Simplify

Step 2: Prove that $\overline{AB} \perp \overline{BC}$ (Theorem 25-F) by showing that $\triangle$ABC is a right triangle and that $\overline{\text{AB}}$ and $\overline{\text{BC}}$ are the legs.

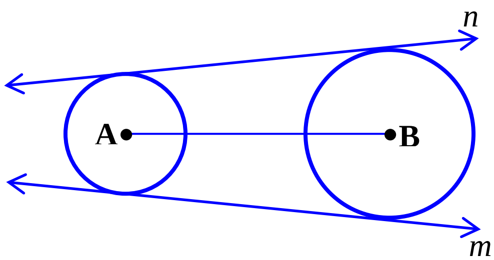
$(7)^2 + (24)^2 = (25)^2$	Pythagorean Theorem ($AB = 7,\ BC = 24,\ AC = 25$)
49 + 576 = 625	Simplify
625 = 625	Simplfiy

Thus, $\triangle ABC$ is a right triangle and segments $\overline{AB}$ and $\overline{BC}$ are the perpendicular legs.

common tangent – A common tangent is a line or line segment that is tangent to two circles in the same plane.

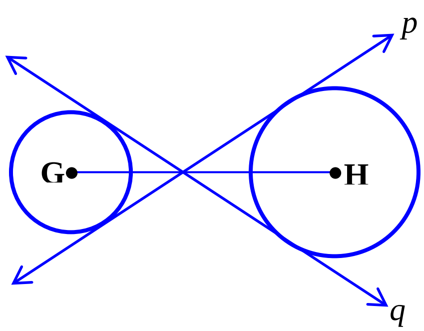
There are two types of common tangents: common external tangents and common internal tangents.

Common External Tangent



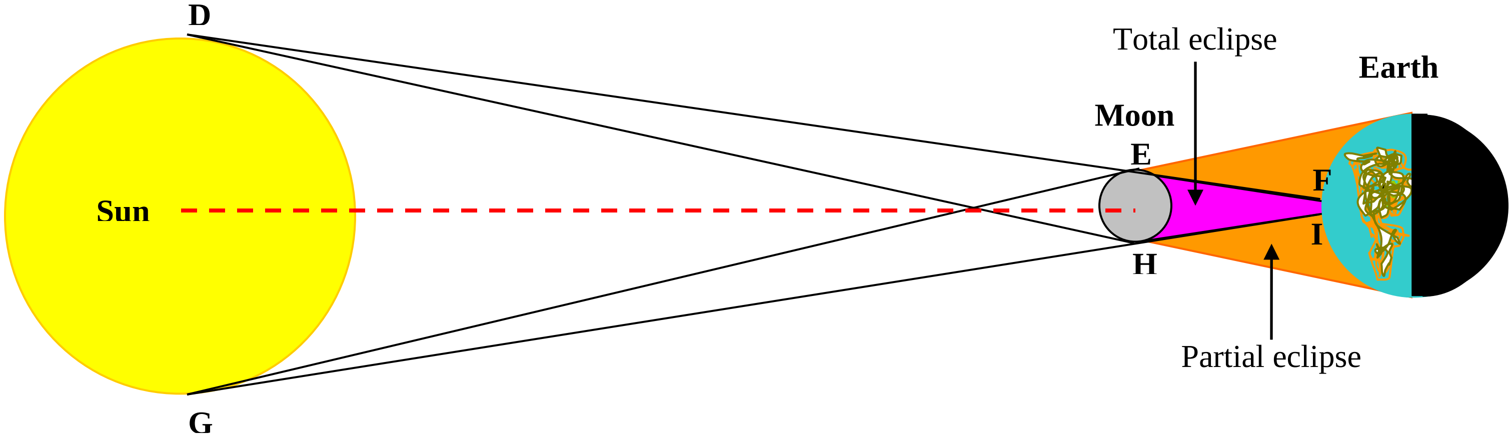
Common external tangents do not intersect the segment that has its endpoints on the centers of the two circles. Lines *n* and *m* are common external tangents for ⊙*A* and ⊙*B*.

Common Internal Tangent



Common internal tangents intersect the segment that has its endpoints on the centers of the two circles. Lines *p* and *q* are common internal tangents for ⊙*G* and ⊙*H*.

Example 4: Let's revisit our diagram of the solar eclipse in example 1. A red-dotted line has been added to represent the line segment that has its endpoints on the centers of the sun and the moon. Identify the common external tangents between the sun and the moon, and then identify the common internal tangents between the sun and the moon.



The common external tangents are $\overline{DE}$ and $\overline{GH}$.

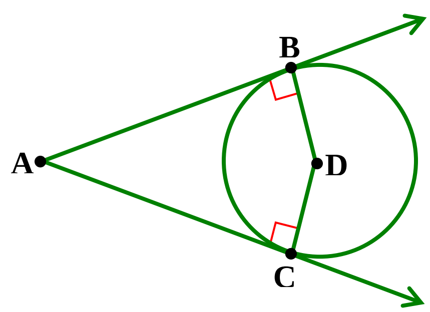
They DO NOT intersect the red dotted line.

The common internal tangents are $\overline{DH}$ and $\overline{GE}$.

They DO intersect the red dotted line.

Theorem 25-G

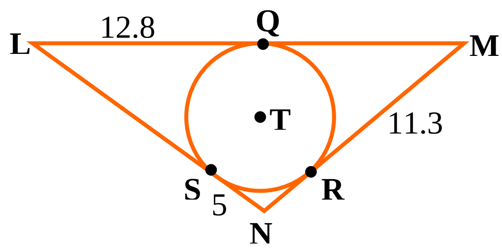
If two segments from the same exterior point are tangent to a circle, then the two segments are congruent.



$\overline{AB} \cong \overline{AC}$

circumscribed polygon – A circumscribed polygon about a circle is a polygon in which all of its sides are tangents to the circle.

Example 5: Triangle *LMN* is circumscribed around circle T. Segments LQ, MR, and SN measure 12.8, 11.3, and 5 centimeters, respectively. Find the perimeter of triangle LMN.



We will apply Theorem 25-G to solve this problem.

$NR = NS = 5$

$MQ = MR = 11.3$

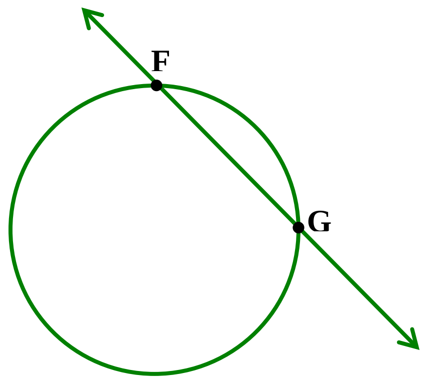
$LS = LQ = 12.8$

$\text{Perimeter} = LQ + LS + NS + NR + MR + MQ$	Definition of Perimeter
$\text{Perimeter} = 12.8 + 12.8 + 5 + 5 + 11.3 + 11.3$	Substitution
$\text{Perimeter} = 2(12.8) + 2(5) + 2(11.3)$	Simplify
$\text{Perimeter} = 58.2$	

The perimeter of $\triangle LMN$ is 58.2 centimeters.

Secants

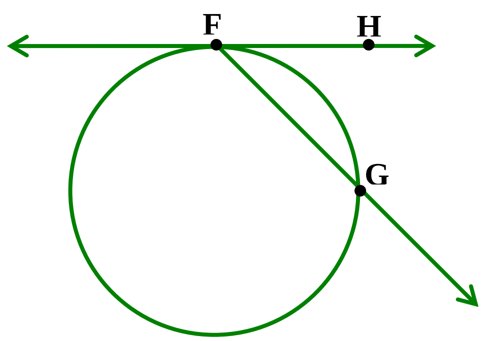
secant – A secant is a line that intersects a circle in exactly two points. A secant of a circle contains a chord of the circle.



$\overleftrightarrow{FG}$ is a secant.

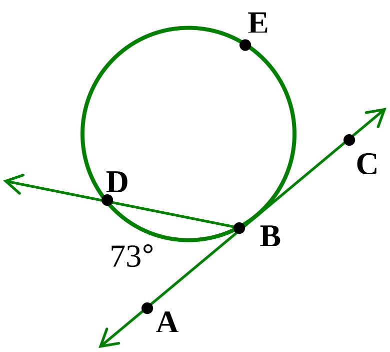
Theorem 25-H

If a secant and a tangent intersect at the point of tangency, then the measure of each angle formed is one-half the measure of its intercepted arc.



$$m\angle HFG = \frac{1}{2} m\overset{\frown}{FG}$$

Example 1: In the given diagram, arc DB measures 73 degrees. Point B is a point of tangency for line AC. Find the measure of angles ABD and CBD.



$m\angle ABD$	$= \frac{1}{2} m\overset{\frown}{DB}$	Theorem 25-H
$m\angle ABD$	$= \frac{1}{2}(73)$	Substitution
$m\angle ABD$	$= 36.5°$	Simplify

$\angle ABD$ and $\angle CBD$ are a linear pair; therefore, they are supplementary.

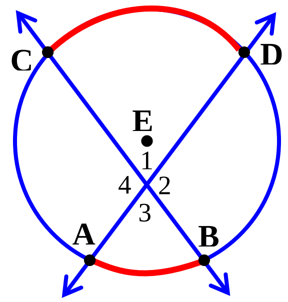
$m\angle ABD + m\angle CBD = 180$	Definition of Supplementary Angles
$36.5 + m\angle CBD = 180$	Substitution
$m\angle CBD = 143.5°$	

Now check the measure of $\angle CBD$ by applying Theorem 25-H.

$m\overset{\frown}{BED}$	$= 360 - m\overset{\frown}{DB}$	Arc Measure
$m\overset{\frown}{BED}$	$= 360 - 73$	Substitution
$m\overset{\frown}{BED}$	$= 287°$	Simplify
$m\angle CBD$	$= \frac{1}{2} m\overset{\frown}{BED}$	Theorem 25-H
$m\angle CBD$	$= \frac{1}{2}(287)$	Substitution
$m\angle CBD$	$= 143.5°$	Simplify

Angle ABD measures 36.5 degrees and angle CBD measures 143.5 degrees.

Theorem 25-I	**If two secants intersect in the interior of a circle, then the measure of an angle formed is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.**



$$m\angle 1 = \frac{1}{2}(m\overset{\frown}{CD} + m\overset{\frown}{AB})$$

$$m\angle 3 = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$$

$$m\angle 2 = \frac{1}{2}(m\overset{\frown}{BD} + m\overset{\frown}{AC})$$

$$m\angle 4 = \frac{1}{2}(m\overset{\frown}{AC} + m\overset{\frown}{BD})$$

Example 2: In circle E above, determine the measure of angle 3 if $\overset{\frown}{CD}$ measures 78 degrees and $\overset{\frown}{AB}$ measures 40 degrees.

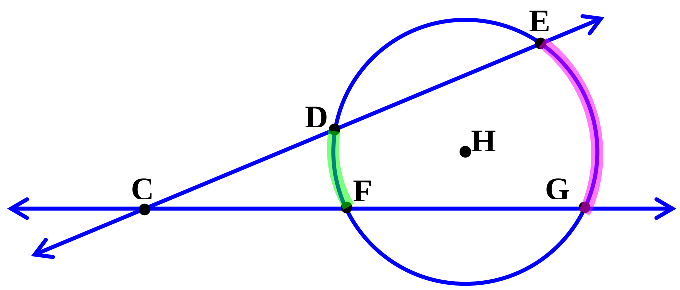
$m\angle 3 = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$	Theorem 25-I
$m\angle 3 = \frac{1}{2}(40 + 78)$	Substitution
$m\angle 3 = 59°$	Simplify

The measurement of angle 3 is 59 degrees.

Theorem 25-J

The measure of an angle formed by two secants, a secant and a tangent, or two tangents intersecting in the exterior of a circle is equal to one-half the positive difference of the measures of the intercepted arcs.

Theorem 25-J (Case 1): Secant-Secant



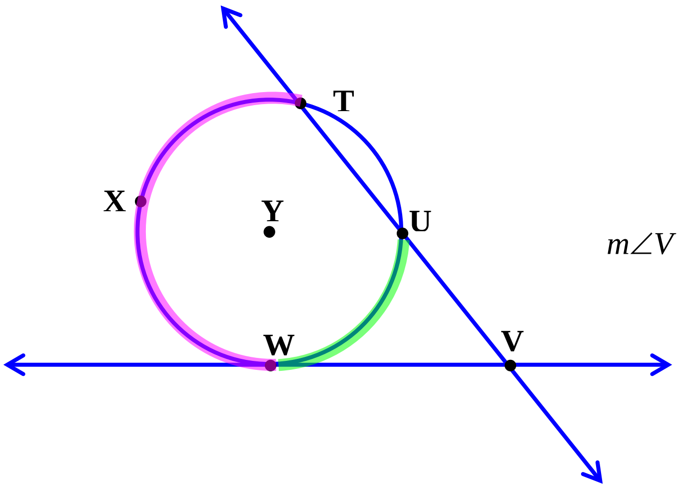
$$m\angle C = \frac{1}{2}(m\overset{\frown}{EG} - m\overset{\frown}{DF})$$

Example 3: In circle H above, find the measure of $\overset{\frown}{DF}$ if angle C measures 23 degrees and $\overset{\frown}{EG}$ measures 100 degrees.

$m\angle C = \frac{1}{2}(m\overset{\frown}{EG} - m\overset{\frown}{DF})$	Theorem 25-J
Let $x = m\overset{\frown}{DF}$.	
$23 = \frac{1}{2}(100 - x)$	Substitution
$46 = 100 - x$	Multiply both sides of the equation by 2.
$-54 = -x$	Subtract 100 from both sides of the equation.
$54 = x$	Muliply both sides of the equation by -1.
$m\overset{\frown}{DF} = 54°$	

The measurement of $\overset{\frown}{DF}$ is 54 degrees.

Theorem 25-J (Case 2): Secant-Tangent



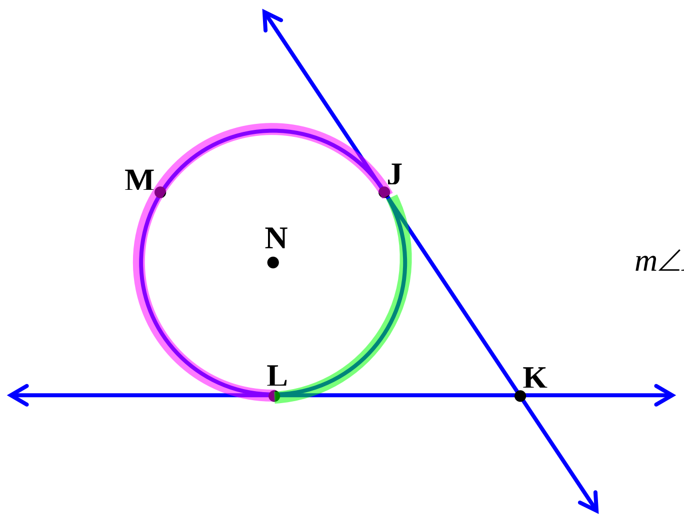
$$m\angle V = \frac{1}{2}(m\overset{\frown}{WXT} - m\overset{\frown}{WU})$$

Example 4: In circle Y above, find the measure of $\overset{\frown}{WXT}$ if angle V measures 49 degrees and $\overset{\frown}{WU}$ measures 93 degrees.

$m\angle V = \frac{1}{2}(m\overset{\frown}{WXT} - m\overset{\frown}{WU})$	Theorem 25-J
Let $x = m\overset{\frown}{WXT}$.	
$49 = \frac{1}{2}(x - 93)$	Substitution
$98 = x - 93$	Multiply both sides of the equation by 2.
$191 = x$	Add 93 to both sides of the equation.
$m\overset{\frown}{WXT} = 191°$	

The measurement of $\overset{\frown}{WXT}$ is 191 degrees.

Theorem 25-J (Case 3): Tangent-Tangent



$$m\angle K = \frac{1}{2}(m\overset{\frown}{LMJ} - m\overset{\frown}{LJ})$$

Example 5: In circle N above, find the measure of angle K if arc LMJ measures 240 degrees.

Step 1: Find the measure of $\overset{\frown}{LJ}$.

$\overset{\frown}{LMJ} + \overset{\frown}{LJ} = 360$	Arc Measure
$240 + \overset{\frown}{LJ} = 360$	Substitution
$\overset{\frown}{LJ} = 120°$	Subtractopm

Step 2: Apply Theorem 25-J to determine the measure of ∠K.

$m\angle K = \frac{1}{2}(m\overset{\frown}{LMJ} - m\overset{\frown}{LJ})$	Theorem 25-J
$m\angle K = \frac{1}{2}(240 - 120)$	Substitution
$m\angle K = 60°$	Simplify

The measurement of angle K is 60 degrees.

Construction of an Inscribed Hexagon

The illustration below shows the steps for inscribing a regular hexagon within a circle.

Step 1: Start by drawing a circle G with a compass.

Step 2: Without changing the setting on the compass, start anywhere on the circle and name the point A. Draw an arc that intersects the circle and name the point of intersection, B.

Step 3: Move the compass metal point to the point of intersection of the arc just drawn and draw another arc. Continue around the circle and draw six arcs. Make sure the setting of the compass does not change.

Step 4: Connect the consecutive points of intersection with line segments.

The polygon created is a regular hexagon.