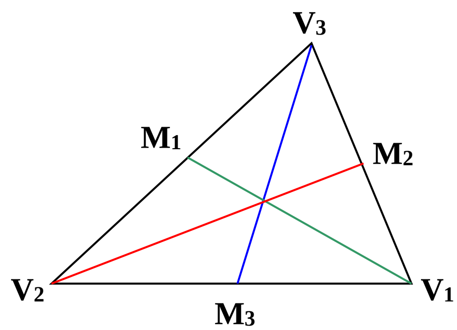
SPECIALTIES WITH TRIANGLES AND INDIRECT PROOFS

In this unit, you will learn about special characteristics in a triangle. You will examine medians which are line segments from vertices to the midpoints of their opposite sides. You will also examine the altitudes of triangles which are perpendicular line segments from the vertices to their opposite sides. You will learn new postulates including the "Hy-Leg Postulate" for right triangle congruency. You will reexamine perpendicular and angle bisectors, and then explore the equidistance theorems about bisectors. The unit will close with creating "indirect" proofs which is a new strategy for proving theorems by stating a contradiction.

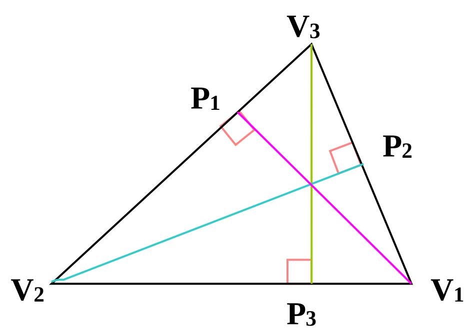
Medians and Altitudes

median – The median of a triangle is a line segment that connects a vertex of a triangle with the **midpoint** of the opposite side.



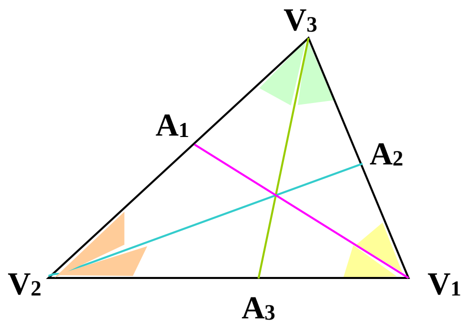
The three medians of a triangle are displayed in the above diagram. Each median is a segment between a vertex and the midpoint of the opposite side.

altitude – The altitude of a triangle is a **perpendicular** line segment between a vertex and its opposite side or an extension of it.



The three altitudes of a triangle are displayed in the above diagram. Each altitude is a perpendicular line segment from a vertex to its opposite side.

angle bisector – The angle bisector is a line segment that **bisects** an angle in the triangle and extends from the vertex to the opposite side.



The three angle bisectors of a triangle are displayed in the above diagram. Each line segment is a bisector of an angle and extends to the opposite side.

Example 1: $\triangle FED$ has vertices F(–4,2), E(5,–2) and D(2,–4). Graph $\triangle FED$, and then construct a line segment that is a median from point E to $\overline{FD}$. Label the midpoint of $\overline{FD}$ as point *M*. Name the median $\overline{EM}$.

Graph $\triangle FED$.

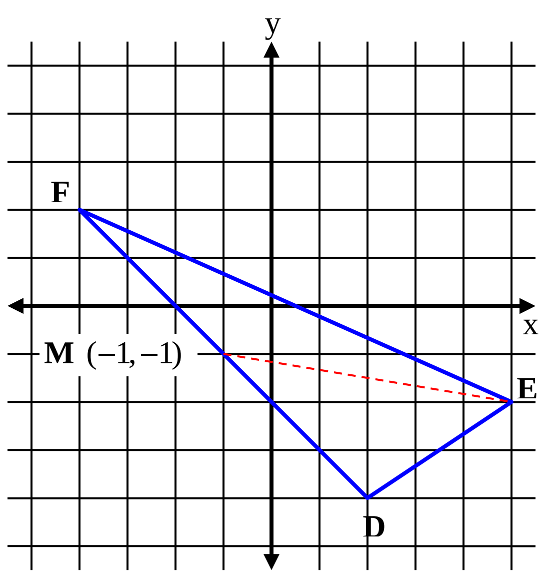
Find the midpoint of $\overline{FD}$. -Recall the midpoint formula. $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$

Midpoint of $\overline{FD}$ = Midpoint of F(–4,2) and D(2,–4)

$$M = (\frac{-4+2}{2}, \frac{2+(-4)}{2})$$

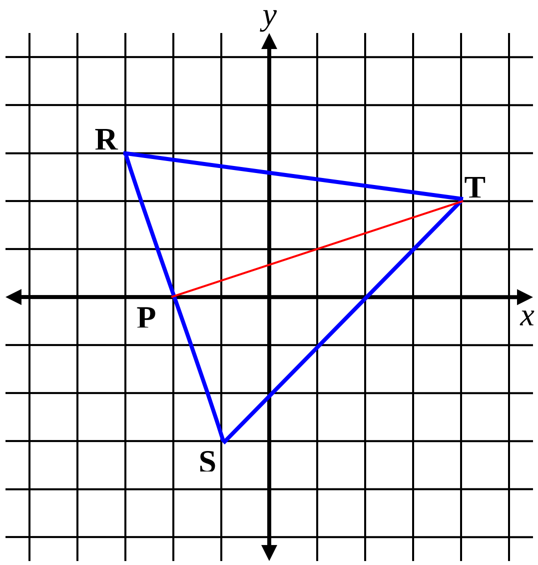
$$M = (-1, -1)$$

Draw a segment from the opposite vertex (Point E) to the midpoint.



$\overline{EM}$ is one median of triangle $\triangle FED$.

Example 2: Determine if $\overline{TP}$ is an altitude for $\triangle RTS$. Triangle *RTS* has vertices of R(–3,3), T(4,2) and S(–1, –3). Point P(–2,0) lies on $\overline{RS}$.



By definition, an altitude is a perpendicular line segment from a vertex to its opposite side.

We will check to see if $\overline{TP} \perp \overline{RS}$.

Recall that the product of the slopes of perpendicular lines is equal to –1; therefore we will find the slopes of $\overline{TP}$ and $\overline{RS}$ and multiply them.

Recall the slope formula: Slope $(m) = \dfrac{\text{rise}}{\text{run}} = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$

Slope of $\overline{PT}$ P(–2, 0) T(4,2)

$$m(\overline{PT}) = \frac{2-0}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

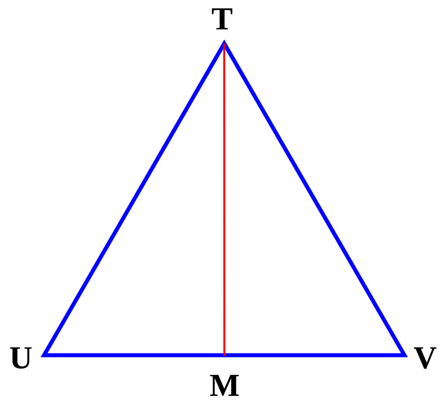
Slope of $\overline{RS}$ R(–3, 3) S(–1, –3)

$$m(\overline{RS}) = \frac{-3-3}{-1-(-3)} = \frac{-6}{2} = -3$$

The product of the slopes of $\overline{PT}$ and $\overline{RS} = \dfrac{1}{3} \times -3 = -1$

Therefore, $\overline{PT} \perp \overline{RS}$ and $\overline{TP}$ is an altitude of $\triangle RTS$.

Example 3: Prove that an angle bisector of an equilateral triangle is also a median of the triangle.



Given: $\triangle TUV$ is an equilateral triangle.
$\overline{TM}$ is an angle bisector of $\angle TUV$.

Prove: $\overline{TM}$ is a median of $\triangle TUV$.

*Recall that a median is a line segment that connects the vertex to the midpoint of the opposite side; therefore, we will prove $\overline{UM} \cong \overline{MV}$.

Statements	**Reasons**
1. $\overline{TM}$ is an angle bisector of $\angle TUV$.	Given
2. $\angle UTM \cong \angle VTM$	Definition of angle bisector.
3. $\overline{TM} \cong \overline{TM}$	Reflexive Property (Postulate 6-A)
4. $\triangle TUV$ is an equilateral triangle.	Given
5. $\overline{TU} \cong \overline{TV}$	Definition of equilateral triangles.
6. $\triangle TUM \cong \triangle TVM$	SAS (Postulate 13-B)
7. $\overline{UM} \cong \overline{VM}$	CPCTC
8. M is the midpoint of $\overline{UV}$	Definition of midpoint.
9. $\overline{TM}$ is the median of $\triangle TUV$	Definition of median.

Let's discuss this proof.

Steps 1 & 2: Since $\overline{TM}$ is given as an angle bisector of $\angle TUV$, we know by the definition of an angle bisector that $\angle UTM \cong \angle VTM$.

Step 3: By the reflexive property, we can state that $\overline{TM} \cong \overline{TM}$.

At this point we have one side congruent and an angle congruent in triangles TUM and TVM.

Steps 4&5: Since $\triangle TUV$ is given as an equilateral triangle, we know that all three of its sides are congruent. Therefore, we can state that $\overline{TU} \cong \overline{TV}$.

Step 6: We have proven that two sides and an included angle of triangles TUM and TVM are congruent (SAS).

Step 7: Thus, we can say that the two triangles are congruent, $\triangle TUM \cong \triangle TVM$, and that all the corresponding parts are congruent (CPCTC). Therefore, $\overline{UM} \cong \overline{VM}$.

Step 8: Once we have proven that $\overline{UM} \cong \overline{VM}$, we can then deduce that point M is a midpoint by definition of midpoint.

Step 9: Finally, by connecting point M with the opposite vertex, we create a median of the triangle $(\overline{TM})$. Segment TM was given as an angle bisector of the triangle.

Thus, through all of these steps, we have shown that a median drawn from one of the angles of an equilateral triangle also bisects that angle.

The HL Postulate

Postulate 14-A

HL Postulate

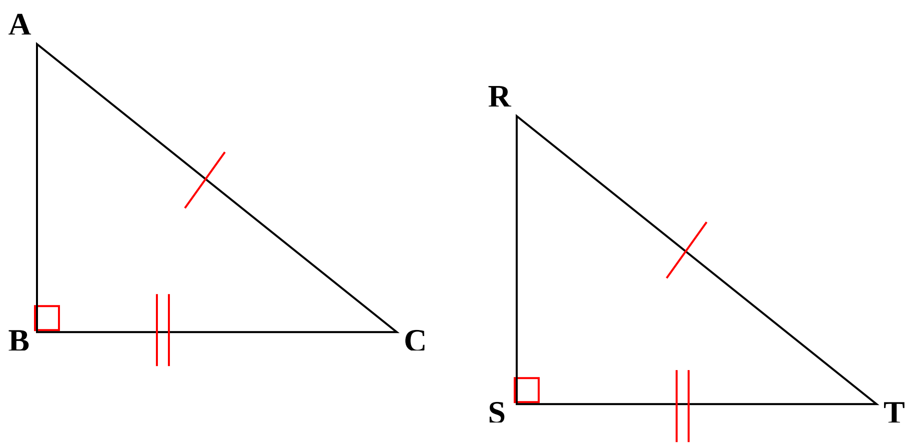
If there exists a correspondence between the vertices of two right triangles such that the hypotenuse and a leg of one triangle are congruent to the corresponding parts of the second triangle, then the two right triangles are congruent.

To demonstrate the HL Postulate, sometimes referred to as the “Hy-Leg” Postulate, we will examine the corresponding congruent parts of the two congruent right triangles below.

Given: $\triangle ABC$ and $\triangle RST$ are right triangles.

Hypotenuse $\overline{AC}$ and hypotenuse $\overline{RT}$ are congruent.

Leg $\overline{BC}$ and leg $\overline{ST}$ are congruent.



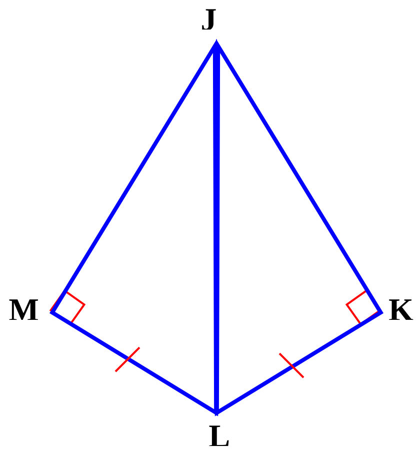
Then, by the HL postulate, we can say that $\triangle ABC \cong \triangle RST$.

*Note: The HL postulate only applies to right triangles.

Example: Apply the HL Postulate to complete the following proof.

Given: $\overline{JM} \perp \overline{ML}$
$\overline{JK} \perp \overline{KL}$
$\overline{ML} \cong \overline{KL}$

Prove: $\triangle JML \cong \triangle JKL$



Statements	**Reasons**
1. $\overline{JM} \perp \overline{ML}$	Given
2. $\angle JML$ is a right angle.	Perpendicular lines intersect to form right angles. (Theorem 10-I)
3. $\overline{JK} \perp \overline{KL}$	Given
4. $\angle JKL$ is a right angle.	Perpendicular lines intersect to form right angles. (Theorem 10-I)
5. $\triangle JML$ and $\triangle$JKL are right triangles.	Definition of right triangles.
6. $\overline{ML} \cong \overline{KL}$ (Leg)	Given
7. $\overline{JL} \cong JL$ (Hypotenuse)	Reflexive Property (Postulate 6-A)
8. $\triangle JML \cong \triangle JKL$	Hy-Leg Postulate (Postulate 14-A)

Discussion of the proof:

In *steps 1 through 5* , we established that the two triangles are right triangles.

In *steps 6 and 7* , we showed that a leg and the hypotenuse are congruent in the right triangles. Thus, we then applied the HL postualte to prove the $\triangle JML \cong \triangle JKL$.

Equidistance Theorems

distance – The distance between two objects is the length of the shortest path joining them.

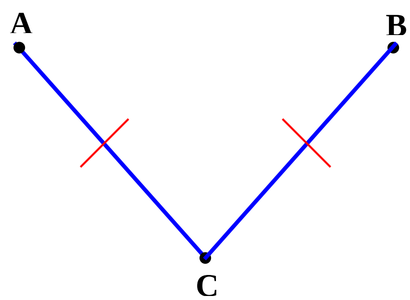
The shortest distance between two points is a straight line.

Postulate 14-B

A line segment is the shortest path between two points.

equidistant – If two points (A and B) are the same distance from a third point (C), then point (C) is said to be equidistant from the two points (A and B).

The meaning of "equidistant" is illustrated in the figure below.



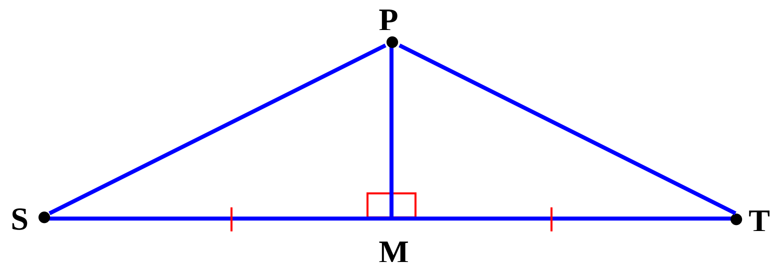
Point C is equidistant from points A and B.

Theorem 14-A

A point on a perpendicular bisector of a segment is equidistant from the endpoints of the segment.

Given: $\overline{PM}$ is a perpendicular bisector of $\overline{ST}$.

Prove: $\overline{SP} \cong \overline{PT}$



Statements	Reasons
1. $\overline{PM} \perp \overline{ST}$	Given
2. $\overline{SM} \cong \overline{MT}$	Definition of perpendicular bisector
3. $\angle PMS$ and $\angle PMT$ are right angles.	Definition of perpendicular bisector
4. $\angle PMS \cong \angle PMT$	Right angles are congruent. (Theorem 10-G)
5. $\overline{PM} \cong \overline{PM}$	Reflexive Property (Postulate 6-A)
6. $\triangle PMS \cong \triangle PMT$	SAS (Postulate 13-B)
7. $\overline{SP} \cong \overline{PT}$	CPCTC

Theorem 14-B

A point that is equidistant from the endpoints of a segment lies on the perpendicular bisector of the segment.

Theorem 14-C

A point on the bisector of an angle is equidistant from the sides of the angle.

Theorem 14-D

A point that is in the interior of an angle and is equidistant from the sides of the angle lies on the bisector of the angle.

Indirect Proofs

Another approach to proving theorems is to prove them indirectly. In this type of proof, you begin by assuming that the conclusion is false. Take a look at the steps below about how to develop an indirect proof.

To prove theorems "indirectly", follow these steps:

(1) Assume that the conclusion is **false**!

(2) Demonstrate that the assumption leads to a contradiction of the hypothesis, theorem, or corollary.

(3) Make the point that the assumption must be false; therefore, the conclusion must be true.

Example 1: Apply the steps of the indirect proof to a prosecutor's point of view of a person on trial who is accused of robbery.

Hypothesis: The prosecutor makes the assumption that the accused person is guilty.

(1) The jury assumes the person to be innocent until proven guilty. (assumes the conclusion is false)

(2) The prosecutor demonstrates substantial and credible ways that the accused cannot be innocent. (steps of the proof to lead to a contradiction of innocence)

(3) The jury concludes that the person is not innocent, but indeed is guilty. (assumption of innocence must be false, therefore the conclusion of guilty must be true)

Example 2: For the following assumptions, make a statement that would start an indirect proof (a false assumption).

Assumption	**False Assumption**
a) $\overline{PQ} \perp \overline{RM}$	$\overline{PQ}$ is not perpendicular to $\overline{RM}$.
b) $m\angle T < m\angle V$	$m\angle T \geq m\angle V$

Example 3: Use an indirect proof to prove that angle F is not congruent to angle M given that angles G and N are congruent, segments GF and NM are congruent, and segments GE and NL are NOT congruent.

Given: $\angle G \cong \angle N$ Prove: $\angle F \ncong \angle M$

$\overline{GF} \cong \overline{NM}$

$\overline{GE} \ncong \overline{NL}$

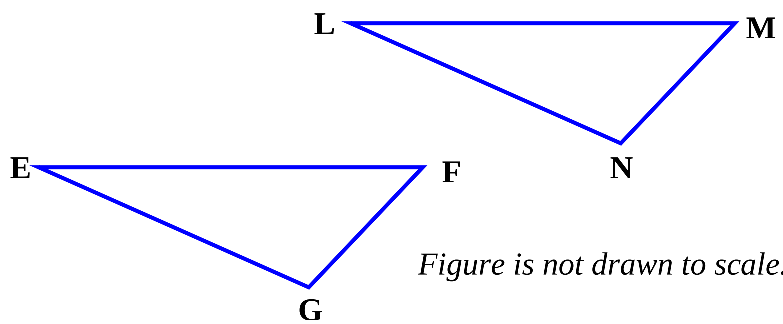


Figure is not drawn to scale.

To use the indirect approach, we will assume the opposite of the conclusion; that is, $\angle F \cong \angle M$.

Statements	**Reasons**
1. $\angle F \cong \angle M$	Assumption
2. $\angle G \cong \angle N$	Given
3. $\overline{GF} \cong \overline{NM}$	Given
4. $\triangle EGF \cong \triangle LNM$	ASA (Postulate 13-C)
5. $\therefore \overline{GE} \cong \overline{NL}$	CPCTC

But this is impossible because $\overline{GE}$ is given as NOT congruent to $\overline{NL}$.
Thus, our assumption ($\angle F \cong \angle M$) is false.

$\therefore \angle F \ncong \angle$M