PROPERTIES AND PROOFS OF SEGMENTS AND ANGLES

In this unit, you will extend your knowledge of a logical procedure for verifying geometric relationships. You will analyze conjectures and verify conclusions. You will use definitions, properties, postulates, and theorems to verify steps in proofs. The proofs in this unit will focus on segment and angle relationships.

Addition Properties

Two-column proof – A two column proof is an organized method that shows statements and reasons to support geometric statements about a theorem.

Theorem 7-A
Addition Property

If a segment is added to two congruent segments, then the sums are congruent.

Let's take a close look at the two-column proof of this theorem. In a two-column proof, both the "given" and "conclusion" are stated at the beginning, a diagram may be drawn as a visual aid, and then statements and their corresponding reasons are listed.

Given: $\overline{MP} \cong \overline{ST}$

Conclusion: $\overline{MS} \cong \overline{PT}$

Statement	Reason
1. $\overline{MP} \cong \overline{ST}$	1. Given
2. $MP = ST$	2. Definition of Congruence
3. $MP + PS = ST + PS$	3. Addition Property of Equality
4. $MP + PS = MS$; $ST + PS = PT$	4. Segment Addition (Postulate 3-B)
5. $MS = PT$	5. Substitution Property of Equality
6. $\overline{MS} \cong \overline{PT}$	6. Definition of Congruence (Remember: definitions are reversible)

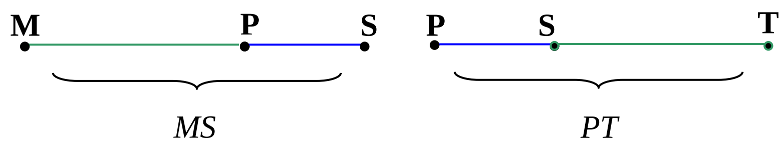
Let's examine each step of the proof closely.

Statement #1: The given information is shown.

Statement #2: This statement is used to show that congruent segments are equal in measure.

Statement #3: This statement applies the addition property of equality; PS is added to both sides of the equation.

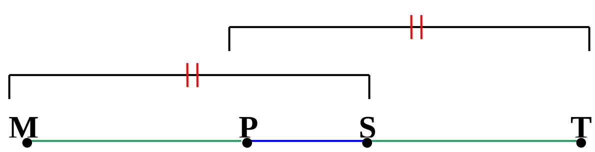
Statement #4: In an earlier unit, we examined segment addition (Postulate 3-B). When two segments share a common endpoint and are opposite each other, they may be combined as one segment.



Statement #5: The property of "substitution of equality" is used to replace the MP + PS with MS and PS + ST with PT in the previous step.

Statement #6: Based on the definition of congruence and that definitions are reversible, segments that have equal measures are congruent.

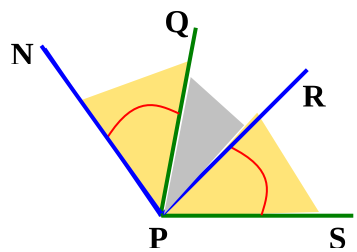
Theorem 7-A is illustrated below.



Now, let's take a look at some other theorems about the addition properties of segments and angles. The theorems are explained briefly with an illustration. Some of the proofs of the theorems will be developed in the exercises.

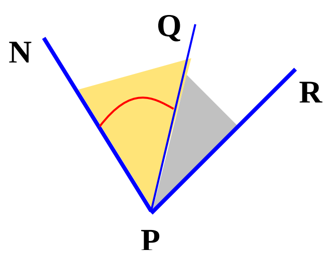
Theorem 7-B
Addition Property

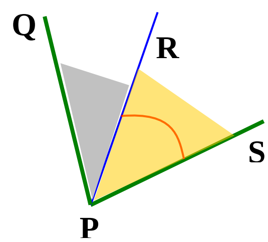
If an angle is added to two congruent angles, then the sums are congruent.



Given: $\angle NPQ \cong \angle RPS$

Conclusion: $\angle NPR \cong \angle QPS$





$$m\angle NPQ + m\angle QPR = m\angle QPR + m\angle RPS$$

$$m\angle NPR = m\angle QPS$$

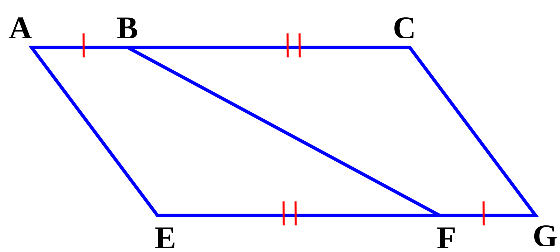
$$\therefore \angle NPR \cong \angle QPS$$

Theorem 7-C
Addition Property

If congruent segments are added to congruent segments, then the sums are congruent.

$$AB + BC = FG + EF$$

$$AC = EG$$



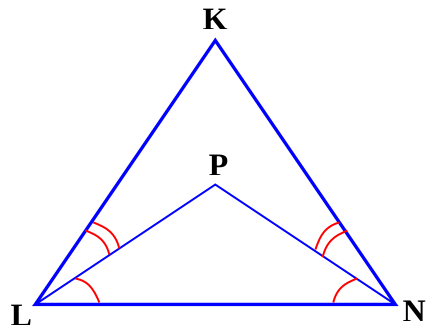
Given: $\overline{AB} \cong \overline{FG}; \overline{BC} \cong \overline{EF}$

Conclusion: $\overline{AC} \cong \overline{EG}$

Theorem 7-D
Addition Property

If congruent angles are added to congruent angles, then the sums are congruent.

$$m\angle KLP + m\angle PLN = m\angle KNP + m\angle PNL$$
$$m\angle KLN = m\angle KNL$$



Given: $\angle KLP \cong \angle KNP; \angle PLN \cong PNL$

Conclusion: $\angle KLN \cong \angle KNL$

Subtraction Properties

Now, let's take a look at some theorems about the subtraction properties of segments and angles. The theorems are explained briefly and may include an illustration. Some of the proofs of the theorems will be developed in the exercises.

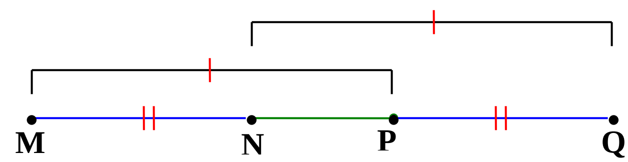
Theorem 7-E Subtraction Property

If a segment is subtracted from congruent segments, then the differences are congruent.

$$MP - NP = NQ - NP$$
$$MN = PQ$$

Given: $\overline{MP} \cong \overline{NQ}$

Conclusion: $\overline{MN} \cong \overline{PQ}$



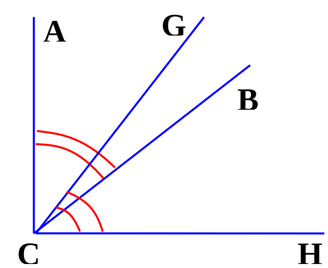
Theorem 7-F Subtraction Property

If an angle is subtracted from congruent angles, then the differences are congruent.

$$m\angle ACB - m\angle GCB = m\angle GCH - m\angle GCB$$
$$m\angle ACG = m\angle BCH$$

Given: $\angle ACB \cong \angle GCH$

Conclusion: $\angle ACG \cong \angle BCH$



Theorem 7-G Subtraction Property

If congruent segments are subtracted from congruent segments, then the differences are congruent.

Theorem 7-H Subtraction Property

If congruent angles are subtracted from congruent angles, then the differences are congruent.

Multiplication and Division Properties

Now, let's take a look at some theorems about the multiplication and division properties of segments and angles. The theorems are explained briefly and may include an illustration. Some of the proofs of the theorems will be developed in the exercises.

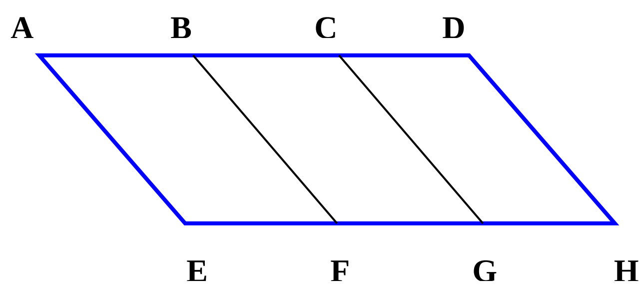
Bisect – Bisect is the division of a geometric shape into two equal parts.

Trisect – Trisect is the division of a geometric shape into three equal parts.

Theorem 7-I Multiplication Property

If segments are congruent, then their like multiples are congruent.

Given: $\overline{AB} \cong \overline{EF}$
Given: $\overline{BF}$ and $\overline{CG}$ trisect $\overline{AD}$ and $\overline{EH}$.
$\overline{AB}, \overline{BC}$, and $\overline{CD}$ are like multiples.
$\overline{EF}, \overline{FG}$, and $\overline{GH}$ are like multiples.
Conclusion: $\overline{AD} \cong \overline{EH}$



Theorem 7-J Multiplication Property

If angles are congruent, then their like multiples are congruent.

Theorem 7-K Division Property

If segments are congruent, then their like divisions are congruent.

Theorem 7-L Division Property

If angles are congruent, then their like divisions are congruent.

Proofs

Proofs are step by step reasons that can be used to analyze a conjecture and verify conclusions. In a formal proof, statements are made with reasons explaining the statements. You begin by stating all the information given, and then build the proof through steps that are supported with definitions, properties, postulates, and theorems.

Proof – A proof is a series of logical mathematical statements that are accepted as true.

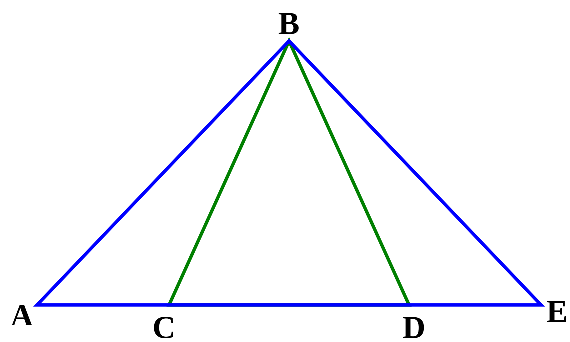
First, we will take a second look at Theorem 7-E to prove its validity. Each statement is supported by a definition or postulate that is presented in previous units. As theorems are presented, proved, and accepted as truthful statements, these theorems will be used as reasons to support geometric statements.

Theorem 7-E Subtraction Property	If a segment is subtracted from congruent segments, the differences are congruent.

Example 1:

Given: $\overline{AD} \cong \overline{CE}$

Prove: $AC \cong DE$



Statement	Reason
1. $\overline{AD} \cong \overline{CE}$	1. Given
2. $AD = CE$	2. Definition of Congruence
3. $AC + CD = AD$; $DE + CD = CE$	3. Segment Addition (Postulate 3-B)
4. $AC + CD = DE + CD$	4. Substitution Property of Equality
5. $AC = DE$	5. Subtraction Property of Equality
6. $\overline{AC} \cong \overline{DE}$	6. Definition of Congruence

Let's take a look at the explanation of each of the statements.

Statement #1: The given information is shown.

Statement #2: This statement is used to show that congruent segments are equal in measure.

Statement #3: In an earlier unit, we examined segment addition. When two segments share a common endpoint and are opposite each other, they may be combined as one segment.

Statement #4: The property of "substitution of equality" is used to replace the AD in statement #2 with AC + CD from statement #3. Substitution is also used to replace CE in statement #2 with DE + CD from statement #3.

Statement #5: Using the "subtraction property of equality", CD is subtracted from both sides of the equation.

Statement #6: Based on the definition of congruence and that definitions are reversible, segments that have equal measures are congruent.

Next we will take a second look at Theorem 7-B to prove its validity.

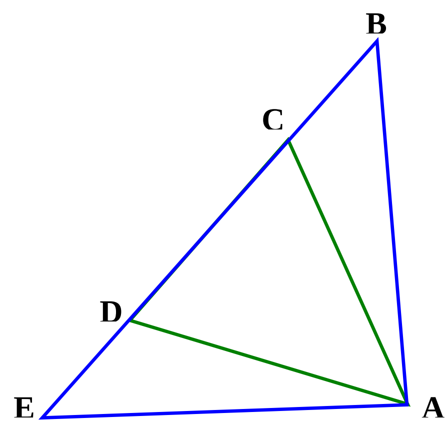
Theorem 7-B
Addition Property

If an angle is added to two congruent angles, then the sums are congruent.

Example 2:

Given: $\angle BAC \cong \angle EAD$

Prove: $\angle BAD \cong \angle EAC$



Statement	**Reason**
1. $\angle BAC \cong \angle EAD$	1. Given
2. $m\angle BAC = m\angle EAD$	2. Defintion of Congruence
3. $m\angle BAD = m\angle BAC + m\angle CAD$	3. Angle Addition (Postulate 2-B)
4. $m\angle BAD = m\angle EAD + m\angle CAD$	4. Substitution ($m\angle BAC = m\angle EAD$)
5. $m\angle EAD + m\angle CAD = m\angle EAC$	5. Angle Addition (Postulate 2-B)
6. $m\angle BAD = m\angle EAC$	6. Transition Property of Equality
7. $\angle BAD \cong \angle EAC$	7. Definition of Congruence

Let’s take a look at the explanation of each of the statements.

Statement #1: The given information is shown.

Statement #2: This statement is used to show that congruent angles are equal in measure.

Statement #3: In an earlier unit, we examined angle addition. When two angles share a common ray and they are non-overlapping angles, then they may be combined as one angle. Thus, measures of angles BAC and CAD may be combined to one angle, BAD.

Statement #4: The property of “substitution of equality” is used to replace $m\angle BAC$ in step #3 with $m\angle EAD$ from Step #2.

Statement #5: In this step, the “Angle Addition” postulate is applied to make one angle, $\angle EAC$, since the two non-overlapping angles share ray AD.

Statement #6: Since the measurement of angle BAD equals the sums of the measures of angles EAD and CAD, and this sum is equal to the measure of angle EAC, then the transitive property may be applied. Thus, the measurement of BAD equals the measurement of EAC. (If a = b and b = c, then a = c.)

Statement #7: Based on the definition of congruence and that definitions are reversible, angles that have equal measures are congruent.