UNCERTAINTY IN MEASUREMENTS

In this unit you will investigate the uncertainty in measurement and consider the difference between accuracy and precision. Since measurements are approximations, calculations with measurements have some degree of error; thus, you will determine absolute and relative error and examine the effects of errors on calculated measurements. You will also find the number of significant digits for calculated measures.

Measurement Uncertainty

Measurement Error

Greatest and Least Possible Error

Calculated Measurements and Significant Digits

Measurement Uncertainty

There is no such thing as a perfect measurement!

Each measurement contains a degree of uncertainty due to the limits of instruments and the people using them.

To compensation for the amount of error, consider the following:

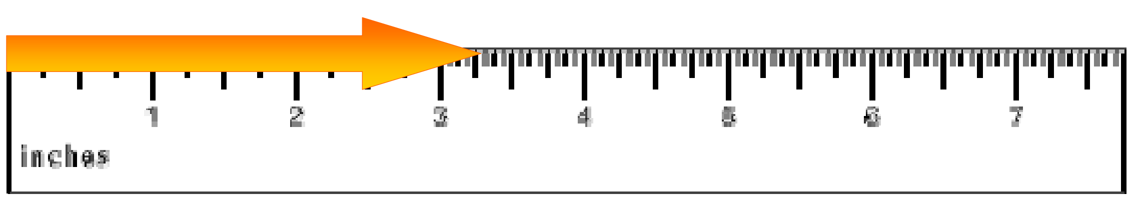
> *Example 1*: What method could be used to measure the length of a book in centimeters with certainty?
>
> Since the measurement desired is centimeters, we can be certain of the measurement if we measure to the nearest half centimeter. Thus, if the book's length is determined to be 18.5 centimeters, then we can be certain of the length of 18 centimeters knowing that that the measurement falls between 18 and 19 centimeters.
>
> Approaching measurement in this manner assures that the length of 18 centimeters in certain, and the final digit, 0.5 centimeters, is the estimate.
>
> *Example 2*: If a person's weight is given as 68.73 kilograms, what part of the measurement is certain and what part is an estimate?
>
> The person's weight is certain for 68.7 kilograms. The 0.03, the final digit, is the estimate.

uncertainty – The uncertainty of measurement is the doubt that exists about the result of any measurement. All measurements are approximations.

Consider this thought...

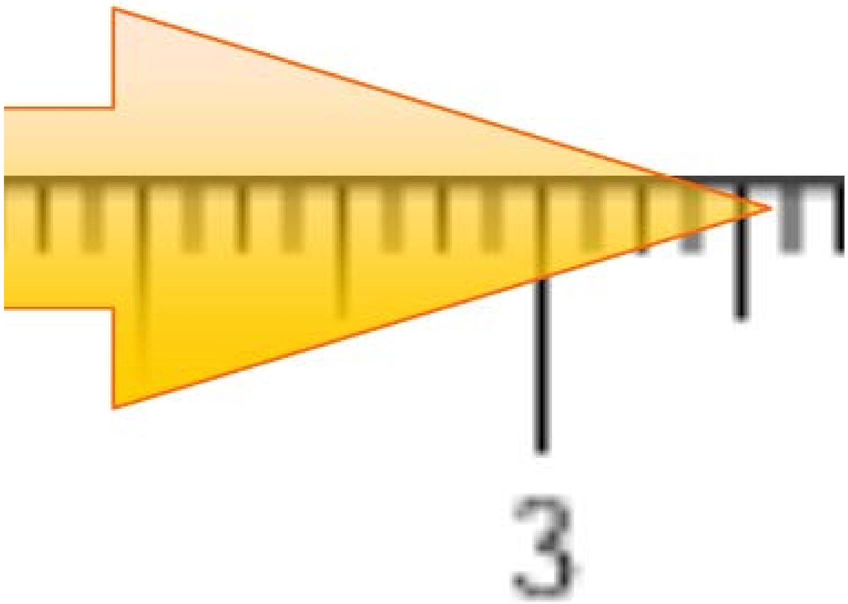
Whenever you are measuring with a ruler, if you are able to examine the item and the ruler close enough, the end of the item you are measuring does not fall exactly on the unit, but rather falls between two units.

The ruler below is a measuring tool with a precision to the nearest 16th inch. The arrow measures $3\frac{5}{16}$ inches.



Does the arrow really measure $3\frac{5}{16}$ inches?

If you zoom in 5 times larger, then you observe that the arrow doesn't quite fall on the $3\frac{5}{16}$ mark.



An estimate may be that the arrow measures $3\frac{9}{32}$ or $3\frac{19}{64}$ inches.

In some cases of measurement, you need to measure with a high degree of accuracy before the uncertainty is noticed. In other cases, the degree of accuracy is relative to the purpose of the measurement.

> *Example 3*: Compare the relevance of the accuracy of the measurement in the following two situations. *Measurement 1*: A mountain's height is listed in a brochure as 3560 feet high in comparison with other mountains in the area. *Measurement 2*: A teen is being measured to the nearest foot for a ride on the "Rocket Drop", a ride in the theme park that has a safety restriction that requires riders to be at least 5 ft 2 in tall. Explain how the measurement of the mountain to the nearest foot is an accurate measure where measuring the rider to the nearest foot could be problematic.
>
> If a mountain is measured at 3560 feet and listed in a brochure to compare its height with other mountains in the area, then its measure to the nearest foot is accurate enough for comparing the height of the mountains in the area.

However, if safety requirements only allow persons 5’ 2” or taller to ride the “Rocket Drop” at a theme park, then measuring the height of a person to the nearest foot would not be accurate enough. Heights of 4 1/2 ft (4 ft 6 in) to 5 1/2 ft (5 ft 6 in) would all round to 5 feet. The safety of the people whose height is actually lesson that 5ft 2 in would be at risk on the ride. Measuring to the nearest foot would not be acceptable. A better measurement would be measuring to the nearest inch.

Accuracy verses Precision

Two, *often misunderstood*, measurement terms are accuracy and precision.

accuracy - The accuracy of a measurement refers to how close the measured value is to the true or accepted value.

precision – The precision of a measurement refers to the degree of specified detail that can be observed.

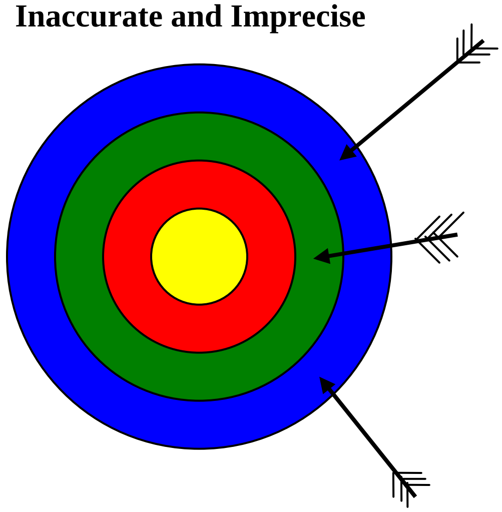
It is quite possible to be very precise and totally inaccurate! In many cases, when precision is high and accuracy is low, the fault may lie with the instrument that is used for the measurement.

What is the difference between accuracy and precision?

Let’s examine two well-know figures to help understand the difference between accuracy and precision.

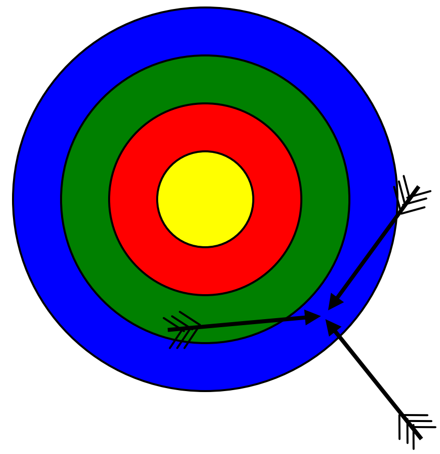
The model below distinguishes the difference between accuracy and precision. In the model, the bull's-eye (center, yellow area) represents the true value of a measurement. The objective is that the three arrows land in the bull’s eye area. There are four possibilities when using this model to represent the accuracy and precision of a measurement.

Inaccurate and Imprecise



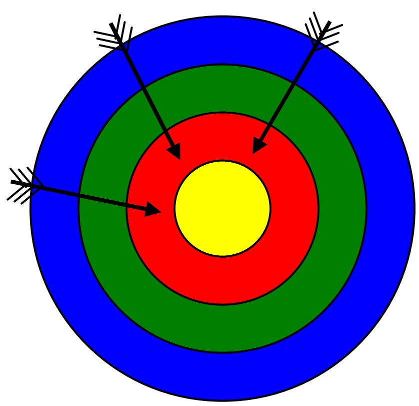
The three areas are not near each other, nor are they near the bull’s eye.

PRECISE but Inaccurate



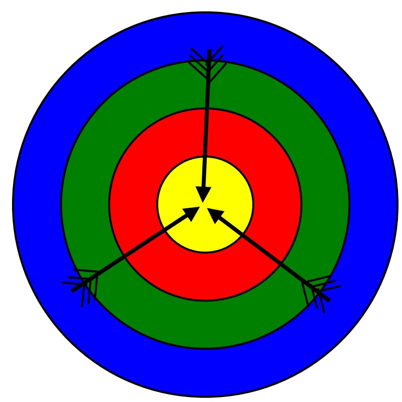
The three arrows are near each other; but, they are not near the bull’s eye.

ACCURATE but Imprecise



The three areas are near the bull’s eye; but, they are not near each other.

PRECISE and ACCURATE



The three arrows hit the bull’s eye every time.

Precision without accuracy is useless. **Accuracy** without precision tells us very little.

Example 4: Determine if the data is accurate and precise in the following scenario: A meteorologist predicted that the temperature for the day would be between 40 and 60 degrees Fahrenheit. The actual reading was 53 degrees.

Is the data accurate? Yes, because the meteorologist provide a true statement about the range of temperature.

Is the data precise? No, because the range of data did not provide enough detail. When planning events related to outdoor temperatures, the general public prefers a more detailed and precise prediction of the temperature.

Example 5: Determine if the data is accurate and precise in the following scenario: A meteorologist predicted that the temperature would be 58.48 degrees Fahrenheit at noon. The actual temperature was 72 degrees.

Is the data accurate? No, the temperature was much warmer than the meteorologist predicted.

Is the data precise? Yes, the meteorologist predicted the temperature to the nearest hundredth of a degree.

Measurement Error

Since measurements are approximations, calculations with measurements have some degree of error.

Absolute Error

The **absolute error** is the **absolute value** of the difference between the accepted value (observed measurement) and the greatest or least possible value of the measurement. This can be written as an equation as shown below.

Absolute Error = | Possible Value - Accepted Value |

$E_a = |P - A|$

E_a represents "absolute error"

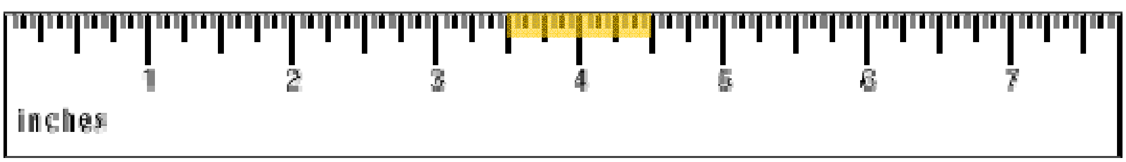
P represents "possible value"

A represents "accepted value"

Example 1: Explain the meaning of the following measurement: A length is measured to $4 \text{ in} \pm 0.5 \text{ in}$.

$$4 \text{ in} - 0.5 \text{ in} = 3.5 \text{ in}$$
$$4 \text{ in} + 0.5 \text{ in} = 4.5 \text{ in}$$

This measurement means that the least or greatest possible length could be anywhere between 3.5 inches and 4.5 inches.



Greatest Possible Value

Possible Value (greatest possible value) = 4.5 in

Accepted Value (observed measurement) = 4 in

$E_a = |P - A|$ $E_a = |4.5 - 4|$ Absolute Error = 0.5 in

*Note: The absolute value of a number is the number's positive value. In this case, the absolute value of 0.5 is 0.5 inches.

Least Possible Value

Possible Value (least possible value) = 3.5 in

Accepted Value (observed measurement) = 13 in

$E_a = |P - A|$ $E_a = |12.5 - 13|$ Absolute Error = 0.5 in

*Note: The absolute value of a number is the number's positive value. In this case, the absolute value of –0.5 is 0.5 inches.

The range of values that a measurement represents is **plus or minus half** the unit used to determine the measurement.

Example 2: Explain the meaning of the following measurement: A room measures 10 feet and is measured to the nearest foot.

Since the measure is to the nearest foot, then the range of values that the measurement represents is $\pm$ a half foot.

This measurement represents $9\frac{1}{2}$ feet to $10\frac{1}{2}$ feet.

Example 3: Explain the meaning of the following measurement: A board measures 8.2 meters and is measured to the nearest tenth of a meter.

Since the measurement is to the nearest tenth of a meter, then the range of values that the measurement represents is $\pm$ a half of a tenth of a meter.

$$\frac{1}{2} \text{ of } 0.1 \quad = \quad \frac{1}{2} \times 0.10 \quad = \quad \frac{0.10}{2} \quad = \quad 0.05$$

To find the range of the values represented by the measurement:

(a) Subtract 0.05 from 8.2

$$8.2 - 0.05 \quad = \quad 8.20 - 0.05 \quad = \quad 8.15 \text{ m}$$

(b) Add 0.05 to 8.2

$$8.2 + 0.05 \quad = \quad 8.20 + 0.05 \quad = \quad 8.25 \text{ m}$$

If a board's measurement is 8.2 meters and has been measured to the nearest tenth of a meter, then the range of values that the measurement represents is 8.15 meters to 8.25 meters.

The range of values represented by 8.2 meters is 8.15 m to 8.25 m or 8.1 ± 0.05 m.

Example 4: What is the range of values represented by a measurement of 75 millimeters measured to the nearest millimeter?

Since the measurement was made to the nearest whole millimeter, the range would be within 0.5 millimeters from the measured unit; 75 mm $\pm$ 0.5 mm.

Thus, this measurement represents 74.5 mm to 75.5 mm.

Example 5: What is the range of values represented by a measurement of 32.4 kilometers measured to the nearest tenth of a meter?

Since the measurement is to the nearest tenth of a kilometer, then the range of values that the measurement represents is $\pm$ a half of a tenth of a kilometer.

$$\frac{1}{2} \text{ of } 0.1 \quad = \quad \frac{1}{2} \times 0.10 \quad = \quad \frac{0.10}{2} \quad = \quad 0.05$$

To find the range of the values represented by the measurement:

(a) Subtract 0.05 from 32.4.

$$32.4 - 0.05 \quad = \quad 32.40 - 0.05 \quad = \quad 32.35 \text{ km}$$

(b) Add 0.05 to 32.4.

$$32.4 + 0.05 \quad = \quad 32.40 + 0.05 \quad = \quad 32.45 \text{ km}$$

Since the measurement was made to the nearest tenth of a kilometer, the range would be within 0.05 kilometers from the measured unit; 32.4 km $\pm$ 0.05 km.

Thus, this measurement represents 32.35 km to 32.45 km.

Relative Error

Relative Error - Relative error is the ratio of the absolute error and the accepted value (observed measurement).

$$\text{Relative Error} = \frac{\text{Absolute Error}}{\text{Accepted Value}}$$

$$E_r = \frac{E_a}{A}$$

E_r represents "relative error"
E_a represents "absolute error"
A represents "accepted value"

The formula for relative error may also be written as:

$$E_r = \frac{|P - A|}{A}$$ Substitution $(E_a = |P - A|)$

Example 6: Calculate the relative error for Example 4; that is, calculate the relative error for the measurement of 75 millimeters measured to the nearest millimeter.

$$E_r = \frac{E_a}{A}$$ Formula for Relative Error

$$E_r = \frac{0.5}{75}$$ Substitution $(E_a = 0.5, \ A = 75)$

$$E_r = 0.006...$$ Simplify

Relative Percentage Error

Relative Percentage Error – The relative percentage error is the relative error expressed as a percent.

$$\text{Relative Percentage Error} = E_r \times 100$$

Example 7: Calculate the percentage error for Example 6 to the nearest hundredth of a percent; that is, calculate the percentage error for the measurement of 75 millimeters measured to the nearest millimeter.

$\text{Relative Percentage Error} = E_r \times 100$	Formula for Percentage Error
$\text{Relative Percentage Error} = 0.006... \times 100$	Substitution ($E_r = 0.006...$)
$\text{Relative Percentage Error} = 0.67\%$	Simplify and round to nearest hundredth of a percent.

Greatest and Least Possible Error

When **adding** or **multiplying** measurements, the **absolute error** is found by using the **greatest possible error**.

Greatest Possible Error (GPE) - The greatest possible error of a measurement is one half of the unit of measure to which the measure is being rounded.

Addition

Example 1: Find the **absolute error** for the following addition problem.

$6 \text{ cm} \pm 0.5 \text{ cm} \quad + \quad 12 \text{ cm} \pm 0.5 \text{ cm} \quad + \quad 10 \text{ cm} \pm 0.5 \text{ cm}.$

Step 1: Ignore the error and add the measurements to find the accepted value.

The total is 6 + 12 + 10 = 28 cm.

Step 2: Now add the measurements using the greatest possible errors for each measurement.

GPE: $6 + 0.5 = 6.5 \quad 12 + 0.5 = 12.5 \quad 10 + 0.5 = 10.5$

The total is 6.5 + 12.5 + 10.5 = 29.5 cm

Step 3: Find the amount of error.

The difference is 29.5 – 28 = 1.5 cm.

The absolute error is 1.5 cm, the sum of the individual errors.

Multiplication

Example 2: Find the **absolute error** for the following multiplication problem.

$15 \text{ km} \pm 0.5 \text{ km} \quad \times \quad 12 \text{ km} \pm 0.5 \text{ km}$

Step 1: Ignore the error and multiply the measurements to find the accepted value.

$15 \text{ km} \quad \times \quad 12 \text{ km} \quad = \quad 180 \text{ km}^2$

Step 2: Now multiply the measurements using the greatest possible errors for each measurement.

GPE: $15 + 0.5 = 15.5 \qquad 12 + 0.5 = 12.5$

The product is $15.5 \times 12.5 = 193.75$ sq km

Step 3: Find the amount of error using GPE.

The difference is 193.75 – 180 or 13.75 sq km.

For this example, we will check the amount of error using the least possible errors for each measurement and compare with the greatest possible error to see which is greater.

Step 4: Multiply the measurements using the **least possible errors** for each measurement.

LPE: $15 - 0.5 = 14.5 \qquad 12 - 0.5 = 11.5$

The product is $14.5 \times 11.5 = 166.75$ sq km

Step 5: Find the amount of error determined using LPE.

The difference is 180 – 166.75 = 13.25 sq km.

To summarize the results, we found that the greatest possible error is 13.75 while the least possible error is 13.25. We will use the greater error to determine the absolute error. Thus the absoluter error is 13.75 square kilometers.

Calculating Absolute Error

(1) When finding the absolute error of a **sum** or **product**, work with the largest measurements,(i.e. the plus (+) errors.)
(2) When finding the absolute error of a **difference** or **quotient**, work with the smallest measurements (i.e. the minus (–) errors.)

Let's examine the following scenario to apply the ideas of absolute and relative errors.

Example 3: The measurements of a rectangular field are 400 meters by 300 meters. The measurements were determined with a trundle wheel which has a relative error of 0.03. Answer the following questions about the possible errors associated with measuring the field with the trundle wheel.

300 m

400 m

a) Calculate the absolute error of each of the measurements.

Since we are given the relative error of the trundle wheel (0.03), and the accepted measurement of the filed, we can determine the absolute error of each dimension of the field using the following formula: $E_r = \frac{E_a}{A}$.

Field Length

$E_r = \frac{E_a}{A}$	Relative Error = $\frac{\text{Absolute Error}}{\text{Accepted value}}$
$0.03 = \frac{E_a}{400}$	Substitution ($E_r = 0.03$, Accepted Length = 400 m)
$E_a = 12$	Cross Multiply

The absolute error of the measurement of the field's length is 12 meters.

That is, the field's length could be 400 m $\pm$ 12 m.

The field's length could measure from 388 meters to 412 meters.

Field Width

$E_r = \frac{E_a}{A}$	Relative Error = $\frac{\text{Absolute Error}}{\text{Accepted value}}$
$0.03 = \frac{E_a}{300}$	Substitution ($E_r = 0.03$, Accepted Width = 300 m)
$E_a = 9$	Cross Multiply

The absolute error of the measurement of the field's width is 9 meters.

That is, the field's width could be 300 m $\pm$ 9 m.

The field's width could measure from 291 meters to 309 meters.

b) Calculate the absolute error in the area of the field.

Area of the Field Using the Accepted Values

$A = l\ w$ Formula for Area of a Rectangle
$A = (400)(300)$ Substitution
$A = 120,000$ Simplify

The accepted area of the field is 120,000 square meters.

Area of the Field Using the Greatest Possible Error (GPE)

$A = l\ w$ Formula for Area of a Rectangle
$A = (412)(309)$ Substitution
(Length with GPE added is 412 meters.)
(Width with GPE added is 309 meters.)
$A = 127,308$ Simplify

The greatest possible area of the field is 127,308 square meters.

Absolute Error of the Area of the Field

$E_a = |P - A|$ Formula for Absolute Error
$E_a = |127,308 - 120,000|$ Substitution
$E_a = 7,308$ Simplify

The absolute error of the area of the field is 7,308 square meters.

c) Calculate the relative error of the area of the field.

$E_r = \frac{E_a}{A}$ Formula for Relative Error
$E_r = \frac{7,308}{120,000}$ Substitution: $E_a = 7,308$, Accepted Area $= 120,000$
$E_r = 0.0609$ Simplify

The relative error of the field's area is 0.0609.

d) Calculate the relative percentage error of the area of the field.

Relative Percentage Error $= E_r \times 100$	Formula for Percentage Error
Relative Percentage Error $= 0.0609 \times 100$	Substitution ($E_r = 0.0609$)
Relative Percentage Error $= 6.09\%$	Simplify

The relative percentage error of the field's area is 6.09%.

Example 2: To gain a better perspective on the kind of error determined in the previous problem, (convert the calculated absolute error of the area of the field from square meters to acres (1 square meter = 0.000247 acres), give an example of how the error may be significant, (state a way to correct the inaccuracy of the measurement.

(a) Multiply the absolute error of the area of the field by the conversion factor and to the nearest hundredth of an acre.

$$7,308 \times 0.000247 = 1.81$$

There was a possible error of 1.81 acres.

(b) Calculate the accepted area of the field in acres.

$$120,000 \times 0.000247 = 29.64$$

The accepted area of the filed was 29.64 acres.

The error would be significant if a developer purchased the land to divide it into 2 acre lots for resale. The possible error could result in a waste of almost 2 acres of land or one lot. He may want to reconsider how he divides the land to get the maximum value for his property.

To correct the error, a laser or surveyor's instrument, with better precision and less relative error, could be used to measure the land's dimensions.

Calculated Measurements and Significant Digits

significant digits – Significant digits are the digits used to record a measurement. The result of a calculated measurement should have the same precision as the least precise measurement. The significant digits in a number may be determined by counting digits from left to right starting with the first nonzero digit.

Rules for Counting Significant Digits

(1) All nonzero numbers are significant.

7.822 has **4 significant digits**	All four digits are nonzero.

(2) Leading zeros, or zeros used to show place value, are NOT significant.

0.0034 has **2 significant digits**	The three zeros are leading zeros used as place holders and not significant.

(3) Zeros that fall between nonzero digits are significant.

607.09 has **5 significant digits**	The two zeros fall between nonzero digits.

(4) In whole numbers or integers (numbers without decimal points, any zeros to the right of nonzero digits are NOT significant

53,000 has **2 significant digits**	The three zeros are to the right of nonzero digits in the whole number.

(5) In a decimal number, all zeros to the right of nonzero digits are significant.

350.000 has **6 significant digits**	The two nonzero digits plus the four zeros to the right of the nonzero digits in the decimal number are significant.

Example 1: Determine the number of significant digits in the number, 0.00902.

The decimal number, 0.00902, has 3 significant digits.

By Rule #2 above, the leading zeros are place holders and thus, not significant.

By Rule #3 above, the zero that falls between the two nonzero digits is significant.

When adding, subtracting, multiplying, or dividing measurements, the results should have the same precision as the least precise measurement determined by the number of significant digits.

Example 2: Find the area of a triangle with a height of 4.01 centimeters and a base of 15.24 centimeters.

$A = \frac{1}{2}bh$	Formula for Area of a Triangle
$A = \frac{1}{2}(15.24)(4.01)$	Substitution $(h = 4.01,\ b = 15.24)$
$A = 30.5562$	Simplify

15.24 has 4 significant digits. (All digits are nonzero.)

4.01 has 3 significant digits. (The zero occurs between two nonzero digts and is a significant digit.)

$\therefore A = 30.6$	Round to three significant digits, the number of digits in the least precise measurement.

The area of the triangle is approximately 30.6 square centimeters.