INEQUALITIES AND TRIANGLE INEQUALITIES

In this unit, you will learn about inequalities and their connection with Geometry. First, you will look closely at the definition of "inequality", and then examine many theorems related to triangles and inequalities. You will also take a very close look at the Triangle Inequality Theorem. The unit will conclude with exploring triangle inequality relationships including the "SAS Inequality Theorem" known better as the "Hinge Theorem" and the "SSS Inequality Theorem".

Inequalities

inequality - An inequality can be defined as follows:

> For any real numbers, a and b, $a > b$ if and only if there is a positive number c, such that $a = b + c$.

Let's think about the definition of inequality.

> If $a = b + c$, and c is a positive number, then a must be greater than b because b has c added to it.

Let's consider several properties for the inequalities of real numbers. For all real numbers a, b, and c, the following properties exist.

Comparison Property	$a < b$, $a = b$, or $a > b$.

Example 1: 7 < 10 8.5 = 8.50 0 > –100

Transitive Property	1. If $a < b$ and $b < c$, then $a < c$. 2. If $a > b$ and $b > c$, then $a > c$.

Example 2: If 7 < 10 and 10 < 15, then 7 < 15.

Example 3: If 20 > –4 and –4 > –25, then 20 > –25.

Addition Property	1. If $a > b$, then $a + c > b + c$. 2. If $a < b$, then $a + c < b + c$.

Example 4: If 6 > 2, and c = 3, then 6 + 3 > 2 + 3 or 9 > 5.

Example 5: If –5 < –1 and c = 4, then –5 + 4 < –1 + 4 or –1 < 3.

Subtraction Property	1. If $a > b$, then $a - c > b - c$. 2. If $a < b$, then $a - c < b - c$.

Example 6: If 6 > 2, and c = 3, then 6 – 3 > 2 – 3 or 3 > –1.

Example 7: If –5 < –1 and c = 4, then –5 – 4 < –1 – 4 or –9 < –5.

Multiplication Properties	1. If $c > 0$ and $a < b$, then $ac < bc$. 2. If $c > 0$ and $a > b$, then $ac > bc$. 3. If $c < 0$ and $a < b$, then $ac > bc$. 4. If $c < 0$ and $a > b$, then $ac < bc$.

Example 8: If $c = 3$ and $7 < 10$, then $7(3) < 10(3)$ or $21 < 30$.

Example 9: If $c = 3$ and $13 > -4$, then $13(3) > -4(3)$ or $39 > -12$.

Example 10: If $c = -3$ and $-5 < 10$, then $-5(-3) > 10(-3)$ or $15 > -30$.
*The **inequality sign**'s direction is **reversed**.

Example 11: If $c = -3$ and $8 > -4$, then $8(-3) < -4(-3)$ or $-24 < 12$.
*The **inequality sign**'s direction is **reversed**.

Division Properties	1. If $c > 0$ and $a < b$, then $\frac{a}{c} < \frac{b}{c}$. 2. If $c > 0$ and $a > b$, then $\frac{a}{c} > \frac{b}{c}$. 3. If $c < 0$ and $a < b$, then $\frac{a}{c} > \frac{b}{c}$. 4. If $c < 0$ and $a > b$, then $\frac{a}{c} < \frac{b}{c}$.

Example 12: If $c = 3$ and $9 < 24$, then $\frac{9}{3} < \frac{24}{3}$ or $3 < 8$.

Example 13: If $c = 3$ and $15 > -6$, then $\frac{15}{3} > \frac{-6}{3}$ or $5 > -2$.

Example 14: If $c = -3$ and $-45 < 27$, then $\frac{-45}{-3} > \frac{27}{-3}$ or $15 > -9$.
*The **inequality sign**'s direction is **reversed**.

Example 15: If $c = -3$ and $-6 > -21$, then $\frac{-6}{-3} < \frac{-21}{-3}$ or $2 < 7$.
*The **inequality sign**'s direction is **reversed**.

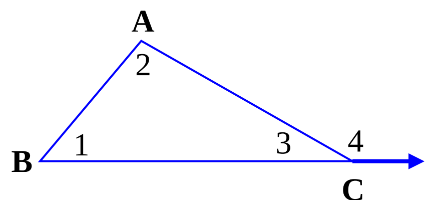
Now let's take a look at how the inequality rules apply to triangles.

Theorem 15-A
Exterior Angle Inequality Theorem

If an angle is an exterior angle of a triangle, then its measure is greater than the measure of either of its remote interior angles.

Given: $\angle 4$ is an exterior angle.

Prove: $m\angle 4 > m\angle 1$; $m\angle 4 > m\angle 2$



We will use an indirect proof to prove $m\angle 4 > m\angle 1$. You will prove the second part, $m\angle 4 > m\angle 2$, in the problem set.

Assumption: $m\angle 4$ is NOT > $m\angle 1$ which can be written as $m\angle 4 \leq m\angle 1$.

$m\angle 4 \leq m\angle 1$ can be split into two cases:

Case 1: $m\angle 4 = m\angle 1$

Case 2: $m\angle 4 < m\angle 1$

Statements	**Reasons**
Case 1:	
1. $m\angle 4 = m\angle 1$	Part of the Assumption
2. $m\angle 4 = m\angle 1 + m\angle 2$	Exterior Angle Theorem (Theorem 12-C)
3. $m\angle 1 = m\angle 1 + m\angle 2$	Substitution Property
4. $m\angle 2 = 0$	Subtraction Property
5. but, $m\angle 2 \neq 0$	$m\angle 2$ cannot equal 0 as it is an angle of a triangle.
6. $\therefore m\angle 4 \neq m\angle 1$	Contradicts "equals to" part of the assumption, $m\angle 4 \leq m\angle 1$.
Case 2:	
7. $m\angle 4 < m\angle 1$	Part of the Assumption
8. $m\angle 4 = m\angle 1 + m\angle 2$	Exterior Angle Theorem (Theorem 12-C)
9. $m\angle 4 > m\angle 1$	Definition of Inequality
10. but, $m\angle 4 < m\angle 1$	Contradictory Assumption
11. $\therefore m\angle 4 > m\angle 1$	Contradicts "less than" part of the assumption, $m\angle 4 \leq m\angle 1$.

Triangle Side and Angle Inequalities

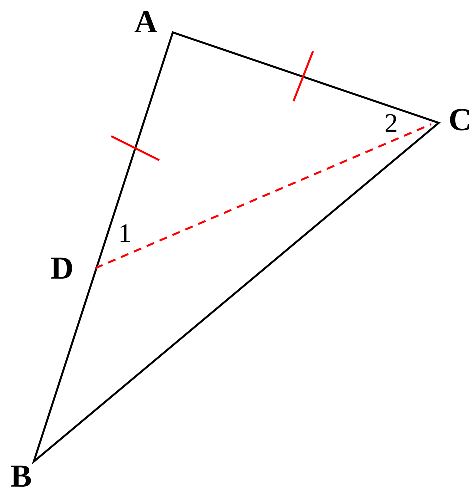
Theorem 15-B

If a side of a triangle is longer than another side, then the measure of the angle opposite the longer side is greater than the measure of the angle opposite the shorter side.

Given: $\triangle ABC$

$AB > AC$

Prove: $m\angle ACB \ > \ m\angle B$



We will use a paragraph proof and auxiliary $\overline{DC}$ to prove this theorem.

Construct $\overline{DC}$ so that segments AD and AC are congruent. Label angles ADC as $\angle 1$ and ACD as $\angle 2$ for easier reference to these angles.

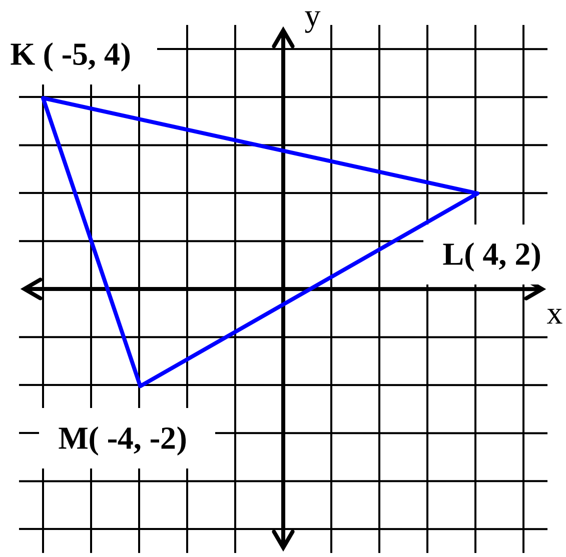
Angles opposite congruent sides are congruent, so angles 1 and 2 are congruent and thus, their measures are congruent.

Angle 1 is an exterior angle of $\triangle$ DBC; thus, its measure is greater than $\angle B$ based on the exterior angle inequality theorem. $m\angle 1 \ > \ m\angle B$

$m\angle ACB = m\angle 2 + m\angle DCB$ based on the angle addition postulate, thus $m\angle 2 < m\angle ACB$ based on the definition of inequality. By substitution, $m\angle 1 < m\angle ACB$ since angles 1 and 2 are congruent or $m\angle ACB > m\angle 1$.

By the transitive property, if $m\angle ACB > m\angle 1$ and $m\angle 1 > m\angle B$, then $m\angle ACB > m\angle B$.

Example: Given $\triangle KLM$ with vertices as shown in the figure below. List the angles of $\triangle KLM$ in order by size from least to greatest.



To solve this problem, we will first determine the length of each side of the triangle using the distance formula, and then apply Theorem 15-B to order the angles.

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$KM = \sqrt{[-5-(-4)]^2 + [4-(-2)]^2}$

$d = \sqrt{(-1)^2 + (6)^2}$

$d = \sqrt{37} \approx 6.1$

$LM = \sqrt{[4-(-4)]^2 + [2-(-2)]^2}$

$d = \sqrt{(8)^2 + (4)^2}$

$d = \sqrt{80} \approx 8.9$

$KL = \sqrt{(-5-4)^2 + (4-2)^2}$

$d = \sqrt{(-9)^2 + (2)^2}$

$d = \sqrt{85} \approx 9.2$

$KL \approx 9.2 > LM \approx 8.9 > KM \approx 6.1$

Thus, based on the Theorem 15-B:

$m\angle L$ (angle opposite $\overline{KM}$) $< m\angle K$ (angle opposite $\overline{ML}$) $< m\angle M$ (angle opposite $\overline{KL}$).

Theorem 15-C	**In a triangle, if the measure of an angle is greater than the measure of a second angle, then the side that is opposite the larger angle is longer than the side opposite the smaller angle.**
Theorem 15-D	**The shortest segment from a point to a line is a perpendicular line segment between the point and the line.**

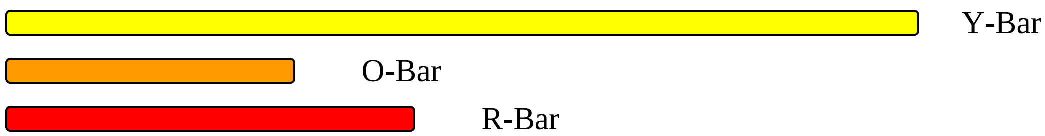
Triangle Inequality Theorem

Consider this statement:

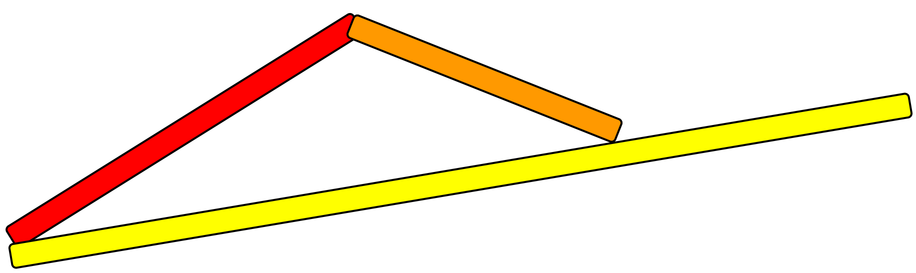
"Not all lengths of three lines segments will make a triangle."

Print out this section and cut out both sets of bars below. Try to make them into a triangle. Try many different arrangements. Can they be made into a triangle where the endpoints of the segments all meet?

Follow along as we examine several possibilities.

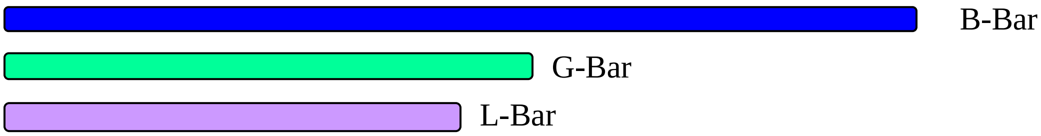


The figure below shows one unsuccessful trial in trying to make a triangle where all endpoints meet. The O-Bar does not reach to the end of the Y-Bar.

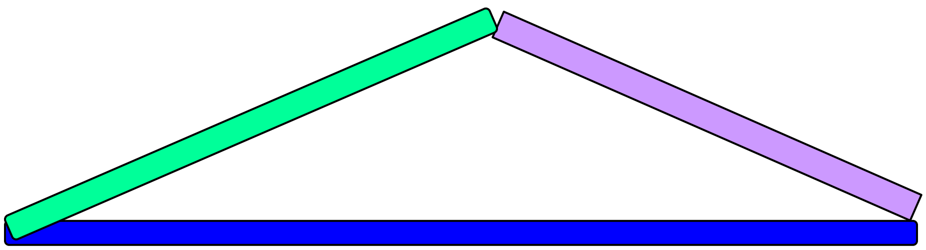


Notice that the sum of the lengths of the O-Bar and the R-Bar segments is not greater than the Y-Bar segment.

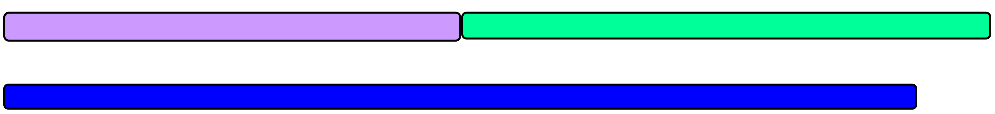
Let's examine a different set of bars with different lengths to see if we can make them into a triangle.



The figure below shows a successful trial in making in which all the endpoints of the segments meet.



Notice that the sum of the lengths of the L-Bar and the G-Bar is greater than the length of the B-Bar.



If you investigate further combinations of three lines segments, you will find that when three segments make a triangle such that all the endpoints meet, the sum of the lengths of any two sides of the triangle is greater than the length of the third side.

Theorem 15-E
Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

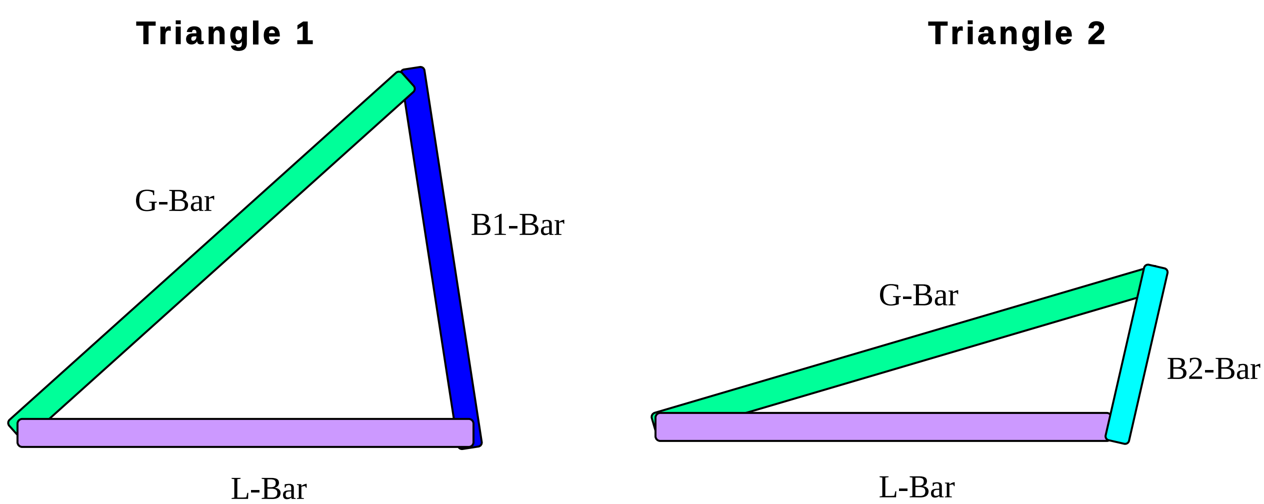
Hinge Theorem (SAS Inequality)

Theorem 15-F
SAS Inequality
(Hinge Theorem)

If two sides of a triangle are congruent to two sides of a second triangle, and if the included angle of the first triangle is greater than the included angle in the second triangle, then the third side of the first triangle is longer than the third side of the second triangle.

Let’s see how the Hinge Theorem works!

Look at the two triangles below. The length of the G-Bars and the L-Bars are congruent. The included angle between the G-Bar and the L-Bar in Triangle 1 is greater than the included angle between the G-Bar and the L-Bar in Triangle 2. Notice that the length of the B1-Bar is greater than the length of the B2-Bar.



This figure depicts the “Hinge Theorem”.

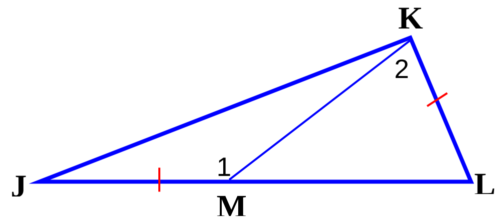
Let’s take a look at how to apply the “Hinge Theorem” in a proof.

Example:

Given: $\triangle KLJ$

$\overline{JM} \cong \overline{KL}$

Prove: $JK > ML$



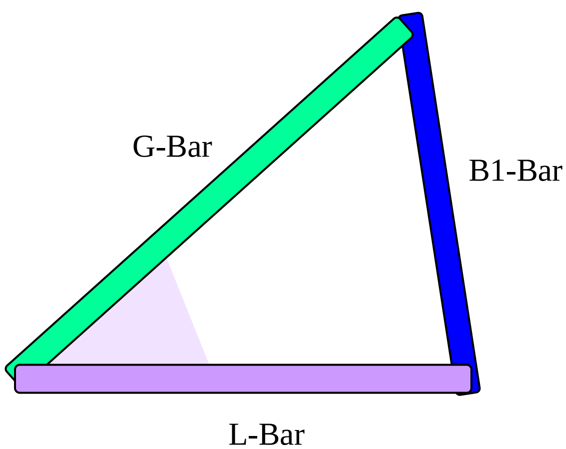
Statements	**Reasons**
1. $\overline{JM} \cong \overline{KL}$	Given
2. $\overline{KM} \cong \overline{KM}$	Reflexive Property (Postulate 6-A)
3. $\angle 1$ is an exterior angle of $\triangle MKL$	Definition of exterior angles
4. $m\angle 1 > m\angle 2$	Exterior Angle Inequality Theorem (15-A)
5. $JK > ML$	SAS Inequality Theorem (Hinge Theorem, 15-F)

SSS Inequality

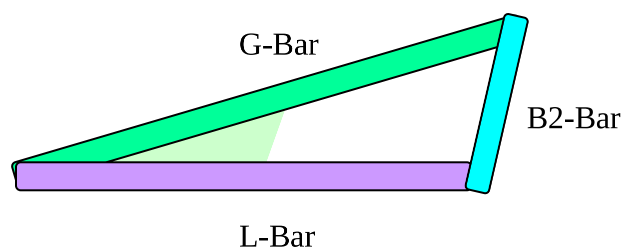
Theorem 15-G **SSS Inequality**	**If two sides of a triangle are congruent to two sides of a second triangle, and if the third side in the first triangle is longer than the third side in the second triangle, then the included angle between the congruent sides in the first triangle is greater than the included angle between the congruent sides in the second triangle.**

Look at the two triangles below. The length of the G-Bars and the L-Bars are congruent. The length of the B1-Bar is greater than the length of the B2-Bar. Note that the included angle between the G-Bar and the L-Bar in Triangle 1 is greater than the included angle between the G-Bar and the L-Bar in Triangle 2.

Triangle 1

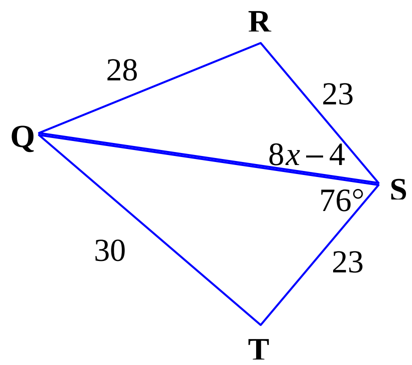


Triangle 2



Now we will apply the SSS Inequality Theorem to solve a problem.

Example: Write an equality to describe a solution for x in the figure shown below, and then solve for x.



We can apply the SSS Inequality Theorem:

$RS = TS$	Given
$QS = QS$	Reflexive Property
$QT > QR$	Given
$m\angle TSQ > m\angle RSQ$	SSS Inequality Theorem (15-G)

Therefore we can write the inequality, $76 > 8x - 4$.

Now, we can solve for x.

$76 > 8x - 4$
$80 > 8x$
$10 > x$

Thus, the solution is $x < 10$.