PROPORTIONAL PARTS OF SIMILAR TRIANGLES

In this unit, you will learn how parallel lines divide triangles into proportional parts, and that three or more parallel lines divide two transversals proportionally. You will examine the proportional relationships of similar triangles' altitudes, medians, angle bisectors, and perimeters. You will use theorems, examples, and proofs to determine similarity.

Parallel Lines and Proportional Parts

Triangle Mid-segment Theorem

Corollaries about Parallel Lines

Proportional Relationships of Parts of Similar Triangles

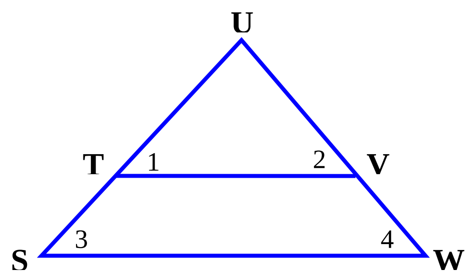
Parallel Lines and Proportional Parts

Theorem 21-A

If a line is parallel to one side of a triangle and intersects the other two sides, then those sides are separated into segments of proportional lengths.

Given: $\overline{TV} \parallel \overline{SW}$

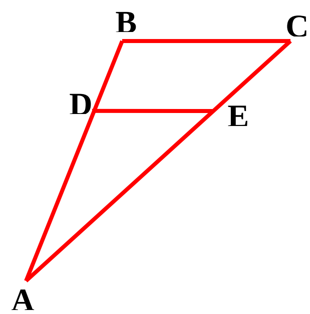
Prove: $\frac{ST}{TU} = \frac{WV}{VU}$



Statements	Reasons
1. $\overline{TV} \parallel \overline{SW}$	Given
2. $\angle 1 \cong \angle 3$ and $\angle 2 \cong \angle 4$.	Corresponding angles of parallel lines are congruent.
3. $\triangle UTV \sim \triangle USW$	AA Similarity (If two angles of one triangle are congruent to two angles of a second triangle, then the triangles are similar. Postulate 20-A)
4. $SU = ST + TU$ and $WU = WV + VU$	Segment Addition (Postulate 3-B)
5. $\frac{SU}{TU} = \frac{WU}{VU}$	Definition of Similar Triangles
6. $\frac{ST+TU}{TU} = \frac{WV+VU}{VU}$	Substitution Property
7. $\frac{ST}{TU} + \frac{TU}{TU} = \frac{WV}{VU} + \frac{VU}{VU}$	Substitution Property (Properties of Real Numbers)
8. $\frac{ST}{TU} + 1 = \frac{WV}{VU} + 1$	Substitution Property (Properties of Real Numbers)
9. $\frac{ST}{TU} = \frac{WV}{WU}$	Subtraction Property

Theorem 21-B	**A line that divides two sides of a triangle proportionally is parallel to the third side of the triangle.**

Example 1: In $\triangle ABC$, $DB = 2.5$ and $AB = 10$. AE is three times as long as EC. Determine if $DE \parallel BC$.



$AB = AD + DB$	Segment Addition
$10 = AD + 2.5$	Substitution
$AD = 7.5$	Subtraction

Let's check to see if the lines are parallel by applying theorem 21-B. We will compare ratios to determine if $\overline{DE}$ divides $\overline{AB}$ and $\overline{AC}$ proportionally, and if so, then $\overline{DE} \parallel \overline{BC}$.

The following values were given or determined:

$DB = 2.5$ $\quad AB = 10$ $\quad AD = 7.5$

Compare the ratios of the two sides of the triangle divided by $\overline{DE}$.

Is $\frac{DB}{AD} = \frac{CE}{AE}$?

Let $x = EC$ and $AE = 3x$ since AE is given as 3 times the length of EC.

Now substitute:

$$\frac{2.5}{7.5} = \frac{x}{3x}$$

$$\frac{\overset{1}{\cancel{2.5}}}{\underset{3}{\cancel{7.5}}} = \frac{\overset{1}{\cancel{x}}}{\underset{3}{\cancel{3x}}}$$

Yes, $\frac{DB}{AD} = \frac{CE}{AE} = \frac{1}{3}$. The sides have proportional lengths; thus, $DE \parallel BC$.

Example 2: If *AE* = 15, through proportional sides, determine the length of *CE*?

$$\frac{DB}{AD} = \frac{CE}{AE}$$

$$\frac{2.5}{7.5} = \frac{x}{15}$$

$$7.5x = 37.5$$
$$x = 5$$

The length of *CE* is 5.

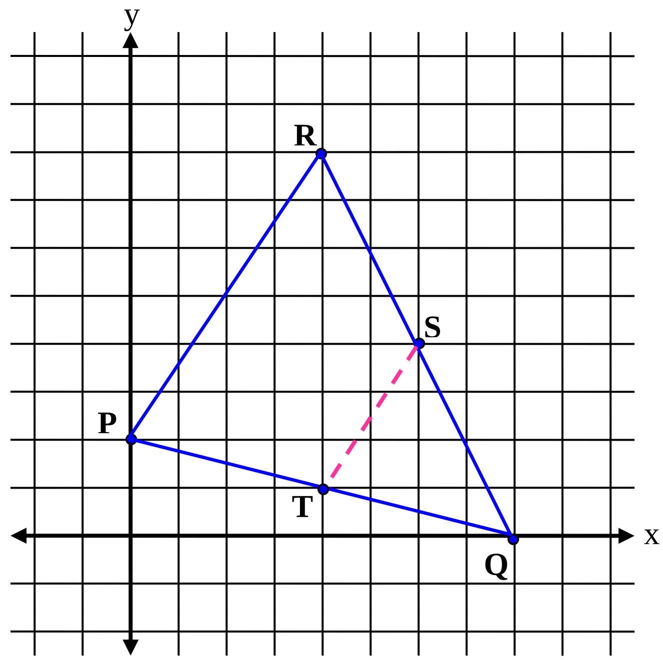
Triangle Mid-segment Theorem

Theorem 21-C Triangle Mid-segment Theorem

If a segment's endpoints are the midpoints of two sides of a triangle, then it is parallel to the third side of the triangle and one-half its length.

Now let's take a closer look at the Triangle Mid-segment Theorem.

Given: $\triangle PQR$ with vertices P(0,2), Q(8,0), and R(4,8)



First we'll find the midpoints of $\overline{PQ}$ and $\overline{RQ}$ and label the points T and S. We will then draw $\overline{TS}$ and examine the theorem.

Use the midpoint formula to determine the midpoints.

Midpoint of $PQ \rightarrow \quad P(0,2) \qquad Q(8,0)$

$$(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}) = (\frac{0+8}{2},\frac{2+0}{2}) = T(4,1)$$

Midpoint of RQ $\rightarrow \quad R(4,8) \qquad Q(8,0)$

$$(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}) = (\frac{4+8}{2},\frac{8+0}{2}) = S(6,4)$$

$\overline{TS}$ has vertices $T(4,1)$ and $S(6,4)$.

Now we'll examine the *first* part of the Triangle Mid-segment Theorem: **If a segment's endpoints are the midpoints of two sides of a triangle**, **then it is parallel to the third side of the triangle** and one-half its length.

*Show that $\overline{TS}$ is parallel to $\overline{PR}$.

Calculate the slope of each line. The slopes of parallel lines are equal.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$T(4,1)$ and $S(6,4)$ | $P(0,2)$ and $R(4,8)$

Slope of $\overline{TS} = \frac{4-1}{6-4} = \frac{3}{2}$ | Slope of $\overline{PR} = \frac{2-8}{0-4} = \frac{-6}{-4} = \frac{3}{2}$

Since the slopes are equal, $\overline{TS} \parallel \overline{PR}$.

Now we'll examine the *second* part of the Triangle Mid-segment Theorem: **If a segment's endpoints are the midpoints of two sides of a triangle**, **then it is** parallel to the third side of the triangle and **one-half its length**.

*Show that $TS = \frac{1}{2} PR$.

Calculate the length of TS and PR using the distance formula, and then compare.

Length of *TS*
$T(4,1)$ and $S(6,4)$

$$TS = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$TS = \sqrt{(6-4)^2 + (4-1)^2}$$
$$TS = \sqrt{(2)^2 + (3)^2}$$
$$TS = \sqrt{13}$$

Length of *PR*
$P(0,2)$ and $R(4,8)$

$$PR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$PR = \sqrt{(0-4)^2 + (2-8)^2}$$
$$PR = \sqrt{(-4)^2 + (-6)^2}$$
$$PR = \sqrt{52}$$
$$PR = \sqrt{4(13)} = 2\sqrt{13}$$

Compare the lengths.

$$\text{PR} = 2 \times \text{TS}$$
$$2\sqrt{13} = 2 \times \sqrt{13}$$

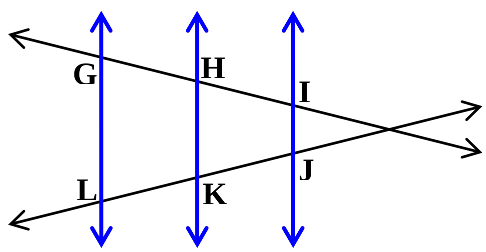
Therefore, $TS = \frac{1}{2}(PR)$.

Corollaries about Parallel Lines

Corollary 21-A-1	**If three or more parallel lines intersect two transversals, then they cut off the transversals proportionally.**

Example 1: State three proportional statements by applying Corollary 19-A-1.

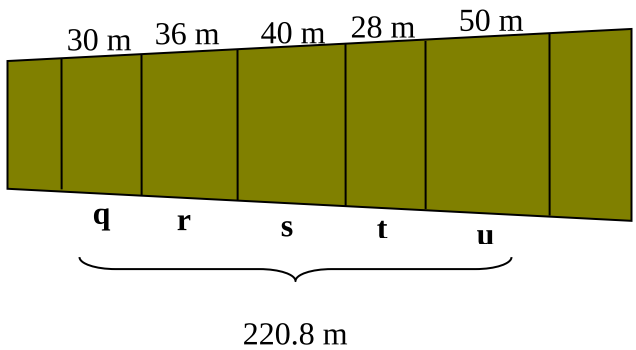
Given: $GL \parallel HK \parallel IJ$



$$\frac{GI}{LJ} = \frac{GH}{LK} \qquad \frac{GI}{LJ} = \frac{HI}{KJ}$$

$$\frac{GH}{LK} = \frac{HI}{KJ}$$

Example 2: Five plots of land are laid out parallel to each other as shown in the diagram. The width of the upper end is given and is shorter than the lower ends. The total length of the lower ends of the five lots is 220.8 meters. Find the lower end length of each lot.



We will find the total length of the upper ends of the five lots, and then apply corollary 19-A-1 to solve the problem.

$$30+36+40+28+50=184$$

Now write a proportion comparing the proportional sides of the lots. Note: The total length of the upper ends of the lots is also proportional to the total length of the lower ends of the lots based on corollary 19-A-1.

$$\frac{q}{30}=\frac{220.8}{184} \qquad \frac{r}{36}=\frac{220.8}{184} \qquad \frac{s}{40}=\frac{220.8}{184} \qquad \frac{t}{28}=\frac{220.8}{184} \qquad \frac{u}{50}=\frac{220.8}{184}$$

$$q=36 \qquad r=43.2 \qquad s=48 \qquad t=33.6 \qquad u=60$$

The widths of the lower ends of the lots are 36 m, 43.2 m, 48 m, 33.6 m, and 60 m.

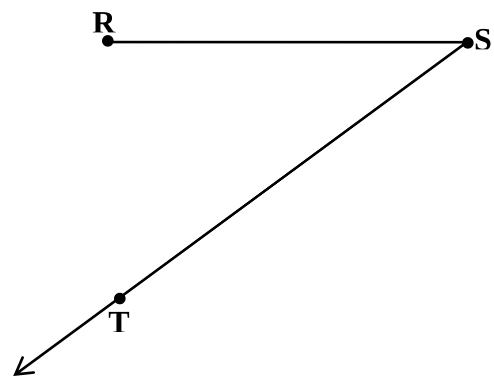
Check: 36 + 43.2 + 48 + 33.6 + 60 = 220.8

Corollary 21-A-2

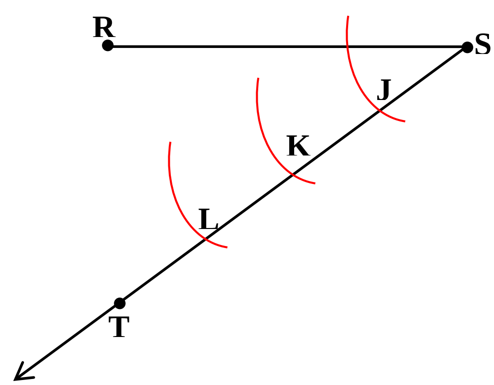
If three or more parallel lines cut off congruent segments on one transversal, then they cut off congruent segments on every transversal.

To investigate this corollary we will make a construction.

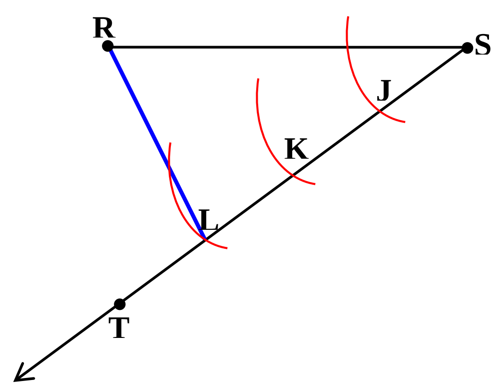
Step 1: Draw $\overline{RS}$ and $\overrightarrow{ST}$.



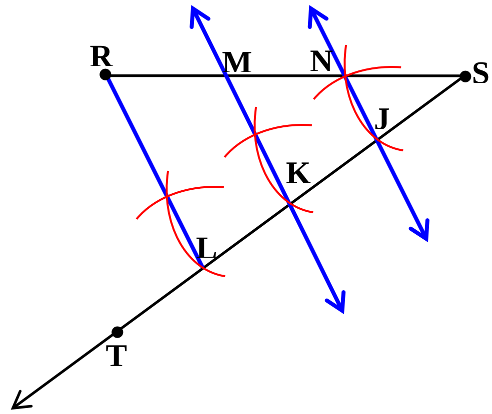
Step 2: With the metal point of the compass set at point S, mark off an arc that intersects $\overrightarrow{ST}$ at point J as illustrated in the diagram. Keeping the setting of the compass the same, move the compass point to point J and mark off an arc that intersects $\overrightarrow{ST}$ at point K. Continue by moving the metal point of the compass to point K and keeping the setting of the compass the same, mark off an arc that intersects $\overrightarrow{ST}$ at point L.



Step 3: Construct $\overline{RL}$ by drawing a segment between points *R* and *L*.



Step 4: Construct lines that are parallel to $\overline{RL}$ and pass through points K and J. Call the intersection points on $\overline{RS}$, points M and N. (Recall that instructions for constructing parallel lines are in a previous unit.)



Apply corollary 21-A-2 to state that $\overline{RM} \cong \overline{MN} \cong \overline{NS}$.

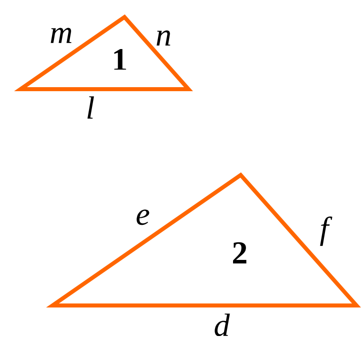
Proportional Relationships of Parts of Similar Triangles

Theorem 21-D

If two triangles are similar, then their perimeters are proportional to the measures of the corresponding sides.

There is a common ratio(r) that exists between two similar triangles.

$$\frac{l}{d} = \frac{m}{e} = \frac{n}{f} = r$$



Therefore,

$$\frac{l}{d} = \frac{r}{1} \qquad \frac{m}{e} = \frac{r}{1} \qquad \frac{n}{f} = \frac{r}{1}$$

$$l = rd \qquad m = re \qquad n = rf$$

Perimeter ($\triangle 1$) $= l + m + n$

Perimeter ($\triangle 2$) $= d + e + f$

Now we will check to see if the ratio of the perimeters of triangles 1 and 2 is equal to r, the common ratio of the similar sides.

$$\frac{P(\triangle 1) \rightarrow}{P(\triangle 2) \rightarrow} \quad \frac{l + m + n}{d + e + f} = \frac{rd + re + rf}{d + e + f}$$ Substitution Property

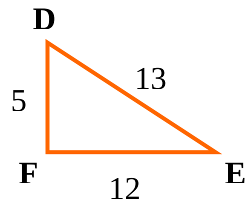
$$= \frac{r(\cancel{d + e + f})}{\cancel{d + e + f}}$$ Distributive Property

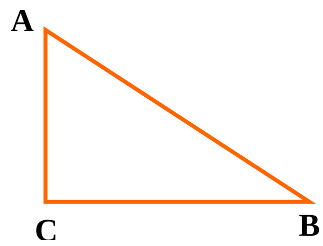
$$= r$$ Simplify

Therefore, $\frac{P(\triangle 1)}{P(\triangle 2)} = r$. The ratio of the perimeters of similar triangles is the same as the ratio between the corresponding sides of similar triangles.

Example 1:   In the figure below, if $AC = 7.5$, find the perimeter of $\triangle ABC$.

Given: $\triangle ABC \sim \triangle DEF$





Let x = Perimeter of $\triangle ABC$.

Perimeter of $\triangle DEF = 5 + 13 + 12 = 30$

Given: $AC = 7.5$, $DF = 5$

$$\frac{AC}{DF} = \frac{\text{Perimeter of } \triangle ABC}{\text{Perimeter of } \triangle DEF}$$ Theorem 21-D

$$\frac{7.5}{5} = \frac{x}{30}$$ Substitution

$$5x = 225$$ Cross products

$$x = 45$$ Division property

The perimeter of triangle *ABC* is 45 units.

Theorem 21-E

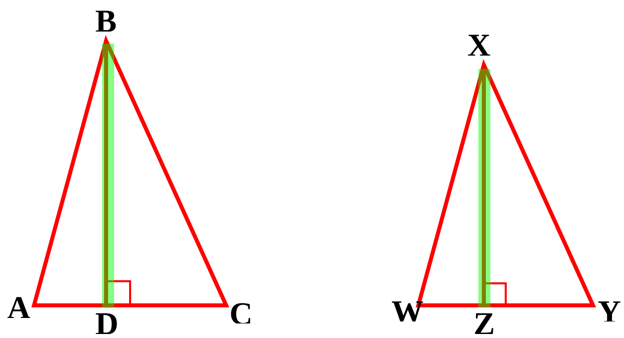
If two triangles are similar, then the measures of the corresponding altitudes are proportional to the measures of the corresponding sides.

Example 2: Fill in the missing segments based on theorem 21-E.

$$\frac{BD}{XZ} = \frac{AB}{?} = \frac{?}{WY} = \frac{BC}{?}$$

Given: $\triangle ABC \sim \triangle WXY$

$\overline{BD}$ and $\overline{XZ}$ are altitudes for their respective triangles.



Since the triangles are similar their corresponding parts and altitudes are proportional.

$$\frac{BD}{XZ} = \frac{AB}{WX} = \frac{AC}{WY} = \frac{BC}{XY}$$

Theorem 21-F

If two triangles are similar, then the measures of the corresponding medians are proportional to the measures of the corresponding sides.

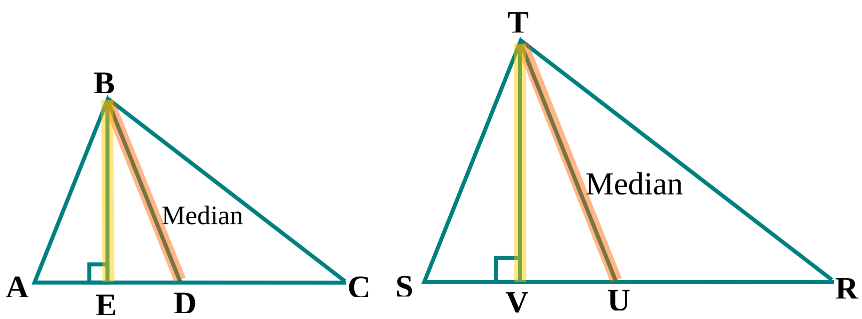
Example 3: Fill in the missing segments based on theorems 21-E and 21-F.

$$\frac{BD}{TU} = \frac{?}{TV} = \frac{BC}{?} = \frac{AC}{?} = \frac{?}{ST}$$

Given: $\triangle ABC \sim \triangle STR$

$\overline{BE}$ and $\overline{TV}$ are altitudes for their respective triangles.

$\overline{BD}$ and $\overline{TU}$ are medians for their respective triangles.



Since the triangles are similar, their corresponding parts, altitudes, and medians are proportional.

$$\frac{BD}{TU} = \frac{BE}{TV} = \frac{BC}{TR} = \frac{AC}{SR} = \frac{AB}{ST}$$

Theorem 21-G

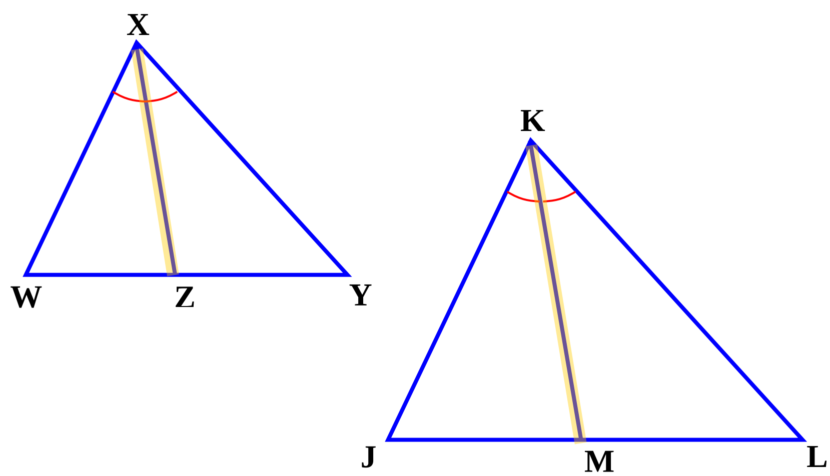
If two triangles are similar, then the measures of the corresponding angle bisectors of the two triangles are proportional to the measures of the corresponding sides.

Given: $\triangle WXY \sim \triangle JKL$

$\overline{XZ}$ is a bisector of $\angle X$

$\overline{KM}$ is a bisector of $\angle K$

Prove: $\dfrac{XZ}{KM} = \dfrac{WX}{JK}$



Statements	Reasons
$\triangle WXY \sim \triangle JKL$	Given
$\angle W \cong \angle J$	Definition of similar triangles (corresponding angles of similar triangles are congruent)
$\angle WXY \cong \angle JKL$	Definition of similar triangles (corresponding angles of similar triangles are congruent)
$\overline{XZ}$ is a bisector of $\angle WXY$	Given
$\overline{KM}$ is a bisector of $\angle JKL$	Given
$\angle WXZ \cong \angle JKM$	Like divisions of congruent angles are congruent. (Theorem 7-L)
$\triangle WXZ \sim \triangle JKM$	AA Similarity (Postulate 20-A)
Therefore, $\frac{XZ}{KM} = \frac{WX}{JK}$	Definition of similar triangles (corresponding parts of similar triangles are proportional)

Proofs that the other corresponding sides are proportional to the corresponding angle bisectors will be completed in the problem set.

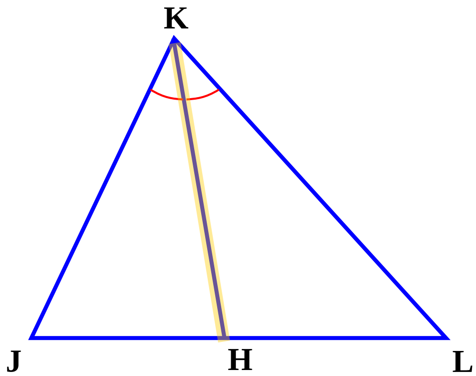
Theorem 21-H
Angle Bisector Theorem

In a triangle an angle bisector separates the opposite side into segments that have the same ratio as the other two sides.

Example 4: Fill in the missing segment based on theorem 21-H.

$$\frac{JH}{?} = \frac{JK}{KL}$$

Given: $\overline{KH}$ is an angle bisector of $\angle K$ in $\triangle JKL$.



According to the angle bisector theorem, $\overline{KH}$, the angle bisector of $\angle K$, separates the opposite side, $\overline{JL}$, into segments ($\overline{\text{JH}}$ and $\overline{\text{HL}}$) that have the same ratios as the other two sides ($\overline{\text{JK}}$ and $\overline{\text{KL}}$).

Thus, $\frac{JH}{HL} = \frac{JK}{KL}$.