COORDINATE PROOFS AND TRANSFORMATIONS

In this unit you will examine proofs of geometry theorems through coordinate geometry. You will then explore mappings of pre-images onto images. You will use the coordinate plane to define transformations. There are four basic types of transformations: reflections, translations, rotations, and dilations.

Coordinate Proofs

Coordinate proofs may be used to prove theorems.

coordinate proof – A coordinate proof is a proof in geometry that is achieved by assigning coordinates to figures, and then using these points and figures to prove the theorems.

First we'll examine some ways to represent figures in the coordinate plane using general terms.

Tips for placing figures on a coordinate plane:

1) Use the origin as a vertex or center.

2) Place at least one of the sides of the polygon on the *x*-axis.

3) Keep as much of the figure as possible in the first quadrant.

4) Keep coordinates as simple as possible.

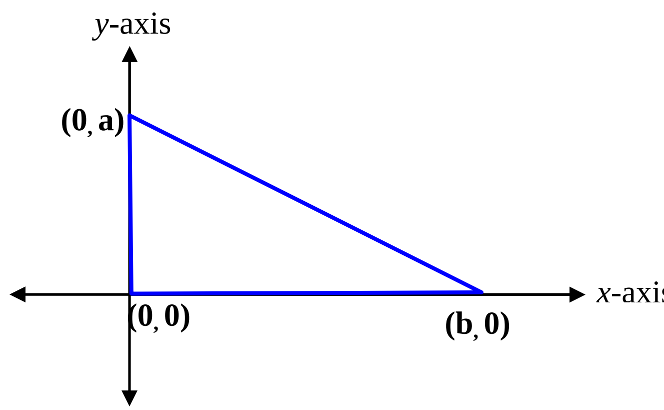
Example 1: Position and label a right triangle so that one side lies on the *x*-axis and the second side lies on the *y*-axis.

Follow the tips above when drawing the figure.

(1)Use the origin as a vertex or center. In the figure on the right, the right angle of the triangle is placed at the origin.

(*2) Place a least one of the sides of the polygon on the x-axis.* One side is placed on the *x*-axis and the other on the *y*-axis.

(3) Keep as much of the figure as possible in the first quadrant. The entire figure lies in the first quadrant.



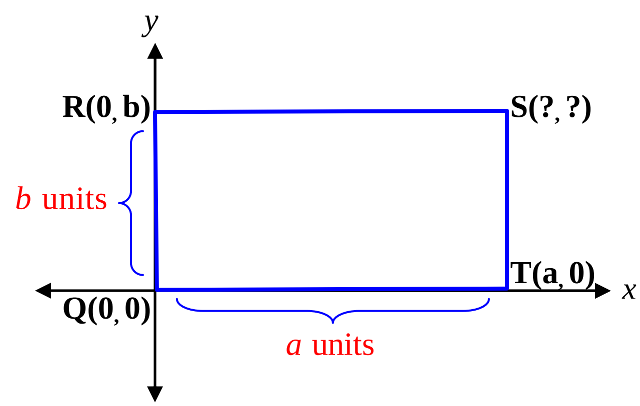
(4) Keep coordinates as simple as possible. The vertices are written as a combination of zero and the letters *a* and *b*.

Example 2: For the figure shown below, state the ordered pair for point S in rectangle QRST.

Properties of a Rectangle: The opposite sides of a rectangle are congruent and the four angles in a rectangle are right angles.

Observation 1: The *x*-coordinate of point S is "a" since QT is the same length as RS and ST is perpendicular to QT.

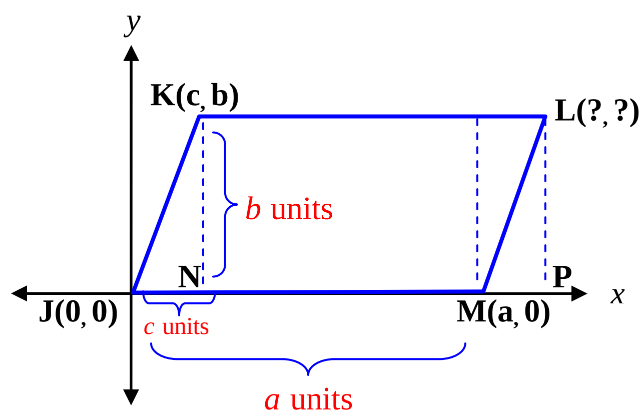
Observation 2: The *y*-coordinate of point S is "b" since QR is the same length as TS.



The ordered pair for point S is (*a*, *b*).

Example 3: State the ordered pair for point L in parallelogram JKLM in terms of *a*, *b*, *c*, or any combination that may be justified through the properties of a parallelogram and any addition proofs.

Properties of a Parallelogram: The opposite sides of a parallelogram are congruent. The altitude of a parallelogram is a perpendicular line between the opposite sides.



Observation 1: The *x*-coordinate of point L is equal to the length of JM + MP along the *x*-axis. Thus, *x*-coordinate (the length of JP) is "a + c".

- The length of JM is "a" because point M is plotted on the *x*-axis at (a,0).

- The length of MP is "c".

- First, note that the length of JN is "c" because point K is plotted at (c,b) and point K lies on a line that is perpendicular to JN.

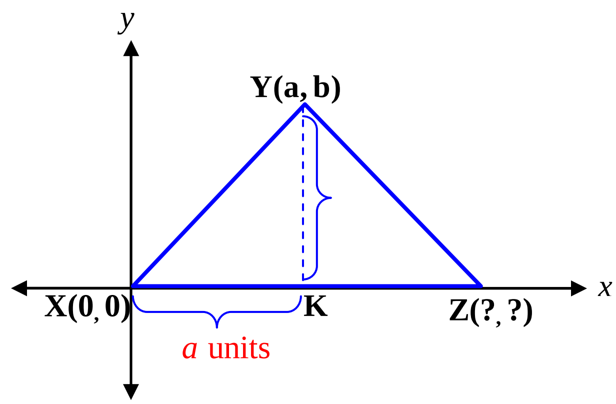
- Second, note that the length of JN and MP are equal, based on CPCTC. (Triangles JKN and MLP are congruent; see paragraph proof below.)

Paragraph Proof: When the altitudes of the parallelogram are drawn from points K to N and points L to M, right triangles JKN and MLP are formed. Since the hypotenuses JK and LM are congruent and the triangles JKN and MLP are right triangles, then right triangles JKN and MLP are congruent by the Hy-Leg Postulate.

Observation 2: The *y*-coordinate of point L is "b". Since NK and LP are altitudes of the parallelogram, they are the same length.

The ordered pair for point L is (*a* + *c*, *b*).

Example 4: What is the ordered pair for point Z in isosceles triangle XYZ?



Properties of an isosceles triangle: Two sides of the triangle are congruent. The altitude of the triangle divides the triangle into two congruent triangles. ((Draw an altitude from vertex Y to segment XZ. Apply the Hy-Leg postulate.)

Observation 1: The *x*-coordinate of point Z is equal to the length of XK + KZ along the *x*-axis. Since XK and KZ are congruent (CPCTC), then the entire length of XZ is twice the length of either XK or KZ. Also, since point Y is on a line perpendicular to XZ, XK = *a* units. Thus, the *x*-coordinate of point Z (the length of XZ) is "2a".

Observation 2: The *y*-coordinate of point Z is zero since point Z lies on the *x*-axis.

Thus, the ordered pair for point Z is (2a, 0).

Example 5: Write a coordinate proof to show that the segments connecting the midpoints of any quadrilateral form a parallelogram.

*Note: After drawing the quadrilateral in the coordinate plane, assign points to twice each variable. This will simplify the proof since calculating midpoints involves dividing by 2.

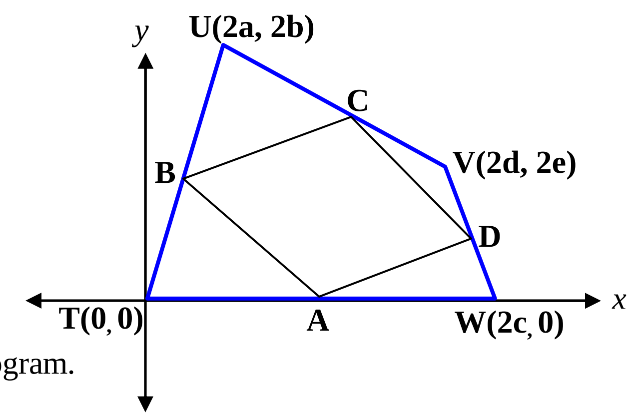
Given: Quadrilateral TUVW

A is the midpoint of $\overline{TW}$.

B is the midpoint of $\overline{TU}$.

C is the midpoint of $\overline{UV}$.

D is the midpoint of $\overline{VW}$.



Prove: Quadrilateral ABCD is a parallelogram.

To prove, make a plan.

First, find the ordered pairs for the midpoints.

Then, recall that a parallelogram is a quadrilateral with both pairs of opposite sides parallel.

- Find the slopes of BC and AD to determine if the segments are parallel.

- Find the slopes of AB and CD to determine if the segments are parallel.

Step 1: Find the ordered pairs for the midpoints.

Midpoint Formula: $(\frac{x_1 + x_2}{2}, \ \frac{y_1 + y_2}{2})$

Midpoint of A: $(\frac{2c+0}{2}, \ \frac{0+0}{2}) = (\frac{2c}{2}, \ \frac{0}{2}) = (c, 0)$

Midpoint of B: $(\frac{2a+0}{2}, \ \frac{2b+0}{2}) = (\frac{2a}{2}, \ \frac{2b}{2}) = (a, b)$

Midpoint of C: $(\frac{2a+2d}{2}, \ \frac{2b+2e}{2}) = (\frac{2(a+d)}{2}, \ \frac{2(b+e)}{2}) = (a+d, b+e)$

Midpoint of D: $(\frac{2c+2d}{2}, \ \frac{0+2e}{2}) = (\frac{2(c+d)}{2}, \ \frac{2e}{2}) = (c+d, e)$

Step 2: Find the slopes of the opposite sides of quadrilateral ABCD to determine if they are parallel.

Slope (m) is $\dfrac{y_2 - y_1}{x_2 - x_1}$

Are opposite sides BC and AD parallel?

Segment BC, B(a,b), C$(a+d, b+e)$

Slope of BC $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{(b+e)-b}{(a+d)-a} = \dfrac{e}{d}$

Segment AD, A$(c,0)$, D$(c+d, e)$

Slope of AD $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{e-0}{(c+d)-c} = \dfrac{e}{d}$

Since the slopes of BC and AD are equal, the line segments are parallel.

Are opposite sides AB and CD parallel?

Segment AB, A$(c,0)$, B(a,b)

Slope of AB $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{b-0}{a-c} = \dfrac{b}{a-c}$

Segment CD, C$(a+d, b+e)$, D$(c+d, e)$

Slope of CD $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{e-(b+e)}{(c+d)-(a+d)} = \dfrac{e-b-e}{c+d-a-d} = \dfrac{-b}{c-a} = \dfrac{(-1)(b)}{(-1)(-c+a)} = \dfrac{b}{-c+a} = \dfrac{b}{a-c}$

Since the slopes of AB and CD are equal, the line segments are parallel.

Therefore, quadrilateral ABCD is a parallelogram because both pairs of opposite sides are parallel.

Mapping Figures

mappings – Mappings are one-to-one correspondences between the vertices and the points that make up an image.

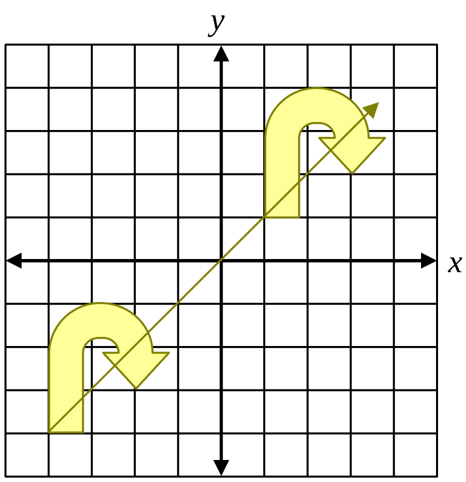
pre-image – A pre-image is the original image.

image – An image is a mapping of a pre-image.

transformation – A transformation is a mapping in which each pre-image point has exactly one image point, and vice versa.

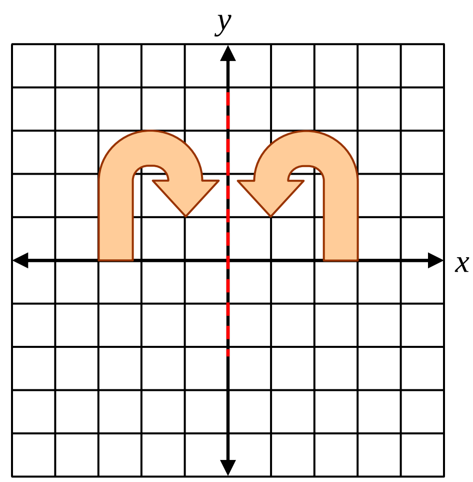
There are four types of transformations:

(1) **translation** – (slide or glide)



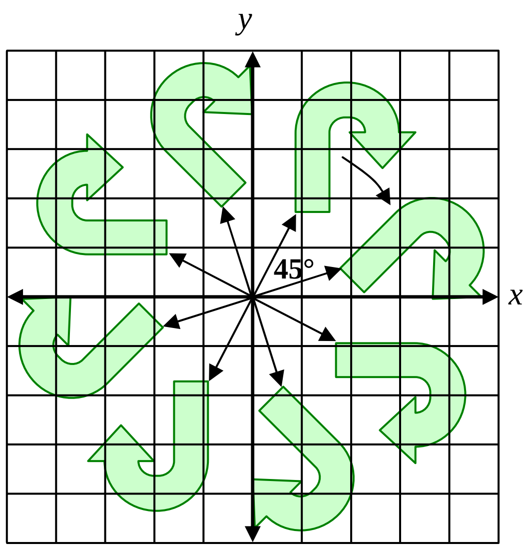
Translation - sliding a figure along a straight line without turning to another location.

(2) **reflection** – (flip)

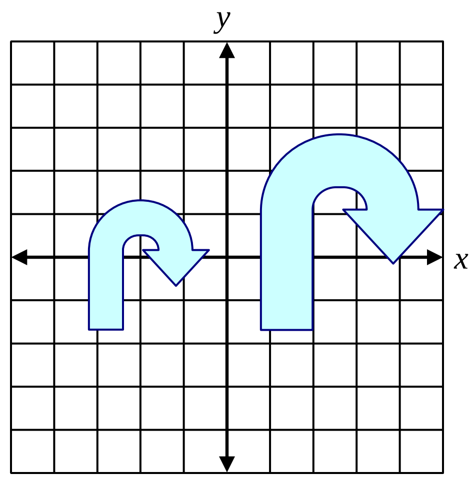


Reflection - flipping a figure over a line of reference creating a mirror image.

(3) **rotation** – (turn)



(4) **dilation** – (reduce or enlarge)



Rotation - turning a figure around a fixed point called the center of rotation.

The **angle of rotation** is formed by rays drawn from the center of rotation through corresponding points on a pre-image and its image.

This picture has been rotated 45° each time around the center of rotation, (0,0).

Dilation - enlarging or reducing a figure.

In geometry a mapping is denoted by the symbol, $\rightarrow$. An example of the use of this symbol follows:

▱DEFG $\rightarrow$ ▱JKLM is read "pre-image parallelogram DEFG maps onto image parallelogram JKLM".

▱DEFG $\rightarrow$ ▱JKLM means that every point in parallelogram DEFG maps onto exactly one point in parallelogram JKLM.

The order of the letters indicates the correspondence of the points in the pre-image to the image.

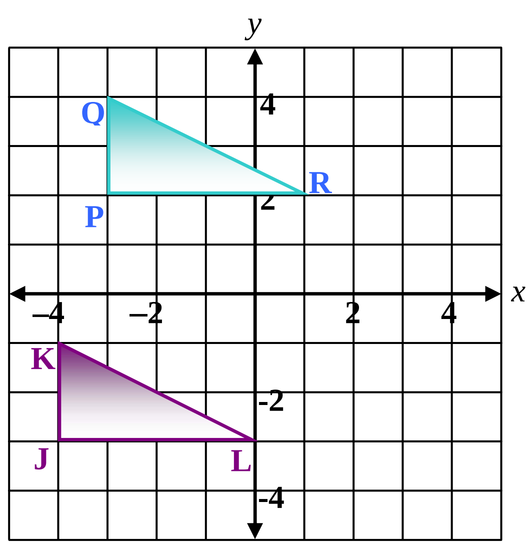
Points D and J are corresponding vertices.

Points E and K are corresponding vertices.

Points F and L are corresponding vertices.

Points G and M are corresponding vertices.

Example 1: For the right triangles shown below, (a) write a symbolic statement about their mappings, (b) write a statement showing how to read the symbolic statement, and (c) explain its meaning. Triangle JKL is the pre-image.



(a) *a symbolic statement*: $\triangle$JKL $\rightarrow$ $\triangle$PQR

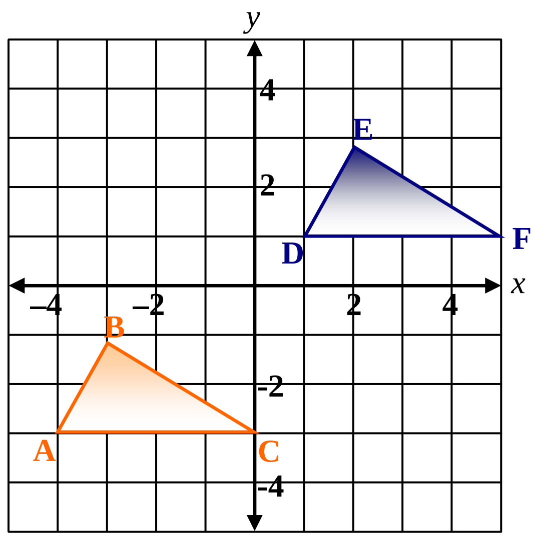
(b) *read the symbolic statement*: " Pre-image triangle JKL maps onto image triangle PQR.

(c) *the meaning of the statement*: Every point in triangle JKL maps onto exactly one point in triangle PQR. Point J maps onto point P, point K maps onto point Q, point L maps onto point R, and each of the other points on triangle JKL have exactly one corresponding point on triangle PQR.

Isometry and Similarity Transformations

isometry (congruence transformation) – Isometry, in a plane, is a transformation that maps every segment to a congruent segment.

Example 1: Show that $\triangle DEF \rightarrow \triangle ABC$ is an isometry.



Let's take a look at the distance formula and the SSS postulate to demonstrate that the triangles are an isometry; that is, a congruent transformation.

Preimage $\triangle$DEF: D(1, 1) E(2, 3) F(5,1)

Image $\triangle$ABC: A(–4,–3) B(–3,–1)

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Are the lengths of DE and AB equal?

$$DE = \sqrt{(1-2)^2 + (1-3)^2} = \sqrt{1+4} = \sqrt{5}$$
$$AB = \sqrt{(-4-(-3))^2 + (-3-(-1))^2} = \sqrt{1+4} = \sqrt{5}$$

Corresponding sides DE and AB are congruent.

Are the lengths of EF and BC equal?

$$EF = \sqrt{(2-5)^2+(3-1)^2} = \sqrt{9+4} = \sqrt{13}$$
$$BC = \sqrt{(-3-0)^2+(-1-(-3))^2} = \sqrt{9+4} = \sqrt{13}$$

Corresponding sides EF and BC are congruent.

Are the lengths of DF and AC equal?

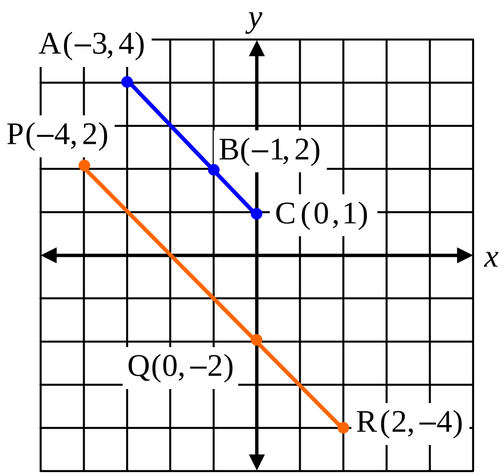
$$DF = \sqrt{(1-5)^2+(1-1)^2} = \sqrt{16+0} = 4$$
$$AC = \sqrt{(-4-0)^2+(-3-(-3))^2} = \sqrt{16+0} = 4$$

Corresponding sides DF and AC are congruent.

All sides are congruent (SSS), thus $\triangle DEF \cong \triangle ABC$.
Therefore, $\triangle DEF \rightarrow \triangle ABC$ is an isometry (congruence transformation).

similarity transformation – A similarity transformation is a mapping in which the pre-image and the image are similar.

Example 2: Show that $\overline{PR} \rightarrow \overline{AC}$ is a similarity transformation.



Plan: We must show that segments PR and AC are proportional in every respect.

(a) Determine the lengths of each segment using the distance formula.

(b) Show that each of the corresponding segments are proportional.

What are the lengths of each part of the segments?

Segment AC

$$AB = \sqrt{[-3-(-1)]^2 + (4-2)^2} \qquad = \sqrt{8} \qquad = 2\sqrt{2}$$

$$BC = \sqrt{(-1-0)^2 + (2-1)^2} \qquad = \sqrt{2}$$

$$AC = \sqrt{(-3-0)^2 + (4-1)^2} \qquad = \sqrt{18} \qquad = 3\sqrt{2}$$

Segment PR

$$PQ = \sqrt{(-4-0)^2 + [2-(-2)]^2} \qquad = \sqrt{32} \qquad = 4\sqrt{2}$$

$$QR = \sqrt{(0-2)^2 + [-2-(-4)]^2} \qquad = \sqrt{8} \qquad = 2\sqrt{2}$$

$$PR = \sqrt{(-4-2)^2 + [2-(-4)]^2} \qquad = \sqrt{72} \qquad = 6\sqrt{2}$$

Are all the corresponding segments proportional?

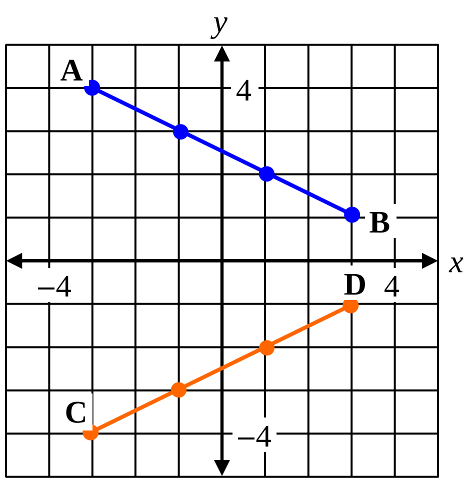
Check:

AB : PQ	BC : QR	AC : PR
$\frac{AB}{PQ}=\frac{2\sqrt{2}}{4\sqrt{2}}=\frac{1}{2}$	$\frac{BC}{QR}=\frac{\sqrt{2}}{2\sqrt{2}}=\frac{1}{2}$	$\frac{AC}{PR}=\frac{3\sqrt{2}}{6\sqrt{2}}=\frac{1}{2}$

Since all of the ratios reduce to 1/2, the segments are proportional.

Thus, $\overline{PR} \to \overline{AC}$ is a similarity transformation.

Example 3: The figure below shows a reflection over the *x*-axis. Create a table of values that represent points on segment AB and segment CD. Look for a pattern in the numbers. Write a rule that would pertain to all reflections over the *x*-axis.



Segment AB	
x	*y*
–3	4
–1	3
1	2
3	1

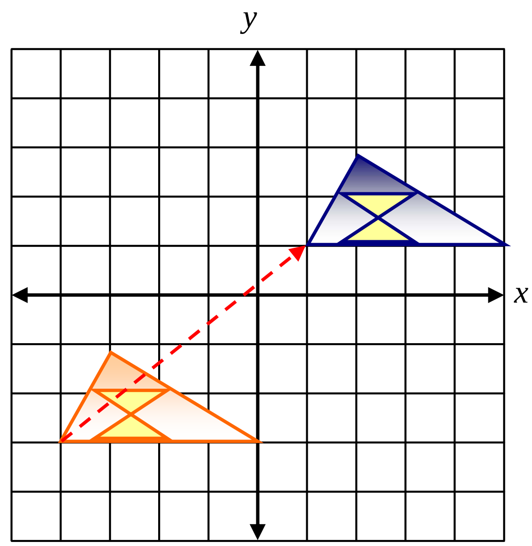
Segment CD	
x	*y*
–3	–4
–1	–3
1	–2
3	–1

For images reflected over the *x*-axis, if the pre-image coordinates are designated as (x, y), then the image's *x*-coordinate is *x*, and the *y*-coordinate is $-y$. The image of (x, y) is $(x, -y)$.

Translations

There are four types of transformations of geometric figures in the coordinate plane. They are translations, reflections, rotations, and dilations. Translations are presented in this section.

A **translation** is a transformation in which each point of the figure moves the same distance in the same direction. A figure and its image are congruent.



A translation slides a figure along a line without turning.

Example 1: Describe the translation in words.

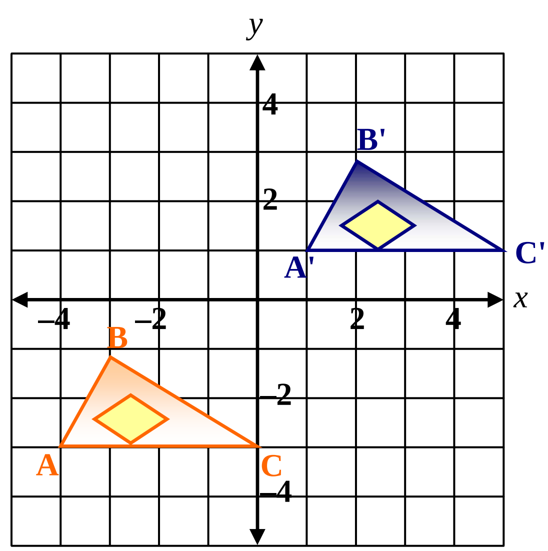


Figure ABC is translated to figure A'B'C'. (A'B'C' is read "A-primed, B-primed, C-primed".) The translation is 5 units to the right, 4 units up.

To check, start at point A and count 5 units to the right and 4 units up. You will end up at point A'. Do the same for the other two pairs of corresponding points.

Notation in the Coordinate Plane

In general, if **(a, b)** describes the translation horizontally "a" units and vertically "b" units, then the image of **(x, y)** is **(x + a, y + b)**.

You can describe a translation of each point (x, y) of a figure using coordinate notation.

Translation: $(x,\ y) \rightarrow (x+a,\ y+b)$

a = how many units a point moves horizontally

b = how many units a point moves vertically

If $a > 0$, the point moves to the right.

If $a < 0$, the point moves to the left.

If $b > 0$, the point moves up.

If $b < 0$, the point moves down.

Let's revisit the translation in Example 1 and write the notation for the translation in the coordinate plane.

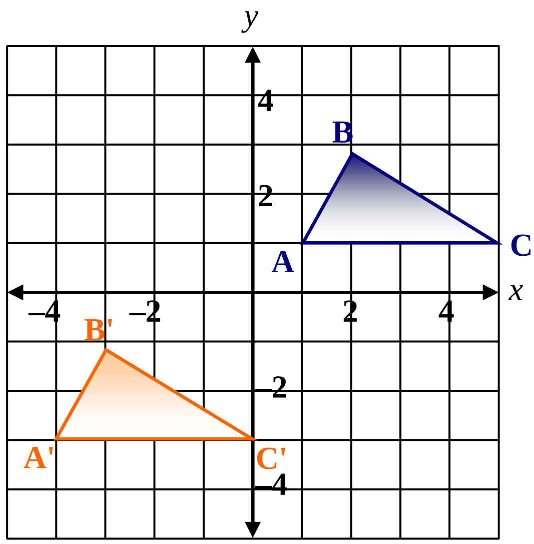
Translation: ABC $\rightarrow$ A'B'C'

a = 5 units to the right (+5)

b = 4 units up (+4)

Notation: $(x,\ y) \rightarrow (x+5,\ y+4)$

Example 2: Describe the translation of triangle ABC to triangle A'B'C' in words, and then in coordinate notation.



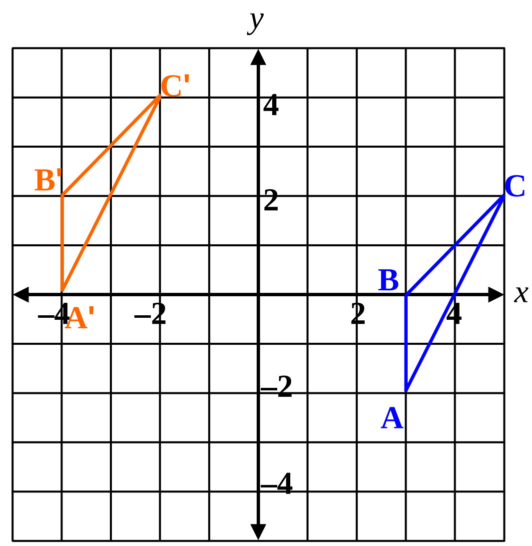
Focus on point A. Count how many units to the left and how many units down it would take to move point A to point A'. Check points B and C. You will find all points move the same number of spaces left and down.

To translate triangle ABC to A'B'C', move horizontally five (5) units to the left and vertically four (4) units down.

Coordinate Notation: $(x, y) \rightarrow (x - 5, y - 4)$

Example 3: Draw triangle ABC with vertices of A(3, –2), B(3, 0), and C(5, 2). Then find the coordinates of the vertices of the image after the translation. Draw the image.

Translation: $(x, y) \to (x-7, y+2)$



Step 1: Draw pre-image triangle ABC with vertices of A(3, –2), B(3, 0), and C(5, 2).

Step 2: Find the coordinates of the vertices of the image. (You could count seven spaces left and two spaces up, but when dealing with larger numbers, it is best to calculate the coordinates first, and then plot them.)

For $x-7$ subtract 7 from each *x*-coordinate.

For $y+2$ add 2 to each *y*-coordinate.

Original		**Image**
(x, y)	$\to$	$(x-7, y+2)$
$A(3,-2)$	$\to$	$A'(-4, 0)$
$B(3, 0)$	$\to$	$B'(-4, 2)$
$C(5, 2)$	$\to$	$C'(-2, 4)$

Step 3: Draw the image, A'B'C'.

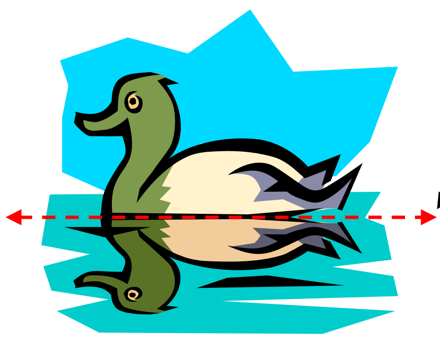
Reflections and Symmetry

There are four types of transformations of geometric figures in the coordinate plane. They are translations, reflections, rotations, and dilations. Reflections are presented in this section.

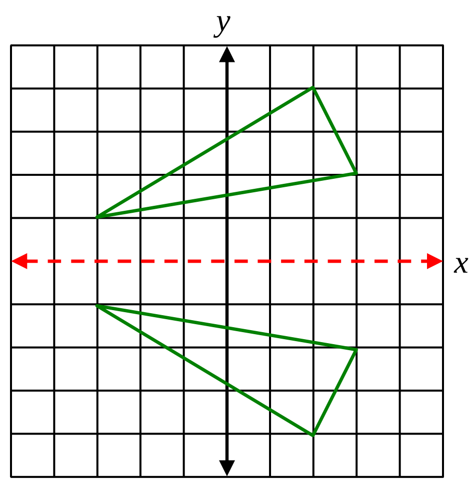
A **reflection** is a transformation of a figure in which the figure is reflected or flipped across the line. The line is called the line of reflection.

line of reflection - A line of reflection is a line over which a pre-image is mapped onto an image.

lines of reflection



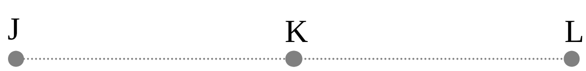
The duck is reflected in a pool of water creating a mirror image.



The *x*-axis is a line of reflection.

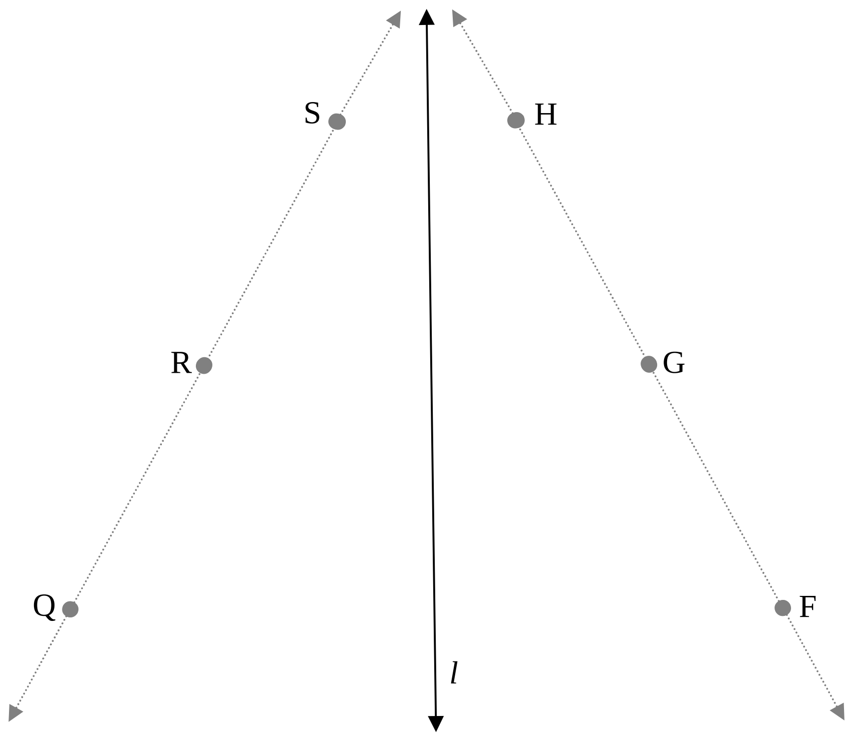
point of reflection – A point may also be referenced as a point over which a pre-image is mapped onto an image.

If the figure below point J is a reflection of point L over point K, and vice versa. We can say, point L is a reflection of point J, with respect to point K. We could also say, point J is a reflection of point L, with respect to point K.



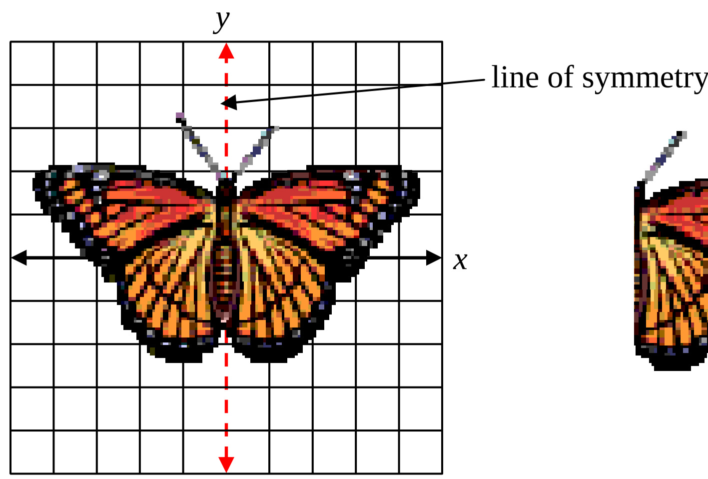
Reflections preserve “collinearity” and “betweenness of points”.

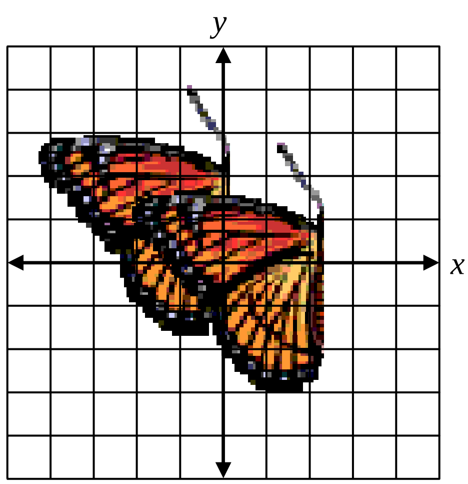
In the figure below, points Q, R, and S are collinear. Points F, G, and H are reflected over line “l” and are also collinear. Point R is between points Q and S. Point G is a reflection of point R and is between points F and H, which are reflections of points Q and S, respectively.



line of symmetry - A figure has **line symmetry** if it can be folded over a line so that one-half of the figure matches the other half. The line over which the figure is folded is called the line of symmetry.

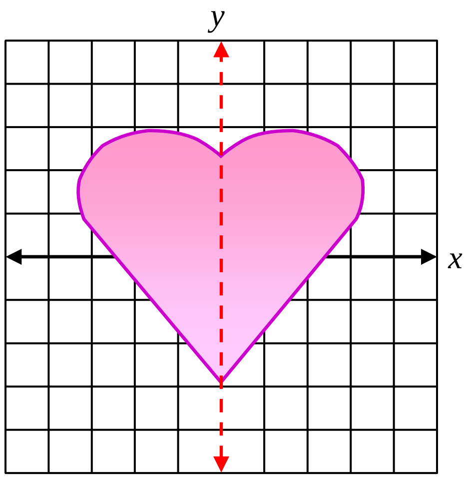
The **line of symmetry** divides the figure into two parts that are reflections of each other. The two sides are mirror images of each other.





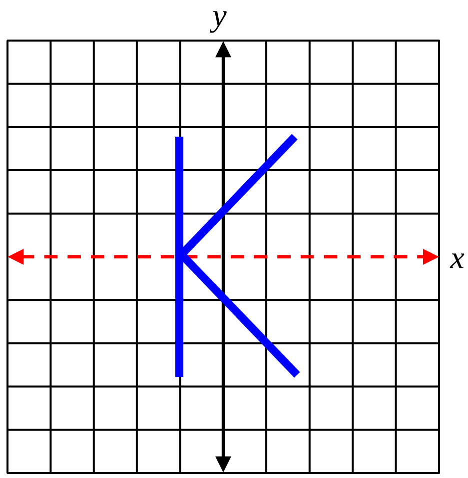
The left side of the butterfly is symmetrical to the right side. Symmetrical objects have parts that are congruent. The wings are congruent.

Samples of Line Symmetry



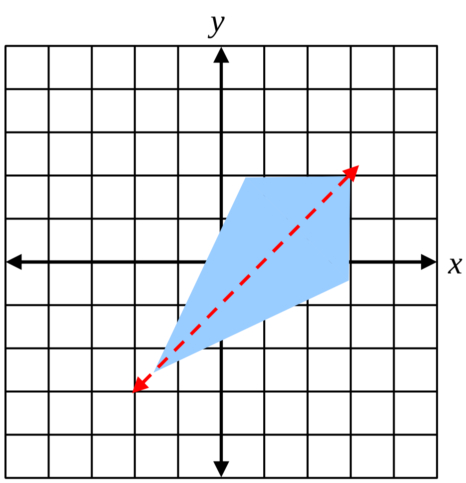
Vertical Line Symmetry

The left side of the heart is a reflection of the right side.

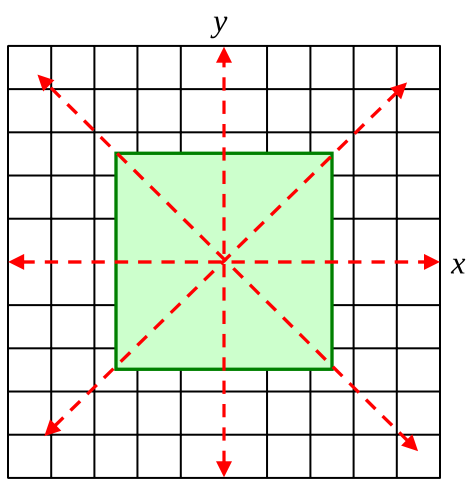


Horizontal Line Symmetry

The top part of the K is a reflection of the bottom part.

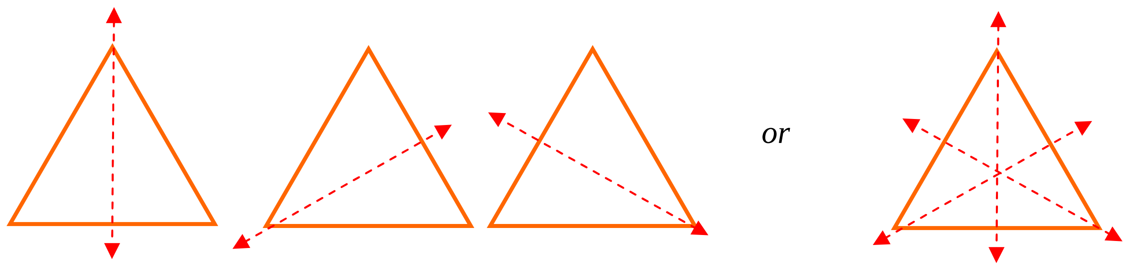


Diagonal Line Symmetry



Four Lines of Symmetry

Example 1: How many lines of symmetry does an equilateral triangle have? Draw each one.



An equilateral triangle has **three lines of symmetry**.

Notation in the Coordinate Plane: Reflections may be described using coordinate notation.

Example 2: Use the coordinates of the points in pre-image segment AB and image segment A'B' to write a rule that would pertain to all reflections over the *y*-axis.

Write the ordered pairs for segment AB and the corresponding ordered pairs for image A'B'.

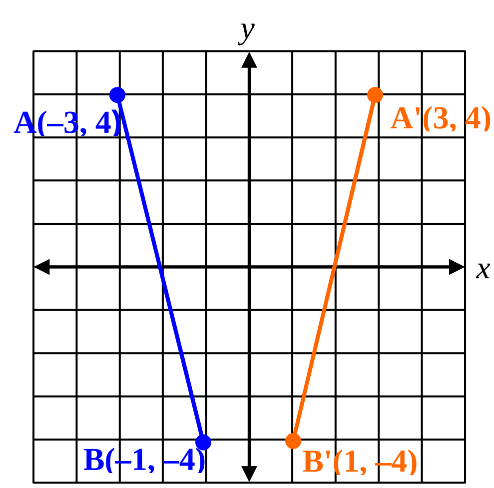
$(-3, 4) \rightarrow \quad (3, 4)$

$(-1, -4) \rightarrow \quad (1, -4)$

Notice that the *x*-values are opposites.

The coordinate notation that summarizes this rule is:

$(x, y) \quad \rightarrow \quad (-x, y)$



When the *x*-coordinate of a point is multiplied by –1, the point is reflected over the *y*-axis. The *y*-axis is the line of symmetry.

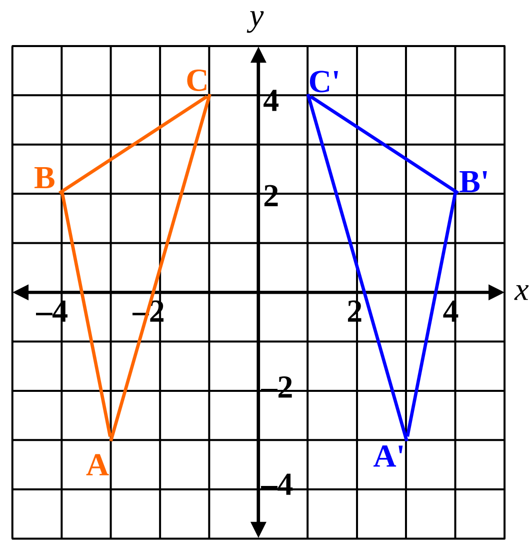
Example 3: Draw triangle ABC with vertices of $A(-3,-3)$, $B(-4,2)$, and $C(-1,4)$. Then find the coordinates of the vertices of the image after a reflection over the y-axis. Draw the image.

Step 1: Draw triangle ABC.

Step 2: Find the coordinates of the vertices of the image by reflecting triangle ABC over the y-axis. .

Original		Image
$A(-3,-3)$	$\rightarrow$	$A'(3,-3)$
$B(-4,2)$	$\rightarrow$	$B'(4,2)$
$C(-1,4)$	$\rightarrow$	$C'(1,4)$

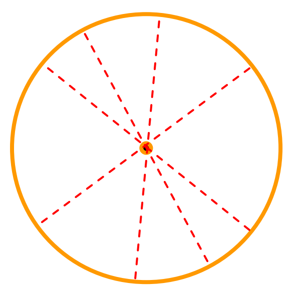
Step 3: Draw the image $A'B'C'$.



Triangle $A'B'C'$ is a reflection of triangle ABC. The y-axis is the line of symmetry.

point of symmetry – A point of symmetry is a point of reflection for all points on a figure. It must be the midpoint for all segments that pass through it and have endpoints on the figure.

Example 4: What is the point of symmetry for a circle?



The center of a circle is the midpoint of all segments that pass through it and have endpoints on the circle. Thus, the center of a circle is its point of symmetry.

Rotations

There are four types of transformations of geometric figures in the coordinate plane. They are translations, reflections, rotations, and dilations. Rotations are presented in this section.

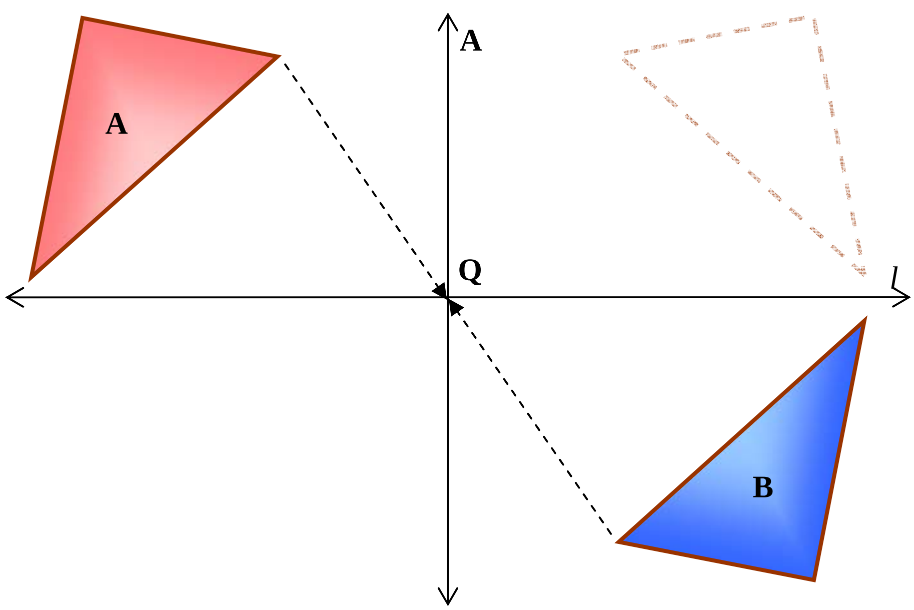
rotation - A **rotation** is a transformation in which a figure is turned about a fixed point, called the center of rotation.

There are two methods by which a rotation may be performed. We will examine both methods through examples.

Method 1: A rotation image can be formed by a composite of two successive reflections over two intersecting lines.

First, we will examine a rotation in with perpendicular intersecting lines.

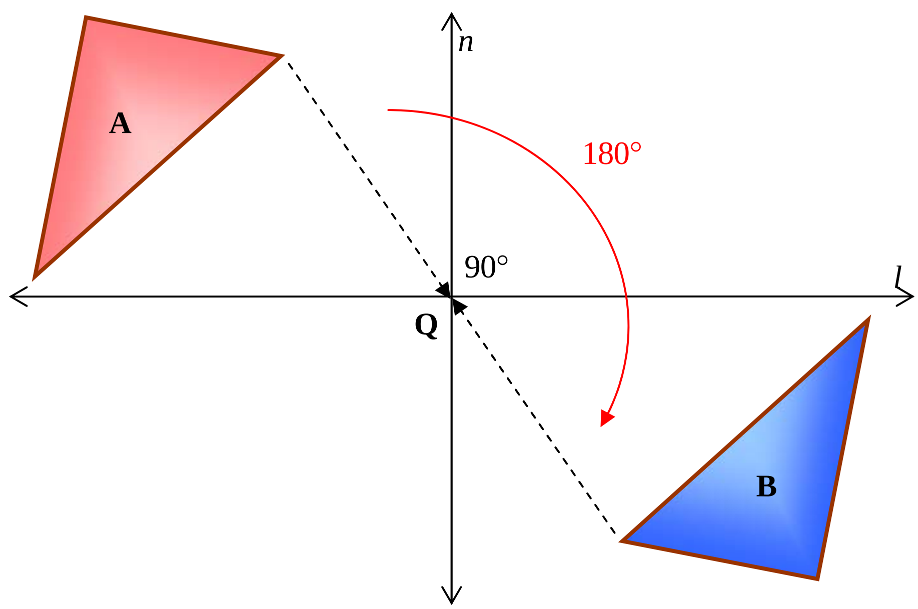
Pre-image triangle A is first reflected over line *n*, and then reflected a second time over line *l*, with point Q as the center of rotation. The result is image B.



Method 2: A rotated image can be determined by using the angle that is formed by two intersecting lines. Twice that angle is called the **angle of rotation**. The point where the two intersecting lines cross is called the **center of rotation**.

Examine the angle formed by the perpendicular lines, line *n* and line *l*. It is a 90-degree angle. The rotation that occurs to rotate pre-image A to image B about point Q is 180 degrees.

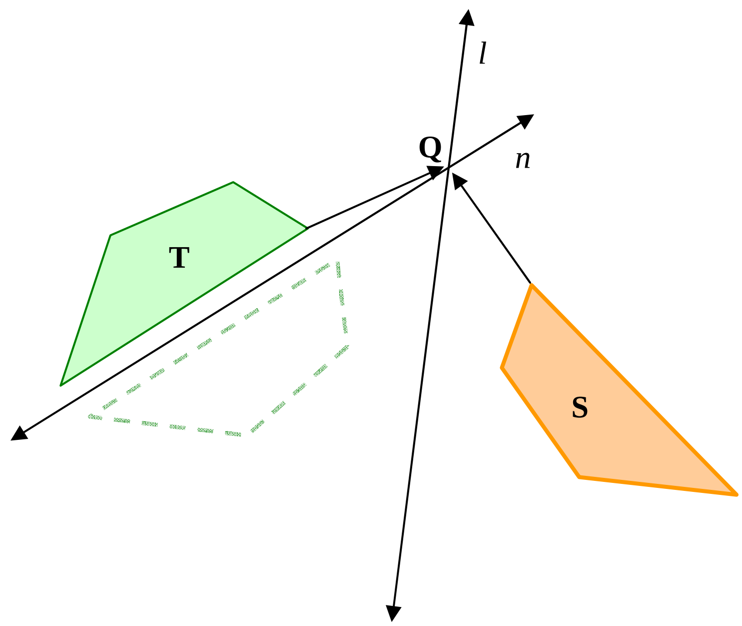
The angle of rotation (180°) is twice the 90-degree angle of formed by the intersecting lines.



Postulate 31	**In a given rotation, if A is the pre-image, B is the image, and Q is the center of rotation, then the measure of the angle of rotation, $\angle$AQB, is twice the measure of the angle formed by the intersecting lines of reflection.**

Now let's examine an example of a rotation where the lines of reflection are not perpendicular.

Example 1: Use *Method 1* to explain the rotation that occurred from pre-image T to image S, rotating counter-clockwise around point Q.

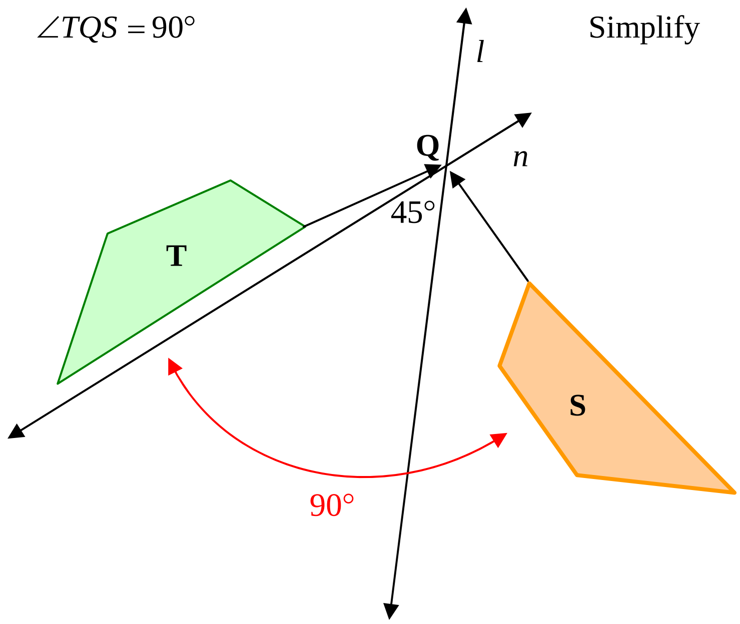


The rotation of pre-image T to image S is a result of TWO reflections. First, pre-image T is reflected over line *n*, and then it is reflected once again over line *l* around point Q, the center of rotation.

Example 2: Use *Method 2* to explain the rotation that occurred from pre-image T to image S, rotating counter-clockwise around point Q.

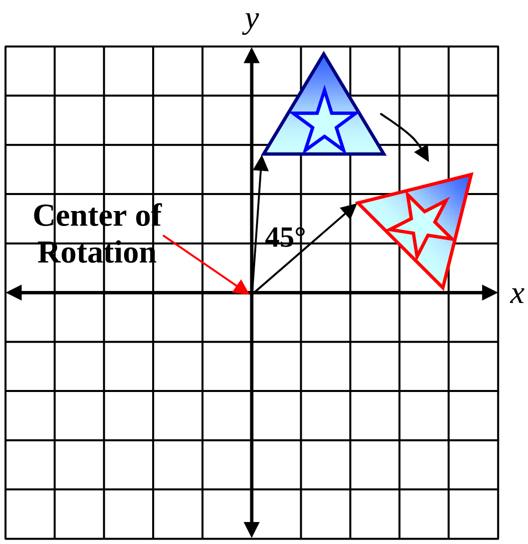
*Note: The size of the angle of the intersecting lines is given.

$\angle TQS = 2(\text{angle of the intersecting lines})$	Postulate 31
$\angle TQS = 2(45°)$	Substitute
$\angle TQS = 90°$	Simplify

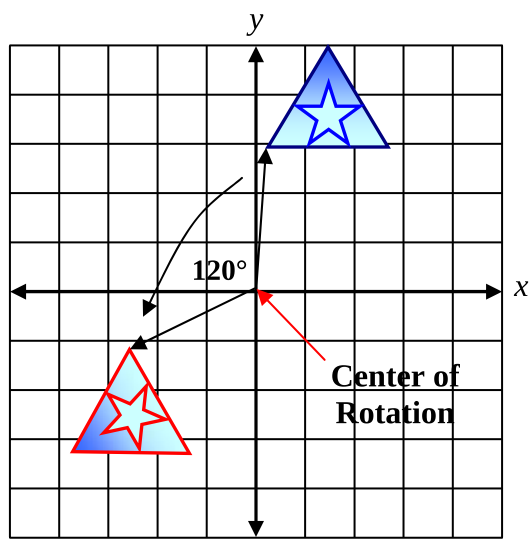


Since the angle formed by the intersecting lines is a 45-degree angle, the angle of rotation is 90 degrees. Pre-image T was rotated around point Q, 90-degrees counter-clockwise, to form image S.

Now, let's examine rotations performed in a coordinate plane. In these instances, the angle of rotation will be given.



45° Clockwise Rotation



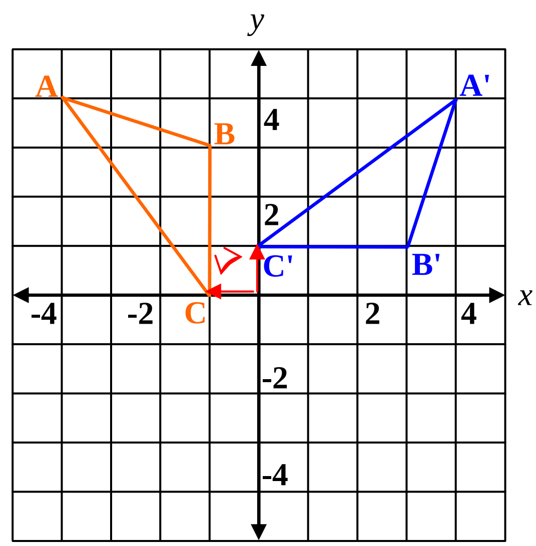
120° Counter-clockwise Rotation

Example 3: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0). Then find the coordinates of the vertices of the image after a $90°$ clockwise rotation around the origin. Draw the image, A'B'C' and write a rule in coordinate notation to describe the rotation.

Step 1: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0).

Step 2: Rotate the image 90 degrees clockwise and draw image A'B'C'.

Step 3: Write the coordinates of the pre-image and the image and compare the changes in the *x*- and *y*-values.



Pre - Image		Image
$A(-4, 4)$	$\rightarrow$	$A'(4, 4)$
$B(-1, 3)$	$\rightarrow$	$B'(3, 1)$
$C(-1, 0)$	$\rightarrow$	$C'(0, 1)$

The coordinate notation for a 90-degree clockwise rotation around the origin is:

$(x, y) \rightarrow (y, -x)$

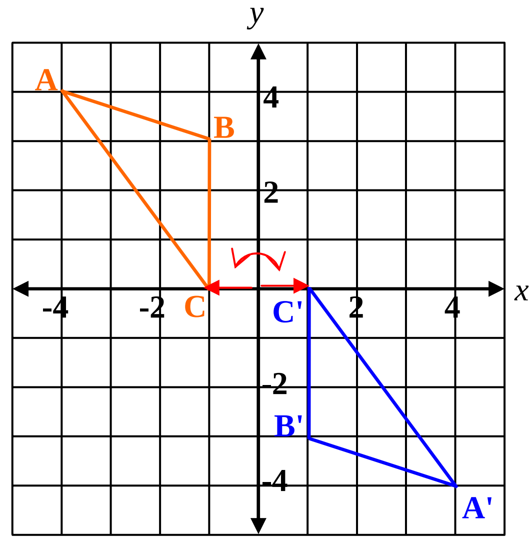
Simply, to write the coordinates of the image, switch the coordinates and multiply the new *y*-coordinate by –1.

Example 4: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0). Then find the coordinates of the vertices of the image after a 180° rotation around the origin. The image is the same whether you rotate the figure clockwise or counterclockwise. Draw the image, A'B'C and write a rule in coordinate notation to describe the rotation.

Step 1: Draw triangle ABC with vertices of A(–4, 4), B(–1, 3), and C(–1, 0).

Step 2: Rotate the image 180 degrees clockwise and draw image A'B'C'.

Step 3: Write the coordinates of the pre-image and the image and compare the changes in the *x*- and *y*-values.



Pre - Image		Image
$A(-4, 4)$	$\rightarrow$	$A'(4, -4)$
$B(-1, 3)$	$\rightarrow$	$B'(1, -3)$
$C(-1, 0)$	$\rightarrow$	$C'(1, 0)$

The coordinate notation for a 180-degree clockwise rotation around the origin is:

$$(x, y) \quad \rightarrow \quad (-x, -y)$$

Simply, to write the coordinates of the image, multiply both the *x*- and the *y*-coordinates by –1.

Dilations

There are four types of transformations of geometric figures in the coordinate plane. They are translations, reflections, rotations, and dilations. Dilations are presented in this section.

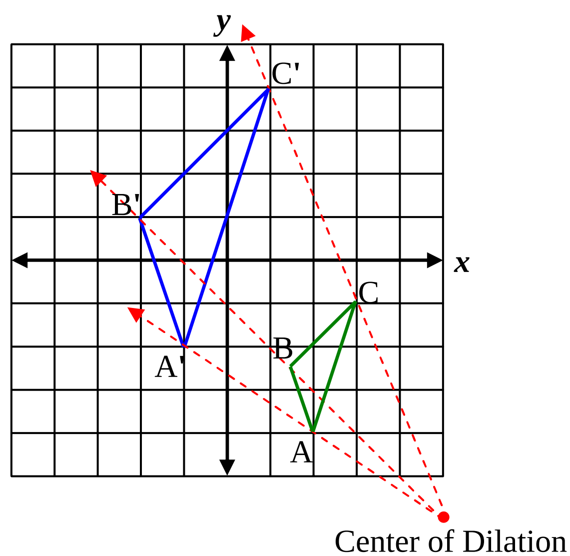
A **dilation** is a transformation in which the size is changed, but not the shape. It can be an enlargement or a reduction of a figure. The dilation of a figure is similar to the original image.

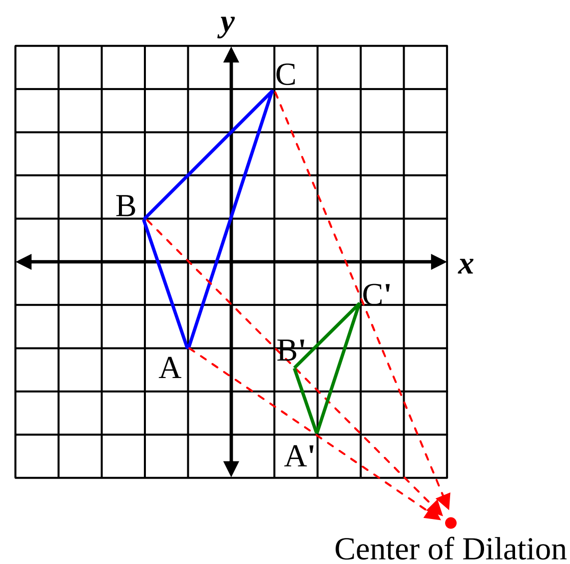
A dilation has a fixed point that is the **center of dilation**. The figure stretches or shrinks with respect to the **center of dilation**. To find the center of dilation, draw a line that connects each pair of corresponding vertices.

The **scale factor** describes how much a figure is enlarged or reduced. It is the ratio of the side length of the image to the corresponding side length of the original figure. A scale factor can be expressed as a decimal, fraction, or percent.

A **dilation** produces an image similar to the original figure.

Multiplying by a scale factor > 1	Enlarges a figure
Multiplying by a scale factor < 1	Reduces a figure



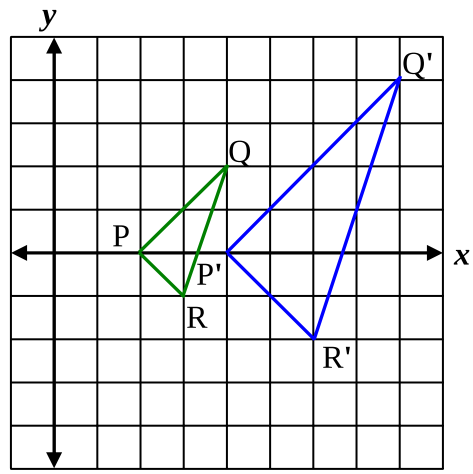


Triangle A'B'C' is a dilation of triangle ABC and has a 100% increase in size. The scale factor is 2.

Triangle A'B'C' is a dilation of triangle ABC and has a 100% decrease in size. The scale factor is 0.5.

Example 1: Draw triangle PQR with vertices of P(2, 0), Q(4, 2), and R(3, –1). Then find the coordinates of the vertices of the image after the dilation having a scale factor of 2. Draw the image, P'Q'R'.

Step 1: Draw triangle PQR with vertices of P(2, 0), Q(4, 2), and R(3, –1).

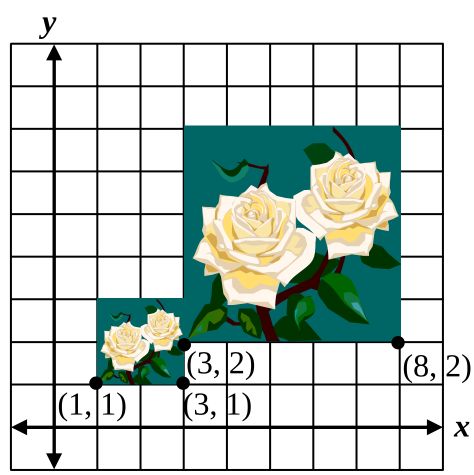


Step 2: Find the coordinates of the vertices of the image. To dilate triangle PQR, multiply the *x*- and *y*-coordinates of each vertex by 2.

Original	Image
(x, y)	(2x, 2y)
P(2, 0)	P'(4, 0)
Q(4, 2)	Q'(8, 4)
R(3, –1)	R'(6, –2)

Step 3: Draw the image, P'Q'R'.

Example 2: An artist uses a computer program to enlarge a design. Find the scale factor of the design.



The scale factor is 2.5.

Step 1: Find the width of the original design.

The width of the original design is the differences of the original design's *x*-coordinates, 3 – 1 = 2 units.

Step 2: Find the width of the image.

The width of the image is the difference of the image's *x*-coordinates, 8 – 3 = 5.

Step 3: Find the scale factor.

The scale factor is:

$$\frac{\text{image}}{\text{original design}} = \frac{5}{2} = 2.5$$

Theorem 31

If a dilation with center C and a scale factor of K maps A onto E and B onto D, then ED = k (AB).

In general, if k is the scale factor for a dilation with center C, then the following is true:

If $k > 0$, then P', the image of point P, lies on $\overrightarrow{CP}$, and CP' = k•CP.

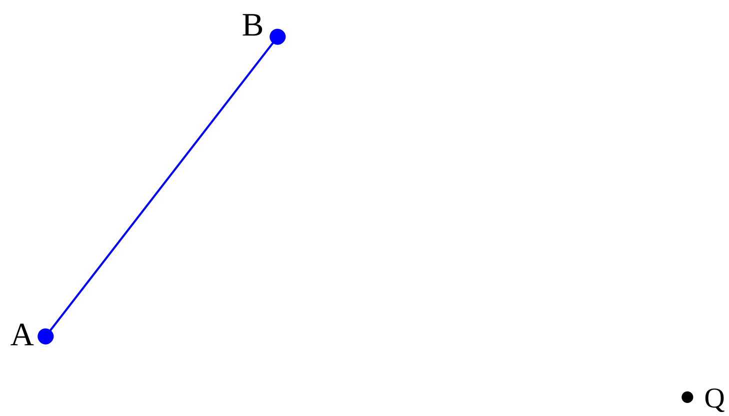
If $k < 0$, then P', the image of point P, lies on the ray opposite $\overrightarrow{CP}$, and CP' = $|k|$•CP.

If $|k| > 1$, then the dilation is an enlargement.

If $0 < |k| < 1$, then the dilation is a reduction.

If $|k| = 1$, then the dilation is a congruence transformation.

Example 3: In the figure shown below, segment AB is the pre-image and point Q is the center of dilation. Sketch segment CD so that it is a dilation segment AB with a scale factor of $k = \frac{2}{3}$.



First, draw a ray from point Q through point A.

Calculate the length from point Q to C. $QC = \frac{2}{3}(QA)$

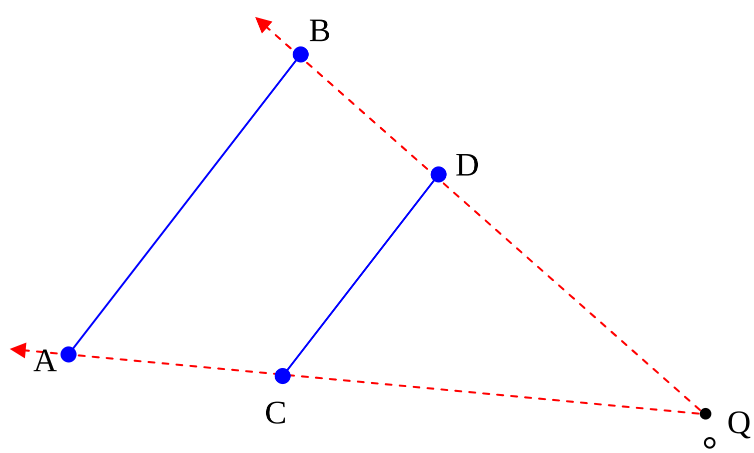
Then, mark a point C on segment QA that is 2/3 the length of QA.

Second, draw a ray from point Q through point B.

Calculate the length from point Q to D. $QD = \frac{2}{3}(QB)$

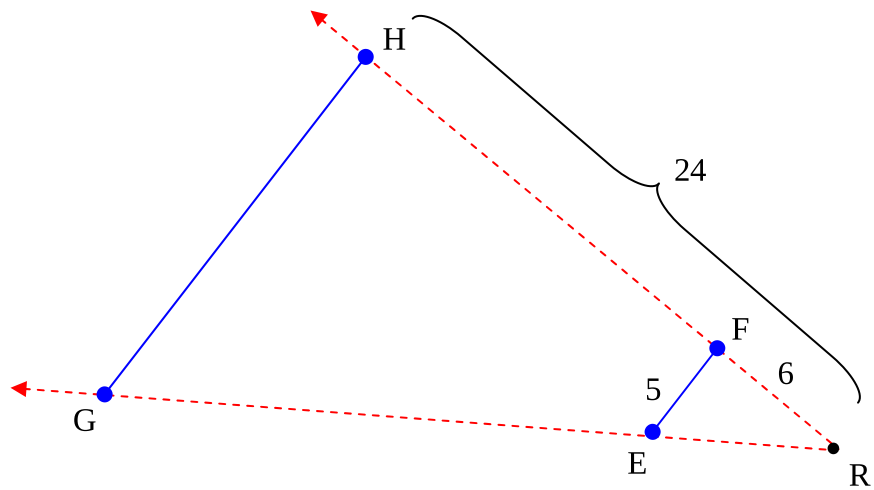
Then, mark a point D on segment QB that is 2/3 the length of QB.

Third, connect points C and D.



The length of segment CD is 2/3 the length of segment AB.

Example 4: In the figure shown below, segment EF = 5. Segment GH is a dilation of segment EF with respect to point R, the center of dilation. The length of segment RF is 6 and the length of segment RH is 24. (a) What scale factor was used to create the dilation? (b) What is the length of GH?



(a) Since GH is an enlargement of EF, the scale factor is greater than one (k>1).

$RH = k(RF)$	Theorem 31
$24 = k(6)$	Substitution
$k = 4$	Divide

The scale factor is 4.

(b) Use the scale factor of 4 to determine the length of GH.

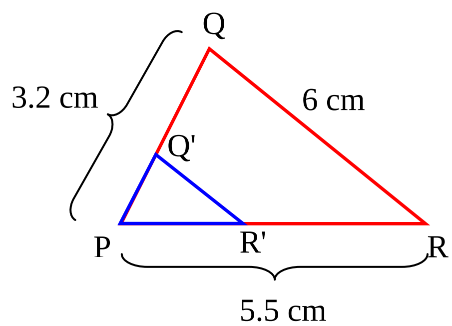
$GH = 4(EF)$	$k = 4$
$GH = 4(5)$	Substitution
$GH = 20$	Simplify

The length of GH is 20.

Example 5: Dilate the figure by a scale factor of 0.4 with P as the center of dilation. Segment PR has a length of 5.5 cm, segment PQ has a length of 3.2 cm and segment QR has a length of 6 cm. P and P' are the same point and the center of dilation. (a) What is the length of PQ'? (b) What is the length of PR'? (c) What is the length of Q'R'?

Since $k = 0.4$, multiply each length by 0.4.

Original (Length)	**Image** ($0.4 \times$ Length)
PQ = 3.2 cm	PQ' = 1.28 cm
PR = 5.5 cm	PR' = 2.2 cm
QR = 6 cm	Q'R' = 2.4 cm



(a) The length of PQ' is 1.28 centimeters. (b) The length of PR' is 2.2 centimeters. (c) The length of Q'R' is 2.4 centimeters.