PARALLEL AND PERPENDICULAR LINES

In this unit, you will examine the relationships of parallel and perpendicular lines. You will begin with taking a close look at the slopes of lines, and then compare the slopes of parallel and perpendicular lines. You will construct angles, parallel lines, and perpendicular lines. Finally, you will define the distance between parallel lines and apply the distance formula to parallel and perpendicular lines in a coordinate plane.

Slope of a Line

When graphing lines, the slope of a line must be considered.

slope - The slope of a line describes the steepness of the line. The slope is the ratio of vertical rise to horizontal run.

$$\text{slope} = \frac{\text{vertical rise}}{\text{horizontal run}}$$

vertical rise - The vertical rise of a line is the change in the *y*-values from one point on the line to another point on the line.

horizontal run – The horizontal run of a line is the change in the *x*-values from one point on a line to another point on the line.

Another way to represent slope is:

$$m = \frac{\Delta y}{\Delta x}$$

Δ is a short way to represent the word "change".
Δ is a Greek symbol and can be read as "delta".

For a line, even though the numbers for the vertical rise and horizontal run may change, the slope of the line remains constant.

To find the slope of a line:

-identify a point on the line

-from that point move up or down until you are directly across from the next point

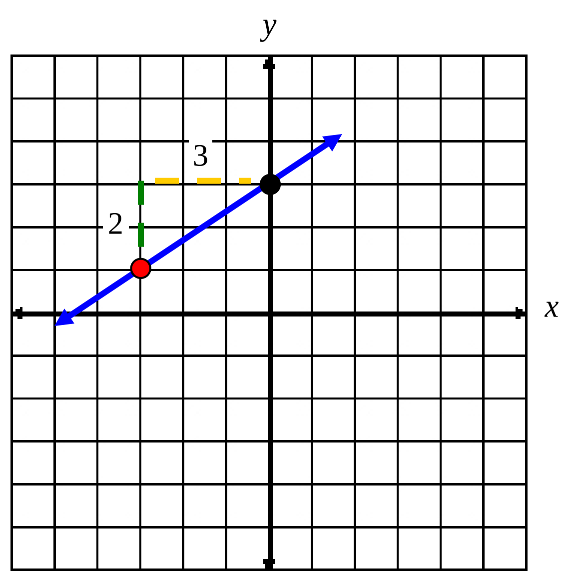
-move left or right to the next point.

Example: Determine the slope of the line shown in the coordinate plane. Each space in the grid represents one unit.

-put your pencil on the red point.

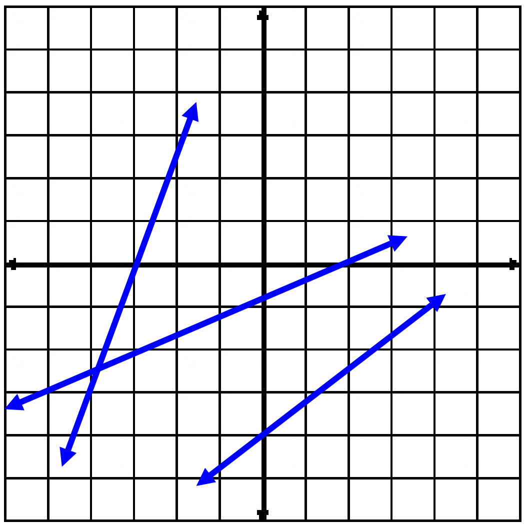
-move straight up (vertical rise) until your pencil is in the same line as the black point, (2 units)

-move right (horizontal run) until you reach the black point. (3 units)

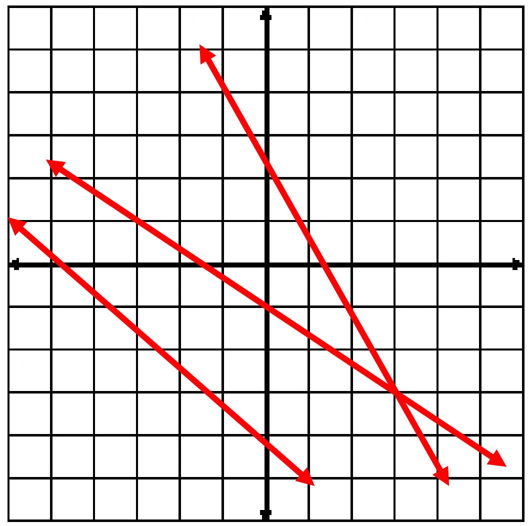


The slope of the line is $\frac{2}{3}$ or $m = \frac{2}{3}$.

On a coordinate plane, there are lines that have positive slopes and lines that have negative slopes.



Lines with positive slopes rise to the **right.**



Lines with negative slopes rise to the **left.**

Finding the Slope of a Line When Given Two Points

The definition of slope states that, given two points (x_1, y_1) and (x_2, y_2), the slope of a line containing the points can be determined using this formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ when } x_2 - x_1 \neq 0$$

*Notice, this is the vertical change over the horizontal change.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{vertical change}}{\text{horizontal change}}$$ *ratio for slope is $\frac{rise}{run}$

Example 1: Find the slope of a line going through the points (3, 5) and (–2, –6).

*It may be easier to pick out the *x* and *y* values if you label them as follows. This will help you in setting up the ratio.

$$(x_1, y_1) \qquad (x_2, y_2)$$
$$(3,5) \qquad (-2,-6)$$

$$x_1 = 3 \quad y_1 = 5 \qquad x_2 = -2 \quad y_2 = -6$$

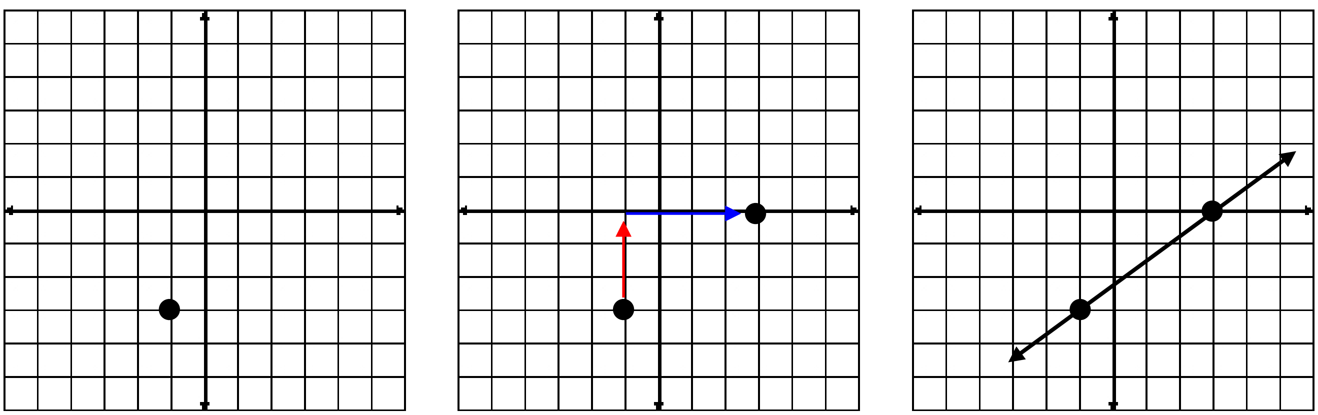
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-5}{-2-3} = \frac{-11}{-5} = \frac{11}{5}$$

Thus, the slope (m) is $\frac{11}{5}$.

Graphing a Line using the Point-Slope Method

When given the slope of a line and a point (location) on the line, the line can then be graphed.

Example 2: Graph the line containing the point (–1, –3) and having a slope of $m = \frac{3}{4}$.



1. Plot the point (–1, –3)

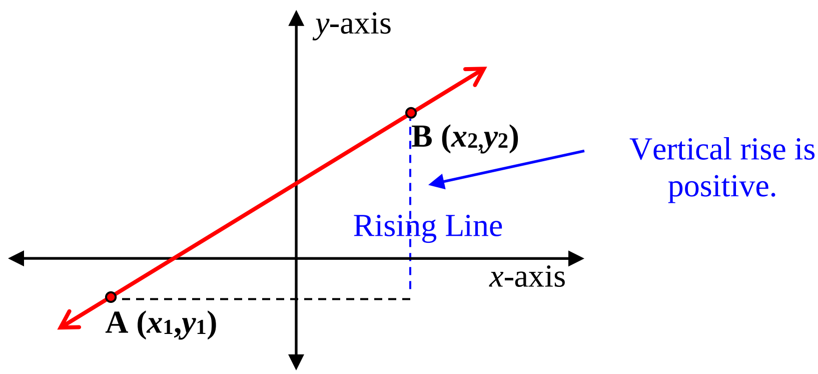
2. Use the rise (3 units) over run (4 units) ratio for slope to plot a second point.

3. Draw a line through the points with a straightedge.

Four Types of Slopes

When working with slopes of lines in a coordinate plane, there are four types of slopes.

Positive Slope

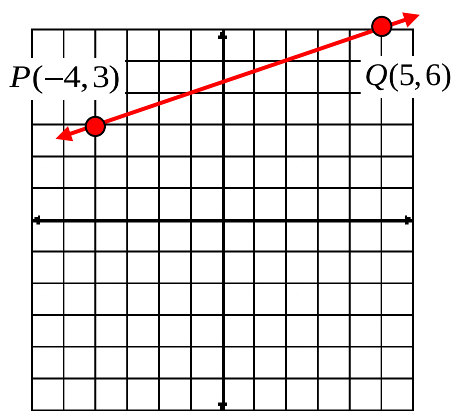


Example 1: Find the slope of the line containing the point $P(-4, 3)$ and $Q(5, 6)$.

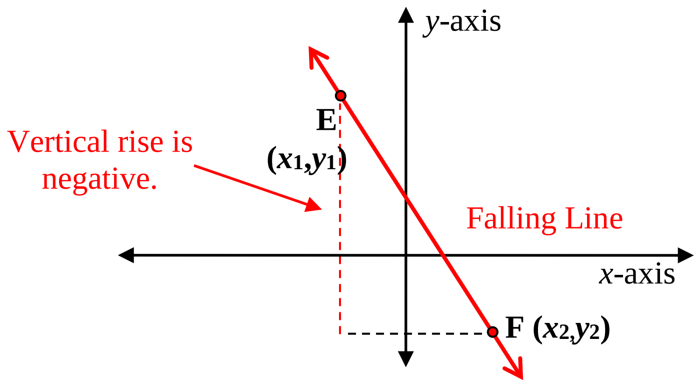
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{6-3}{5-(-4)}$$

$$m = \frac{3}{9} \text{ or } \frac{1}{3}$$



Negative Slope

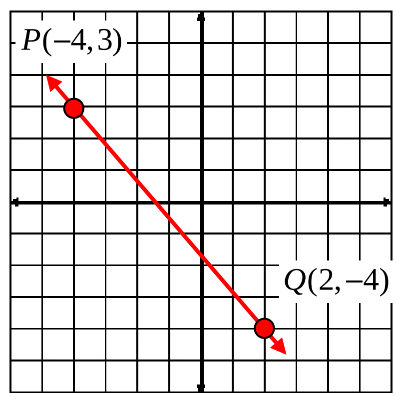


Example 2: Find the slope of the line containing the point $P(-4, 3)$ and $Q(2, -4)$.

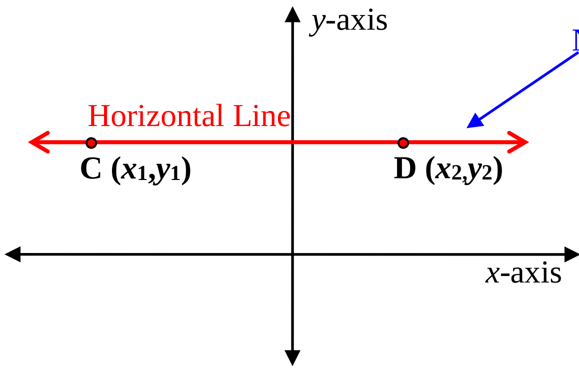
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-4 - 3}{2 - (-4)}$$

$$m = \frac{-7}{6} \text{ or } -\frac{7}{6}$$

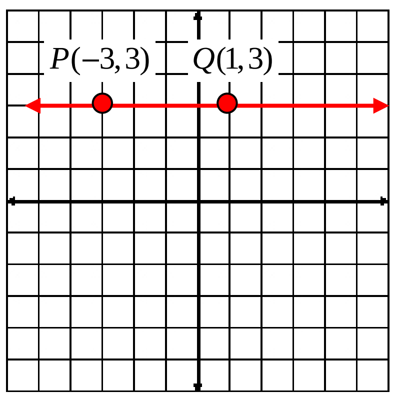


Zero Slope



$$\text{slope} = \frac{0}{\text{horizontal run}} = 0$$

Example 3: Find the slope of the line containing the point $P(-3, 3)$ and $Q(1, 3)$.



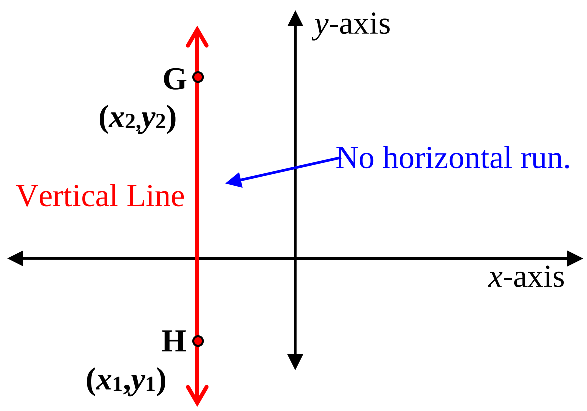
The y-coordinate for every point on a horizontal line is the same.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{3-3}{-3-1} = \frac{0}{-4} = 0$$

*Note: The slope of EVERY horizontal line in a coordinate plane is zero (0).**

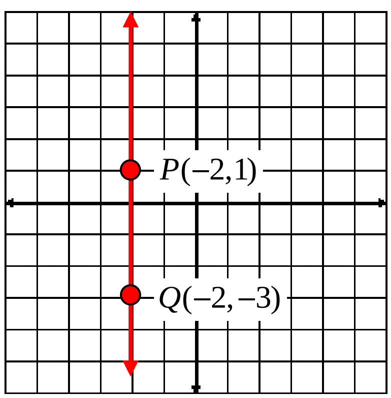
No Slope



$$\text{slope} = \frac{\text{vertical rise}}{0} = \text{undefined}$$

**Division by 0 is undefined.*

Example 4: Find the slope of the line containing the point $P(-2,1)$ and $Q(-2,-3)$.



The *x*-coordinate for every point on a vertical line is the same.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{1-(-3)}{-2-(-2)} = \frac{4}{0} = \text{undefined}$$

$$m = \text{undefined or "no" slope}$$

***Note: The slope of EVERY vertical line in a coordinate plane is "undefined".**

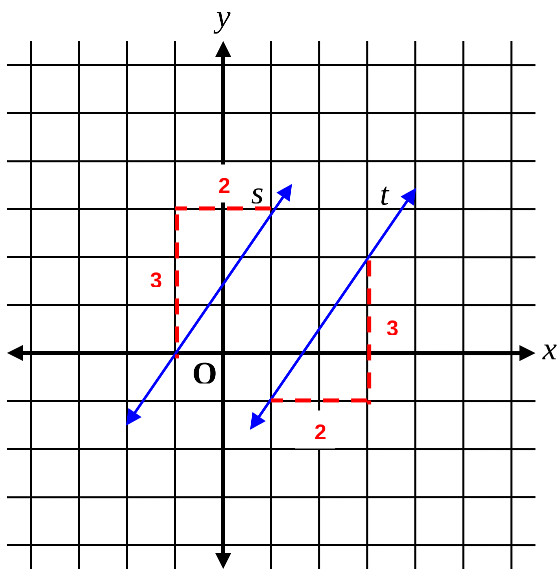
Slopes of Parallel and Perpendicular Lines

Slope of Parallel Lines

Lines have the same slope if they are parallel. Conversely, if lines are parallel, they have the same slope. Since both the original statement and its converse are true, we can condense the statement into one statement using the phrase "if and only if".

Postulate 11-A	**Two non-vertical lines have the same slope if and only if they are parallel.**

Example 1: Are lines *s* and *t* parallel?



The slope of line *s* is $\frac{2}{3}$. The slope of line *t* is $\frac{2}{3}$. Since the lines have the same slope, they are parallel.

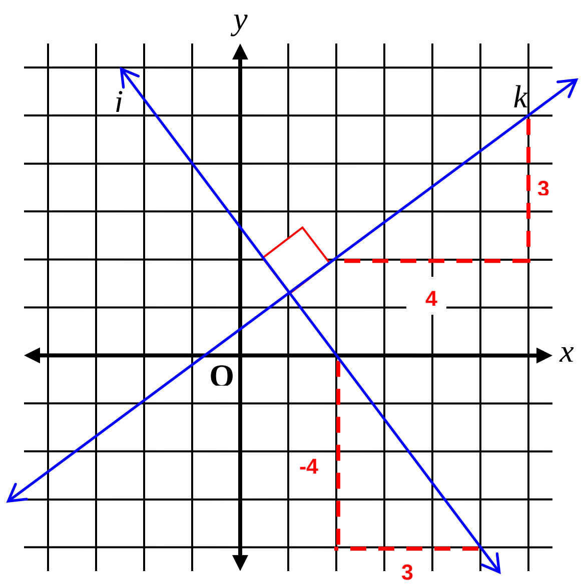
Slope of Perpendicular Lines

If two lines are perpendicular, then the product of their slopes is –1. Conversely, if the product of the slopes of two lines is –1, then the lines are perpendicular. Since both the original statement and its converse are true, we can condense the statement into one statement using the phrase "if and only if".

Postulate 11-B

Two non-vertical lines are perpendicular if and only if the product of their slopes is –1.

Example 2: Are lines *j* and *k* perpendicular?



The slope of line *j* is $-\frac{4}{3}$ and the slope of line *k* is $\frac{3}{4}$.

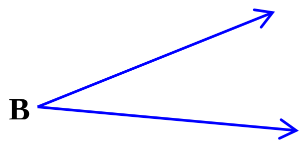
The product of $-\frac{4}{3}\times\frac{3}{4}=-1$; therefore, lines *j* and *k* are perpendicular.

Constructions of Angles and Parallel Lines

In this section, we will first look at constructing an angle from a given angle, and then apply that technique to constructing parallel lines.

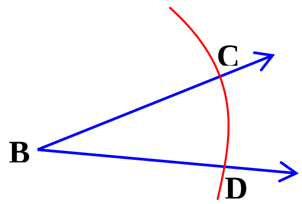
Constructing an Angle from a Given Angle

Step #1: On paper, start with an angle and name it angle B $(\angle B)$.

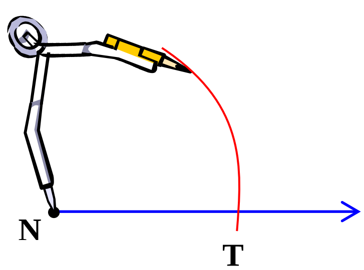


Step #2: Move away from the angle and draw a ray $(\overrightarrow{N})$ (make sure you use a ruler or straight-edged tool).

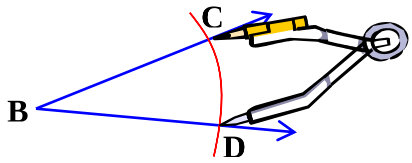
Step #3: Move back to $\angle B$ and place the metal point of the compass at the vertex the angle. Draw an arc that passes through both rays. Name the points of intersection, points C and D.



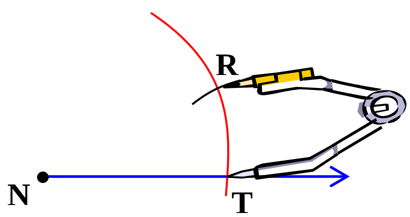
Step #4: Keep the setting of the compass the same and move over to $(\overrightarrow{N})$. Place the metal point of the compass on point N and draw the same-sized arc as in Step #3. Name the intersection between the arc and the ray, point T.



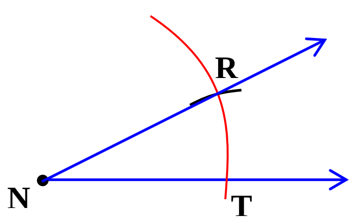
Step #5: Move over to $\angle B$ and place the metal tip of the compass on point D. Adjust the compass setting so that the pencil point is on point C.



Step #6: Keep the setting of the compass the same and move over to $\overrightarrow{NT}$ and place the metal point of the compass at point T. Draw a small arc that intersects the larger arc. Name the point of intersection, point R.



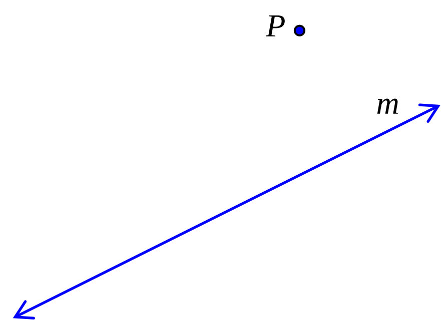
Step# 7: Use a ruler or straight-edged tool and draw $\overrightarrow{NR}$ through point R.



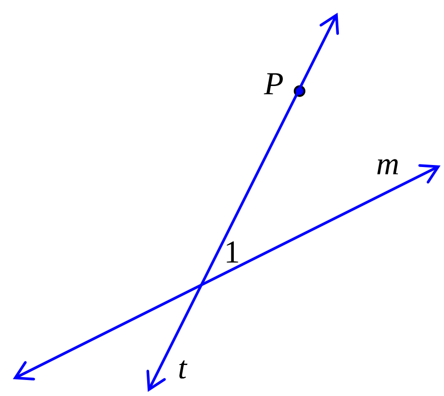
$\angle RNT \cong \angle B$

Constructing Parallel Lines using Corresponding Angles

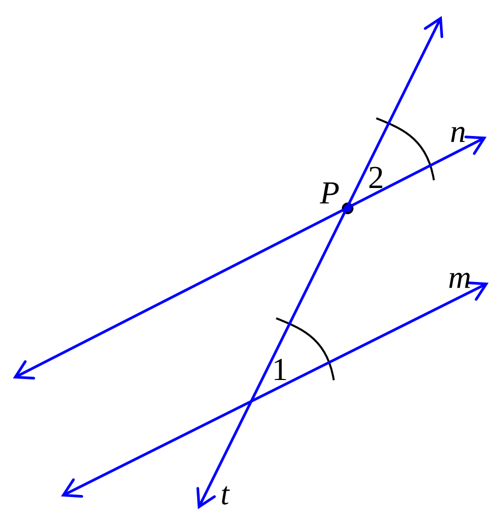
Step #1: On paper, draw a line (*m*) and a point (P) that is not on the line.



Step #2: Draw line *t* through point P so that it intersects line *m* and forms ∠1.



Step #3: Turn you paper to construct ∠2 at point P so that is congruent to ∠1.



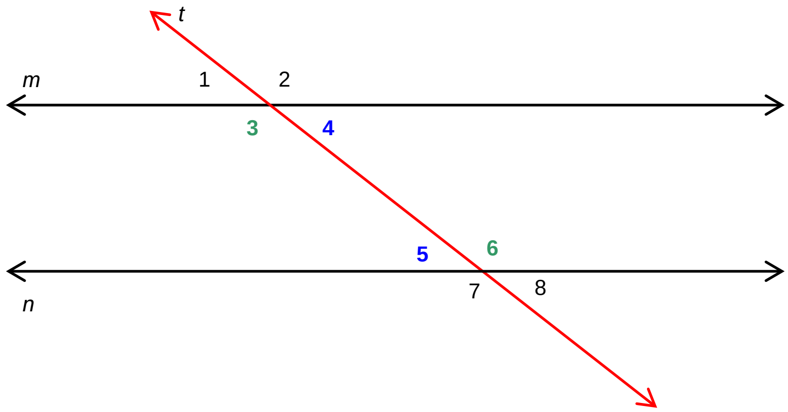
Since $\angle 1$ and $\angle 2$ are congruent corresponding angles and line *t* is a transversal over lines *n* and *m*, lines *n* and *m* are parallel.

Postulate 11-C	**If two lines in a plane are cut by a transversal and the corresponding angles are congruent, then the lines are parallel.**
Postulate 11-D	**If there is a line and a point that is not on the line, then there exists exactly one line that passes through the point that is parallel to the given line.**

Theorems about Parallel Lines

Theorem 11-A

If two lines in a plane are cut by a transversal and the alternate interior angles are congruent, then the lines are parallel.



Given: $\angle 3 \cong \angle 6$; $\angle 4 \cong \angle 5$

[If two lines in a plane are cut by a transversal and the alternate interior angles are congruent,
$\angle 3 \cong \angle 6$ $\angle 4 \cong \angle 5$]

Prove: $m \parallel n$

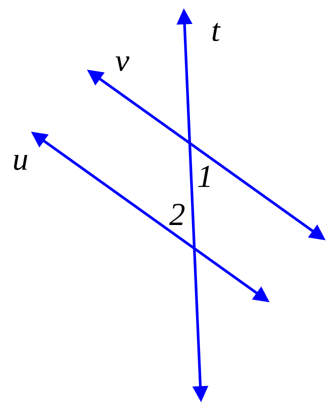
[then line lines are parallel.
$m \parallel n$]

Statements	**Reasons**
1. $\angle 3 \cong \angle 6$	Given
2. $\angle 6 \cong \angle 7$	Vertical angles are congruent. (Theorem 10-H)
3. $\angle 3 \cong \angle 7$	Transitive Property (Theorem 10-A)
4. $\therefore m \parallel n$	If lines in a plane are cut by a transversal and the corresponding angles are congruent, then the lines are parallel. (Postulate 11-C)

*Note: We could also have used congruent angles 4 and 5 along with vertical angle 8 to show the lines were parallel.

Example: Refer to the figure below to find the value of x that will make lines v and u parallel. Also find the measures of angles 1 and 2.

Given: $m\angle 1 = 7x + 34$ $\quad$ $m\angle 2 = 13x + 22$



Apply Theorem 11A, "If two lines in a plane are cut by a transversal and the alternate interior angles are congruent, then the lines are parallel".

$\angle 1 \cong \angle 2$, therefore $m\angle 1 = m\angle 2$ by definition of congruence.

Set $m\angle 1 = m\angle 2$.

$7x + 34 = 13x + 22$	
$12 = 6x$	-subtraction property
$2 = x$	-division property

$\angle 1 = 7x + 34$	
$\angle 1 = 7(2) + 34$	-substitution
$\angle 1 = 48$	-simplify

Since the two angles are congruent, their measures should be the same. Let's check.

$\angle 2 = 13x + 22$	
$\angle 2 = 13(2) + 22$	-substitution
$\angle 2 = 48$	-simplify

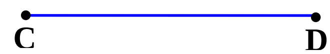
Therefore, $x = 2$ and $m\angle 1 = m\angle 2 = 48°$.

Theorem 11-B	**If two lines in a plane are cut by a transversal and the alternate exterior angles are congruent, then the lines are parallel.**
Theorem 11-C	**If two lines in a plane are cut by a transversal and the consecutive interior angles are supplementary, then the lines are parallel.**
Theorem 11-D	**If two lines in a plane are perpendicular to the same line, then the lines are parallel.**

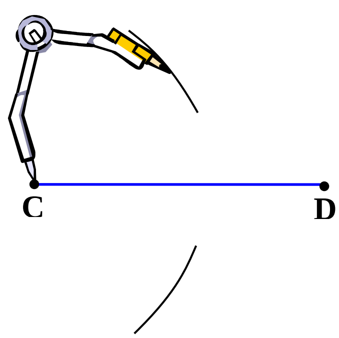
Constructions with Perpendicular Lines

Constructing a Perpendicular Line to a Line Segment

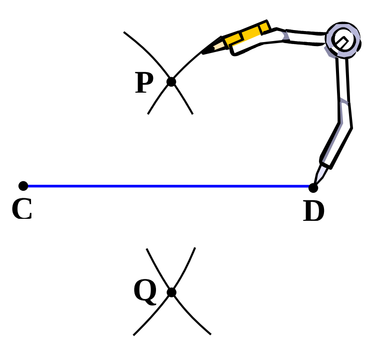
Step #1: On paper draw $\overline{CD}$.



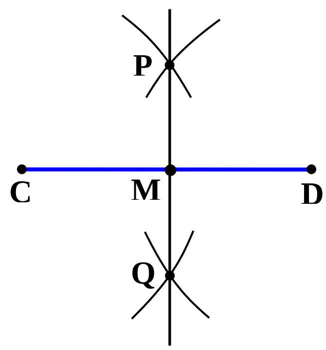
Step #2: Place the metal point of a compass on one of the endpoints of the segment. Adjust the setting of the compass so that it is greater than halfway across the segment. Draw an arc above and below the segment.



Step #3: Place the metal point of the compass on the other end point and, without changing the setting of the compass, draw an arc above and below the segment so that the arcs intersect. Mark the points of intersection as points P and Q.



Step #4: Draw a line that passes through the two points of intersection, points P and Q.



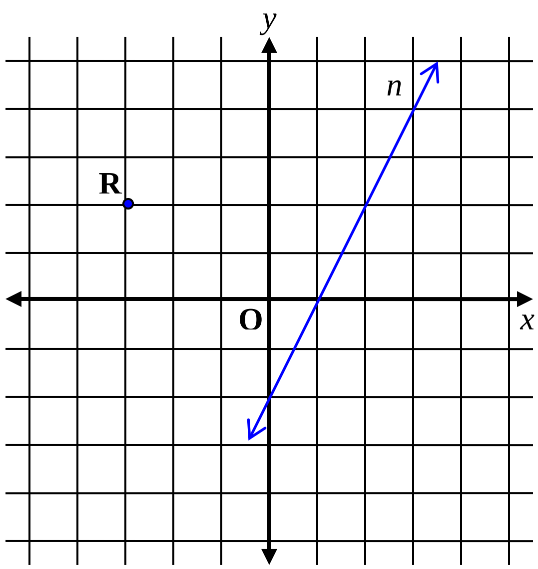
$\overleftrightarrow{PQ} \perp \overline{CD}$

In addition, $\overleftrightarrow{PQ}$ is a perpendicular bisector of $\overline{CD}$.

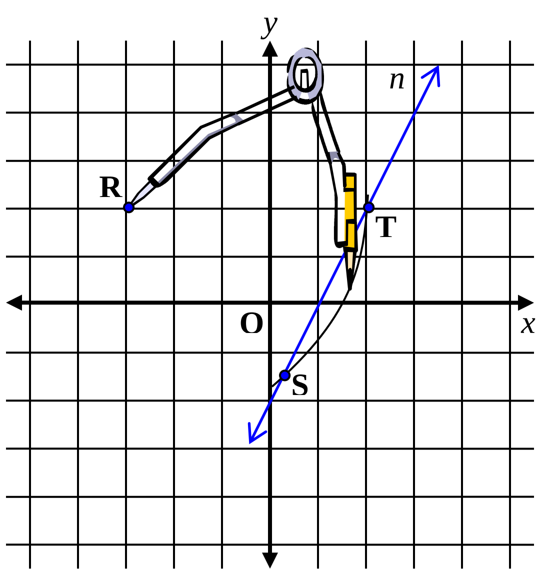
*Note: The symbol for "is perpendicular to" is $\perp$.

Constructing a Perpendicular Line from a Point to a Line

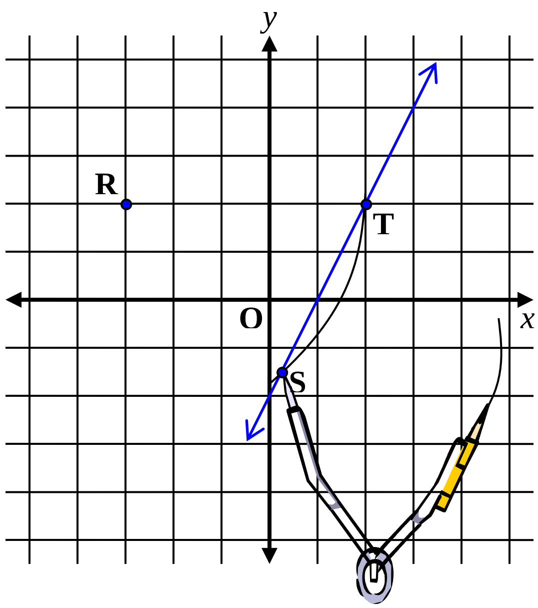
Step #1: On graph paper draw line *n* and point R (–3, 2)

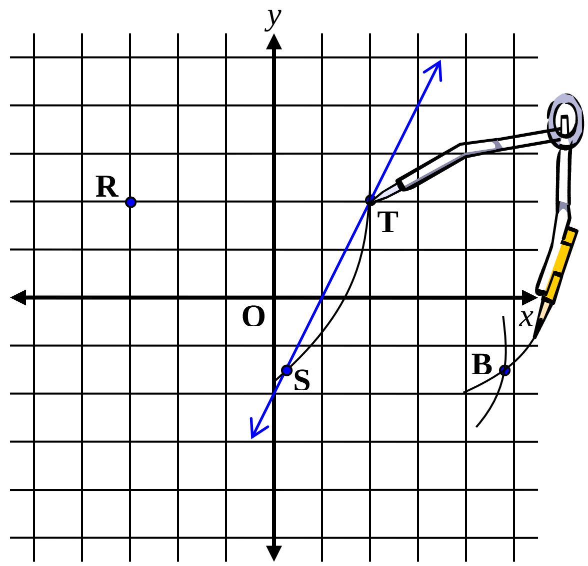


Step #2: Place the metal point of the compass at point R and draw a large arc that passes through line *n* and mark the points of intersection as points S and T.

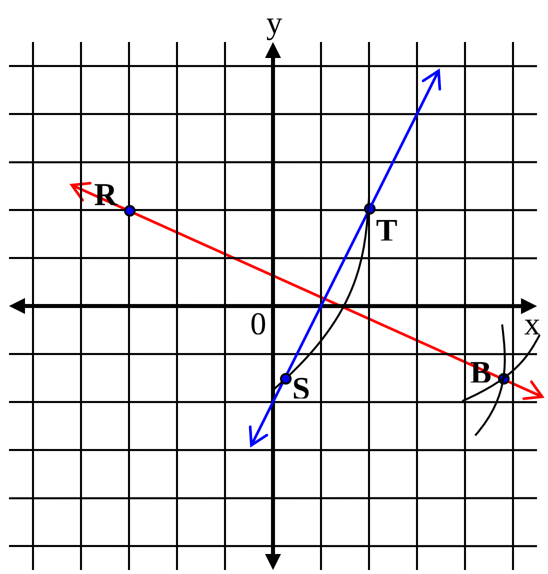


Steps #3 and #4: Keep the setting of the compass the same. First put the metal point at point S and make a large arc. Then move the metal point of the compass to point T and make a second large arc. Name the point of intersection of the two arcs point B.





Step #5: Draw a line through points B and R. $\overleftrightarrow{BR} \perp \overleftrightarrow{ST}$.



You can verify that the lines are perpendicular by determining the slopes of each line by counting the rise and run and checking to see if the product of the slopes equals –1.

The slope of $\overleftrightarrow{RB}$ is –1/2.
The slope of $\overleftrightarrow{ST}$ is 2/1 or 2

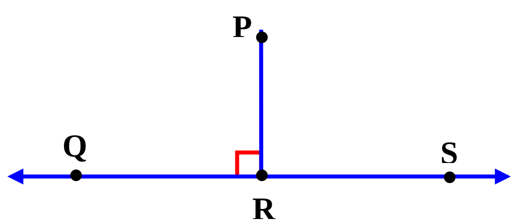
$$\frac{-1}{2} \times 2 = -1$$

Verified: Since the product of the two slopes is –1, the lines are indeed perpendicular.

Distance between a Point and a Line

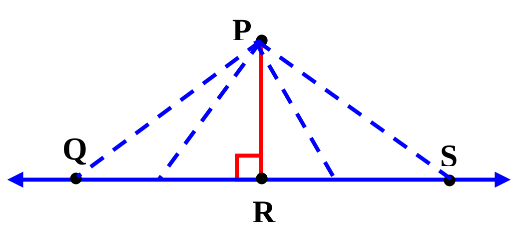
Consider this statement and diagram.

The distance from a point, which is not on a line, and a line is the length of a line segment that is perpendicular from the point to the line.



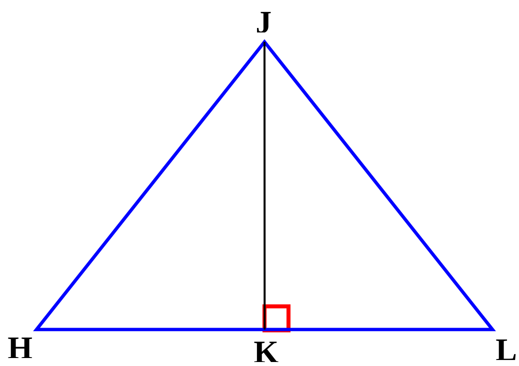
The distance from point P to $\overleftrightarrow{QS}$ is equal to the length of $\overline{PR}$.

Common sense suggests that there are many different lengths from a point to a line; however, in geometry, when we refer to the distance between a point and a line, we are referring to the length of a line segment that starts at the point and is perpendicular to the line.



The distance from Point P to $\overleftrightarrow{QS}$ is equal to the length of $\overline{PR}$.

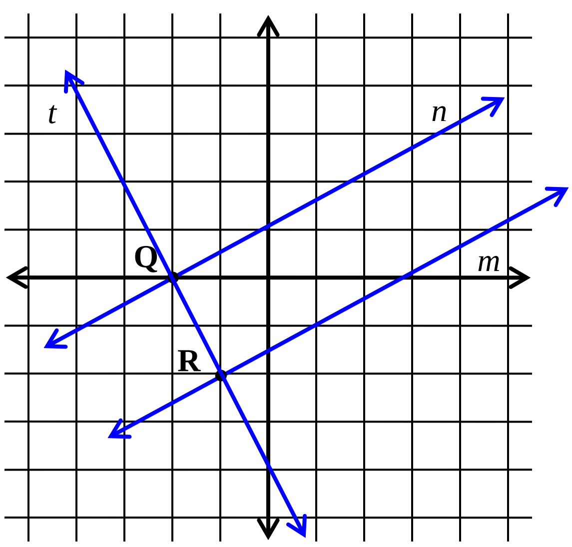
Example 1: State which line segment represents the distance between point J and $\overline{HL}$ and explain your reasoning.



Segment JK represents the distance between Point J and Segment HL because it starts at Point J and is perpendicular to Segment HL.

The distance between two parallel lines is the distance between one line and any point on the other line.

Example 2: Find the distance between the two parallel lines, *n* and *m*. Point Q (–2,0) is a point on line *n*, point R (–1, –2) is on line *m* and both points are on line *t*.



It appears that line *t* is perpendicular to lines *n* and *m*; but, first we must verify that this is true.

The slope of line *n* is $\frac{1}{2}$. (count rise / run $\rightarrow$ 1 unit up, 2 units over)

The slope of line *t* is $\frac{-2}{1}$. (count rise / run $\rightarrow$ 2 units down, 1 unit over)

Now, check the products of the slopes: $\frac{1}{2} \times -\frac{2}{1} = -1$

Thus, lines *n* and *t* are perpendicular.

Now that we know lines *n* and t are perpendicular, we may use the distance formula to find the distance betwe the point and the line. The distance between a point and a line is the length of a perpendicular line segment between them.

We will now find the distance between points Q (–2,0) and R (–1, –2).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$d = \sqrt{[-1-(-2)]^2 + (-2-0)^2}$$
$$d = \sqrt{(1)^2 + (-2)^2}$$
$$d = \sqrt{(5}$$

Graph Paper