

VOLUME

In this unit you will examine the volume of several solids. You will develop and apply formulas for the volume of cubes, prisms, cones, cylinders, pyramids, and spheres. You will investigate Cavalieri's Principle and its applications to right and oblique solids. You will also examine congruent and similar solids.

Volume of Right Prisms

Finding the Cube Root

Volume of Right Cylinders

Volume of Right Cones and Pyramids

Cavalieri's Principle

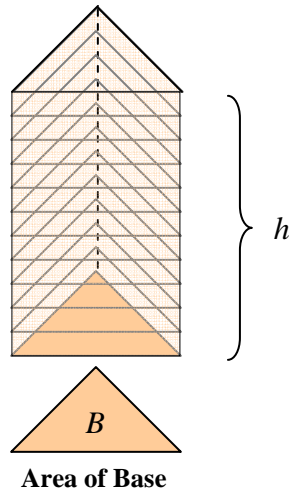
Volume of Spheres

Similar and Congruent Solids

Volume and Surface Area Formulas

Volume of Right Prisms

Triangular Prism



The **volume of a right prism** is the amount the prism can hold. Volume is measured in cubic units. To calculate the volume of a prism, multiply the area of the base times the height.

$$V = \text{Area of the Base} \times \text{Height}$$

Volume of a Right Prism

The volume (V) of a right prism is the product of the area of the base (B) and the height (h) of the prism.

$$V = Bh$$

Example 1: Find volume of a right triangular prism with a height of 16 inches, and the dimensions of the triangular base are 6 inches (length of the triangle's base) and 4 inches (height of the triangular base).

Plan:

- (1) Calculate the area of the base.
- (2) Calculate the volume.

Step 1: Calculate the area of triangular base.

$$A = \frac{1}{2}bh \quad \text{Formula for Area of a Triangle}$$

$$A = \frac{1}{2}(6)(4) \quad \text{Substitution}$$

$$A = 12 \quad \text{Simplify}$$

The area of the triangular base is 12 square inches.

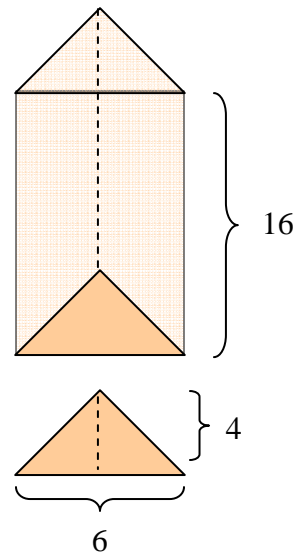
Step 2: Calculate the volume of the triangular prism.

$$V = Bh \quad \text{Formula for the Volume of a Prism}$$

$$V = (12)(16) \quad \text{Substitution}$$

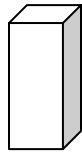
$$V = 192 \quad \text{Simplify}$$

The volume of the triangular prism is 192 cubic inches (in^3).

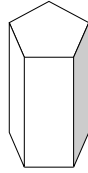


There are many kinds of prisms. They are named for the shape of their bases. The volume formula, $V = Bh$, applies to all prisms.

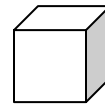
Here are some examples of the various prisms:



Rectangular Prism



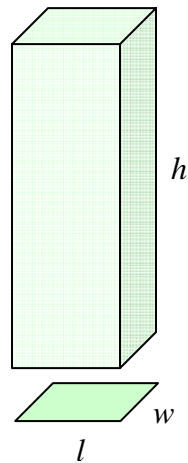
Pentagonal Prism



Cube

Some prisms are common and used so often that they have a formula specific to their shape. The rectangular prism (box) and cube are two commonly used prisms.

Rectangular Prism



To calculate the volume of a **rectangular prism**, multiply the length times the width times the height.

$$V = Bh$$

$$V = (\text{length} \times \text{width})h$$

$$V = lwh$$

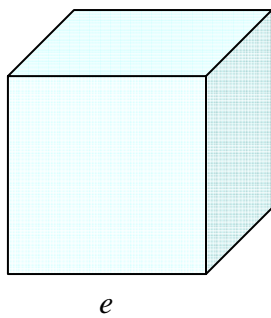
Formula for Volume of a Prism

Formula for Area of a Rectangle ($A = lw$)

(A rectangular prism has a rectangular base.)

Simplify

Cube



To calculate the volume of a **cube**, multiply the edge times itself three times.

$$V = lwh$$

$$V = \text{edge} \times \text{edge} \times \text{edge}$$

$$V = e \times e \times e$$

$$V = e^3$$

Volume of a Rectangular Prism

The length, width, and height (edges)

all measure the same in a cube.

e represents edge.

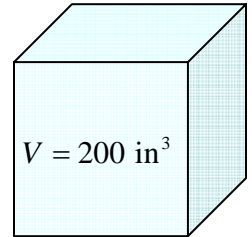
Simplify

Example 2: Calculate the volume of a rectangular prism with a length of 11 feet, a width of 8 feet, and a height of 23 feet.

$V = lwh$	Volume of a Rectangular Prism
$V = (11)(8)(23)$	Substitution
$V = 2024$	Simplify

The volume of the rectangular prism is 2024 cubic feet.

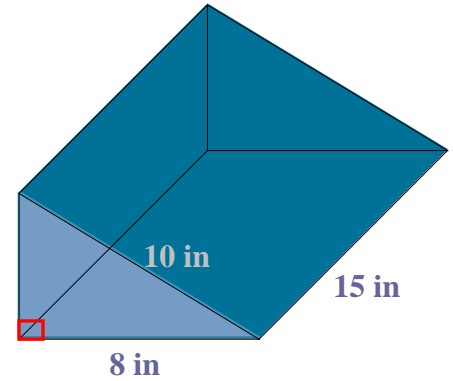
Example 3: If the volume of a cube is 200 cubic inches, what is the length of one edge?



$V = e^3$	Volume of a Cube
$200 = e^3$	Substitution
$\sqrt[3]{200} = \sqrt[3]{e^3}$	Find the third root of both sides of the equation.
$\sqrt[3]{200} = e$	$\sqrt[3]{e^3} = \sqrt[3]{e \cdot e \cdot e} = e$
$e \approx 5.85$	See the unit link to "Finding the Cube Root".

The length of one edge of the cube is approximately 5.85 inches.

Example 4: Find the volume a right prism with a height of 15 inches and a base which is a right triangle. The base of the right triangle measures 8 inches and the hypotenuse measures 10 inches.



Plan:

- (1) Find the height of the right triangle.
- (2) Find the area of the triangular base.
- (3) Find the volume of the right triangular prism.

Step 1: Use the Pythagorean Theorem to find the height of the right triangle.

$a^2 + b^2 = c^2$	Pythagorean Theorem
$h^2 + 8^2 = 10^2$	Substitution ($h = a$, $b = 8$, $c = 10$)
$h^2 + 64 = 100$	Simplify
$h^2 = 36$	Subtract
$h = 6$	Take the square root of both sides.

The height of the right triangular base is 6 inches.

Step 2: Calculate the area of the base.

$$A = \frac{1}{2}bh \quad \text{Area of a Triangle}$$

$$A = \frac{1}{2}(8)(6) \quad \text{Substitution}$$

$$A = 24 \quad \text{Simplify}$$

The area of the triangular base is 24 square inches.

Step 3: Calculate the volume of the triangular prism.

$$V = Bh \quad \text{Volume of a Prism}$$

$$V = 24(15) \quad \text{Substitution}$$

$$V = 360 \quad \text{Simplify}$$

The volume of the triangular prism is 360 cubic inches.

Finding the Cube Root

To find the cube root (third root) of 200, we will explore two methods.

Method 1: Use approximation.

List the first ten perfect cubes:

$1^3 = 1$	$2^3 = 8$	$3^3 = 27$	$4^3 = 64$	$5^3 = 125$
$6^3 = 216$	$7^3 = 343$	$8^3 = 512$	$9^3 = 729$	$10^3 = 1000$

Find the two perfect cubes that 200 falls between.

$$125 < 200 < 216$$

$$5^3 < 200 < 6^3$$

First approximate the cube root to the nearest tenth.

200 is closer to 216...

Try 5.7 as the root ... $5.7 \times 5.7 \times 5.7 = 185.193$... too low.

Try 5.8 as the root ... $5.8 \times 5.8 \times 5.8 = 195.112$... too low, but closer.

Try 5.9 as the root ... $5.9 \times 5.9 \times 5.9 = 205.379$... too high.

Now approximate the cube root to the nearest hundredth.

Try 5.85 as the root ... $5.85 \times 5.85 \times 5.85 = 200.201625$... too high, but close.

Try 5.84 as the root ... $5.84 \times 5.84 \times 5.84 = 199.176704$... too low, but close.

We'll pick 5.85 since 200.2 is closer to 200 than 199.2. (200.2 is 2 tenths away from 200 while 199.2 is 8 tenths away from 200.)

$$\therefore \sqrt[3]{200} \approx 5.85$$

Method 2: Use a scientific calculator.

There are many types of scientific calculators. You will need to consult the manual of your personal calculator to find out how to calculate cube roots. The explanation below is based on the calculator that is found on most computers in the accessories group. (Start / Programs / Accessories / Calculator).

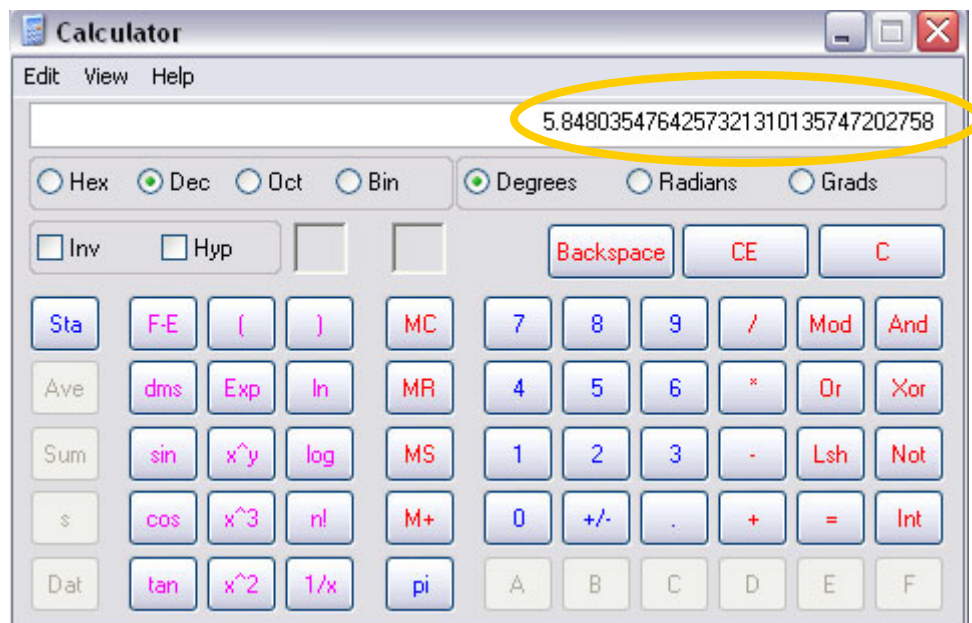
- 1) Type 200 in the white textbox.
- 2) Select "Inv" (A check in the checkbox will be displayed when selected.)

3) Click on x^3 .

*Note: Finding the cube root is the inverse (Inv) of finding x-cubed.



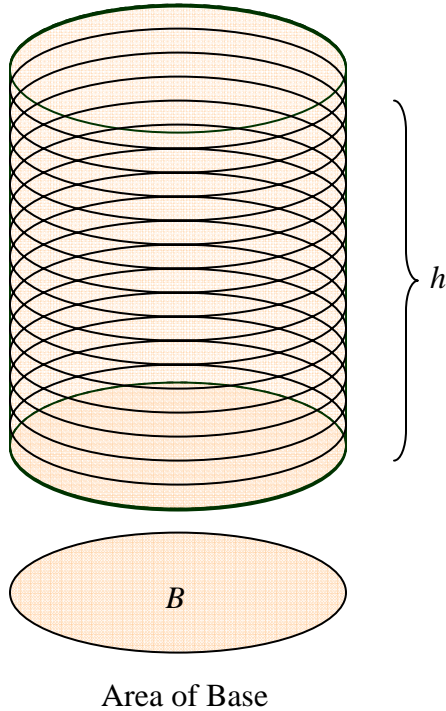
4) The answer will be displayed in the white textbox.



5) $\sqrt[3]{200} \approx 5.85$

The cube root of 200 is approximately 5.85.

Volume of Right Cylinders



The **volume of a right cylinder** is the amount a cylinder can hold. Volume is measured in cubic units. To calculate the volume of a cylinder, multiply the area of its base times its height.

$$V = Bh$$

Volume Formula

$$V = (\pi r^2)h$$

Substitution ($B = A = \pi r^2$)

The area of the base is circular.

$$V = \pi r^2 h$$

Volume of a Cylinder

Volume of a Right Cylinder

The volume (V) of a right cylinder is the product of the area of its base (πr^2) and the height (h) of the cylinder.

$$V = \pi r^2 h$$

Example 1: Find volume of a cylinder with a radius of 5 inches and a height of 9 inches.



$$V = \pi r^2 h$$

Volume of a Cylinder

$$V = (3.14)(5^2)(9)$$

Substitution ($r = 5$, $h = 9$)

$$V = (3.14)(25)(9)$$

Simplify

$$V = 706.5$$

Simplify

Reminder: Volume is measured in cubic units.

The volume of the cylinder is 706.5 cubic inches.

Example 2: A cylindrical tank holds 300 gallons of liquid. One gallon is about 0.13 cubic foot. If the tank is 5.5 feet tall, what is the length of the diameter of the tank? Round the answer to nearest tenth, when appropriate.



Plan:

- (1) Convert the volume of the tank from gallons to cubic feet.
- (2) Find the radius of the tank, and then find the diameter of the tank.

Step 1: Write a ratio to convert the volume of the tank from gallons to cubic feet.

$$\frac{1 \text{ gal}}{0.13 \text{ cu ft}} = \frac{450 \text{ gal}}{x}$$

Write a proportion to convert.

(Given: 1 gallon = 0.13 cubic feet)

$$x = 450(0.13)$$

Cross Multiply

$$x = 58.5$$

Simplify

The volume of the tank is 58.5 cubic feet.

The volume of the tank is 58.5 cubic feet.

Step 2: Calculate the radius of the tank.

$$V = \pi r^2 h$$

Formula for volume of a cylinder.

$$58.5 = (3.14)r^2(5.5)$$

Substitution

$$58.5 = 17.27r^2$$

Simplify

$$3.39 = r^2$$

Division

$$r = 1.8 \text{ ft}$$

Take the square root of both sides.

$$d = 2(1.8)$$

Two opposite radii equal one diameter. ($d = 2r$)

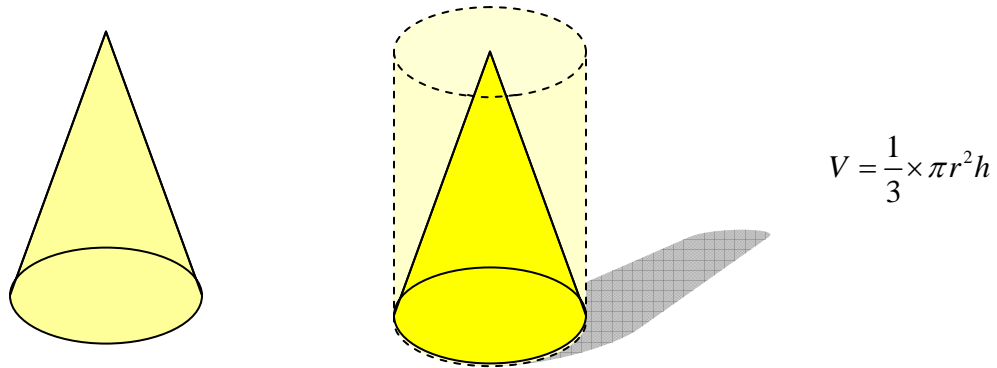
$$d = 3.6$$

The diameter of the tank is approximately 3.6 feet.

Volume of Right Cones and Pyramids

Volume of a Right Cone

The volume of a right cone can be found similar to finding the volume of a cylinder. The volume of a cone is equal to one-third the volume of a cylinder with the same base area and height.



Volume of a Right Cone

The volume (V) of a right cone is one-third of the product of the area of its base (πr^2) and the height (h) of the cone.

$$V = \frac{1}{3} \pi r^2 h$$

Example 1: Find the volume of a right cone with a radius of 5 inches and the height is 9 inches.

$$V = \frac{1}{3} \pi r^2 h \quad \text{Volume of a Right Cone}$$

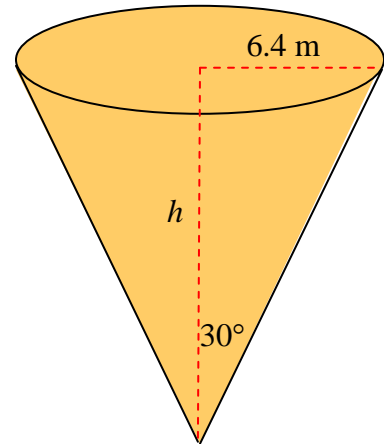
$$V = \frac{1}{3} (3.14)(5^2)(9) \quad \text{Substitution}$$

$$V = \frac{1}{\cancel{3}} (3.14)(25)(\cancel{9})^3 \quad \text{Cancel and Simplify}$$

$$V = 235.5 \quad \text{Simplify}$$

The volume of the cone is 235.5 cubic inches.

Example 2: Find the volume of a right cone with a radius of 6.4 meters, and the slant height and the height of the cone form a 30-degree angle at the vertex of the cone. Round the calculations to nearest tenth, when appropriate.



Plan:

- (1) Recall the relationship between the sides of a 30-60-90 degree triangle (If x equals the length of the shorter leg, then $2x$ equals the length of the hypotenuse, and $x\sqrt{3}$ equals the length of the longer leg.
- (2) Calculate the volume of the cone.

Step 1: Find the height of the cone.

$h = x\sqrt{3}$	Length of the Longer Leg (30-60-90-degree triangle)
$h = 6.4\sqrt{3}$	Substitution ($h = 6.4$)
$h \approx 11.1$	Simplify

The height of the cone is approximately 11.1 meters.

Step 2: Calculate the volume of the cone.

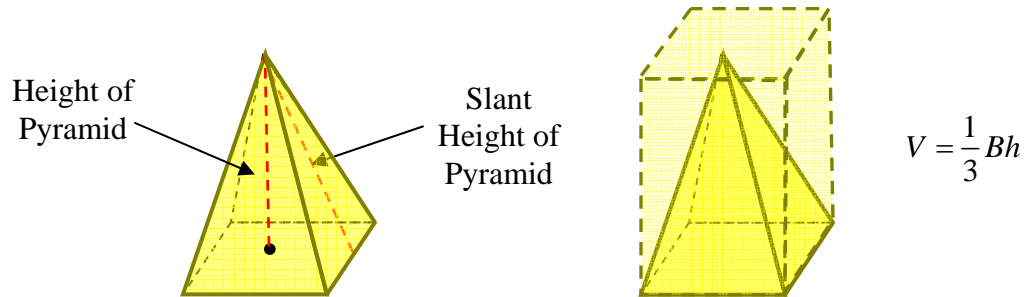
$V = \frac{1}{3}\pi r^2 h$	Volume of a Right Cone
$V = \frac{1}{3}(3.14)(6.4)^2(11.1)$	Substitution
$V \approx 475.9$	Simplify

The volume of the cone is approximately 475.9 cubic meters.

Volume of a Right Pyramid

The volume of a right pyramid can be calculated in a similar manner to the volume of a right cone. The volume of a right pyramid is one-third the volume of a rectangular prism with the same base area and height.

**Note:* Be sure to use the height of the pyramid, not the slant height; and make sure to use the area of the base of the pyramid, not just the length an edge of the base.



Volume of a Right Pyramid

The volume (V) of a right pyramid is one-third of the product of the area of its base (B) and the height (h) of the pyramid.

$$V = \frac{1}{3} Bh$$

Example 2: Find the volume of a right square pyramid with a base edge (of the square) that measures 8 feet and a height of 12 feet.

Plan:

- (1) Find the area of the square base.
- (2) Calculate the volume of the pyramid.

Step 1: Find the area of the square base.

$$A = s^2$$

Area of a Square

$$A = 8^2$$

Substitution ($s = 8$)

$$A = 64$$

Simplify

Step 2: Calculate the volume of the square pyramid.

$$V = \frac{1}{3}Bh \quad \text{Volume of a Right Pyramid}$$

$$V = \frac{1}{3}(64)(12) \quad \text{Substitution}$$

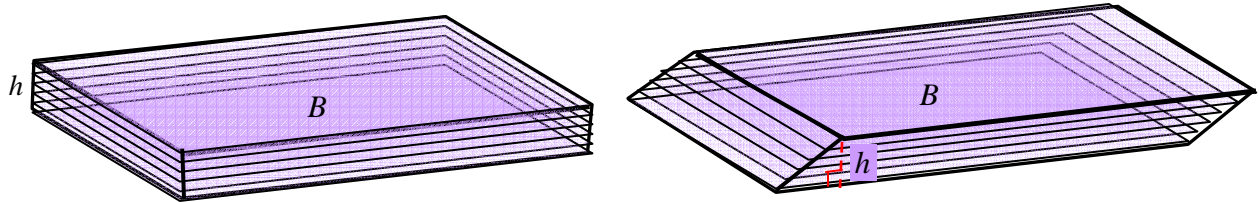
$$V = 256 \quad \text{Simplify}$$

The volume of the pyramid is 256 cubic inches.

Cavalieri's Principle

Cavalieri's Principle applies to oblique shapes.

Shown below is a right rectangular prism and an oblique rectangular prism. Each has the same base area (B) and same height (h). Notice that the cross sections (slices parallel with the bases) in each figure has the same area as its base area.



In the 17th century, Cavalieri noticed that with this occurrence, the volumes will remain the same. This principle is applied to right and oblique shapes.

If a prism has a base with an area of B and a height of h , then its volume is Bh , whether it is right or oblique. This principle also applies to cylinders, cones, and pyramids. Thus, the volume formulas hold true if the solids are right or oblique.

Cavalieri's Principle

If two solids have the same height and same cross-sectional areas at every level, then they have the same volume.

Volume of Oblique Solids:

Prisms: $V = Bh$

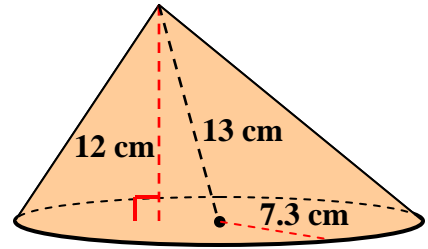
Cylinders: $V = \pi r^2 h$

Pyramids: $V = \frac{1}{3} Bh$

Cones: $V = \frac{1}{3} \pi r^2 h$

* h = height of the solid (not slanted edges)

Example: Find the volume of an oblique cone that has a radius of 7.3 centimeters and a height of 12 centimeters. Round the answer to the nearest tenth.



$$V = \frac{1}{3}\pi r^2 h$$

Volume of a Right Cone

$$V = \frac{1}{3} \times 3.14 \times 7.3^2 \times 12$$

Substitution ($h = 12$)

$$V = 669.3$$

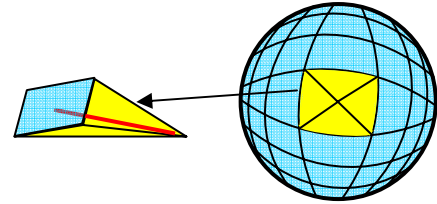
Simplify

The volume of the cone is 669.3 cubic centimeters.

Volume of a Sphere

The volume of a sphere is found by multiplying four-thirds times “pi” times the cube of the radius. Let’s explore the derivation of this volume formula.

First imagine the spherical solid divided up into many pyramids, with the base of the pyramid being part of the surface of the sphere and the vertex of the pyramid being the center of the sphere. The height of each of the pyramids would be a radius of the sphere.



Each of the pyramids’ volumes would be $V = \frac{1}{3}Bh$.

The volume of the sphere would be the sum of all the volumes of these pyramids.

The following math statement represents the sum of an infinite number of pyramids that make up the sphere’s volume.

$$V = \frac{1}{3}B_1h_1 + \frac{1}{3}B_2h_2 + \frac{1}{3}B_3h_3 + \frac{1}{3}B_4h_4 + \dots \frac{1}{3}B_nh_n$$

Since the height of each of the pyramids would equals the radius of the sphere, we will substitute “ r ” for each of the “ h ’s” in the formula shown above.

$$V = \frac{1}{3}B_1r + \frac{1}{3}B_2r + \frac{1}{3}B_3r + \frac{1}{3}B_4r + \dots \frac{1}{3}B_nr \quad h_1 = r, h_2 = r, h_3 = r, h_4 = r, \dots h_n = r$$

Using the distributive property, we can factor out $(1/3 r)$.

$$V = \frac{1}{3}r(B_1 + B_2 + B_3 + B_4 + B_5 + \dots B_n)$$

The sum of the areas of the bases of the infinite number of pyramids is equal to the surface area of the sphere. So, we will substitute the formula of the surface area of a sphere for the sum of the areas of the infinite number of pyramids.

$$V = \frac{1}{3}r(4\pi r^2) \quad B_1 + B_2 + B_3 + B_4 + B_5 + \dots B_n = 4\pi r^2$$

Using the Commutative Property of Multiplication, we can rearrange the factors of the product.

$$V = \frac{1}{3} \cdot 4\pi r \cdot (r^2) \quad \text{Commutative Property}$$

Now, simplify and we have derived the formula for finding the volume of a sphere.

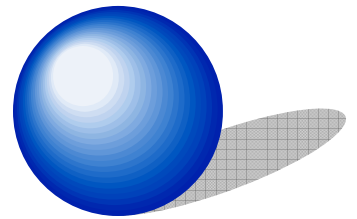
$$V = \frac{4}{3}\pi r^3 \quad \text{Simplify}$$

Volume of a Sphere

The volume (V) of a sphere is four-thirds of the product of “pi” and the radius-cubed.

$$V = \frac{4}{3}\pi r^3$$

Example 1: Find the volume of a sphere with a radius of 7 centimeters. Round the answer to the nearest tenth.



$$V = \frac{4}{3}\pi r^3 \quad \text{Volume of a Sphere}$$

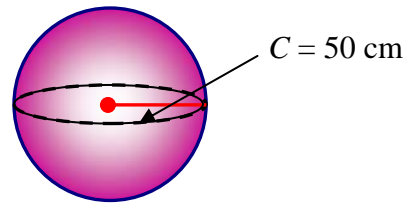
$$V = \frac{4}{3}(3.14)(7)^3 \quad \text{Substitution}$$

$$V = \frac{4}{3}(3.14)(343) \quad \text{Simplify}$$

$$V = 1436.0 \quad \text{Simplify}$$

The volume of the sphere is approximately 1436.0 cubic centimeters.

Example 2: Calculate the volume of a sphere if the circumference of its great circle is 50 centimeters. Express the value of the radius in terms of “pi”. Round the final answer to the nearest tenth.



Plan:

- (1) Use the formula for circumference to determine the radius of the sphere.
Express the answer in “pi”.
- (2) Calculate the volume of the sphere.

Step 1: Find the radius of the great circle (which is also the radius of the sphere) using the formula for circumference.

$C = 2\pi r$	Circumference of a Circle
$50 = 2\pi r$	Substitution ($C = 50$)
$\frac{50}{2\pi} = r$	Division
$\frac{25}{\pi} = r$	Simplify

The radius of the sphere is $25/\pi$ centimeters.

Step 2: Calculate the volume of the sphere.

$V = \frac{4}{3}\pi r^3$	Volume of a Sphere
$V = \frac{4}{3}\pi\left(\frac{25}{\pi}\right)^3$	Substitution ($r = \frac{25}{\pi}$)
$V = \frac{4}{3}\cancel{\pi}\left(\frac{25}{\cancel{\pi}}\right)\left(\frac{25}{\pi}\right)\left(\frac{25}{\pi}\right)$	Cancel
$V = \frac{4}{3}\left(\frac{25}{1}\right)\left(\frac{25}{\pi}\right)\left(\frac{25}{\pi}\right)$	Simplify
$V \approx \frac{62500}{29.5788}$	Simplify
$V \approx 2113.0$	Simplify

The volume of the sphere is approximately 2,113.0 cubic centimeters.

Similar and Congruent Solids

similar solids – Similar solids have the same shape, but not the same size.

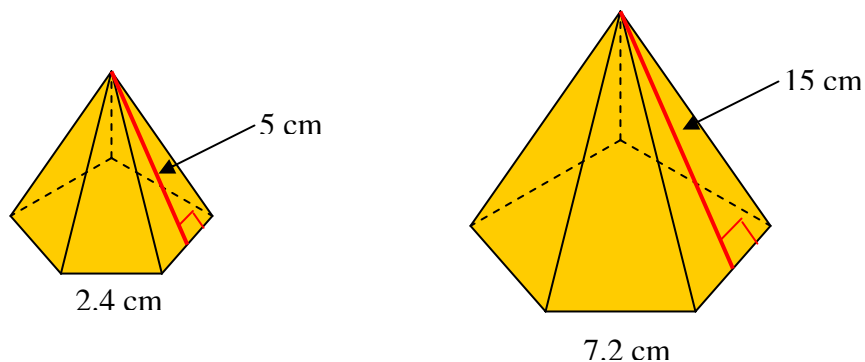
scale factor – The scale factor of similar solids is the ratio of corresponding linear measures.

If the corresponding linear measures of polyhedra have the same ratio, they are similar. In addition, the corresponding faces of similar polyhedra are similar and their edges are proportional.

The regular pentagonal pyramids shown below are similar. Compare the slant heights and we have 5 : 15 which simplifies to 1/3. Compare the base edge of the pentagons and we have 2.4 : 7.2 which simplifies to 1/3.

The scale factor between the linear measurements is 1:3.

$$\frac{5}{15} = \frac{2.4}{7.2} = \frac{1}{3}$$

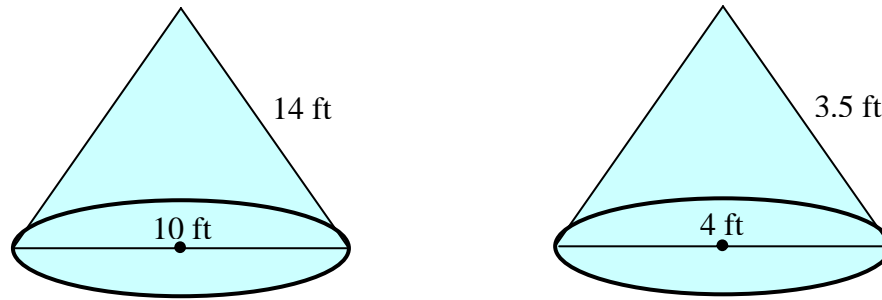


congruent solids - Solids are congruent if the ratio of their corresponding measures is 1:1.

Solids are congruent if...

- the corresponding edges are congruent.
- the corresponding angles are congruent.
- the areas of corresponding faces are congruent.
- the volumes are congruent.

Example: Determine if the cones are similar, congruent, or neither. State the letter of the correct answer. The diagram is not drawn to scale.



Similar, Congruent, or Neither?

Congruent? No. We can eliminate congruency since the corresponding edges (lateral height) are not equal. Thus, the scale factor is not 1:1.

Similar? We must check the ratio of the slant heights and the ratio of the diameters. Do these corresponding linear measures have the same ratios?

Ratio of Slant Heights

$$\frac{14}{3.5} = \frac{140}{35} = \frac{4}{1}$$

Ratio of Diameters

$$\frac{10}{4} = \frac{5}{2}$$

No. We can eliminate similarity because the corresponding linear measures have different ratios.

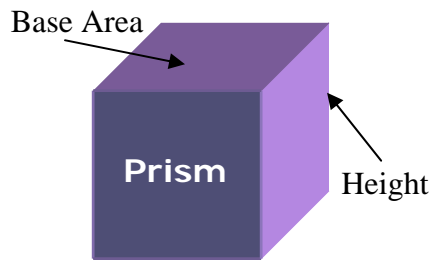
Thus, the cones are neither similar nor congruent.

With further investigation of similar solids, we find the following theorem to be true. We will explore this theorem in the problem set.

Theorem 29

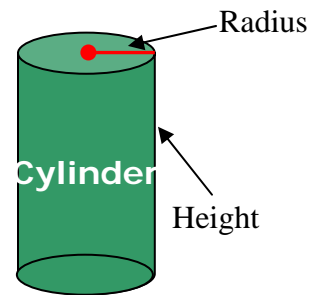
If two solids are similar with a scale factor of $a : b$, then the surface areas have a ratio of $a^2 : b^2$ and the volumes have a ratio of $a^3 : b^3$.

Volume and Surface Area Formulas



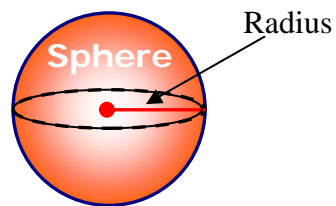
$$V = Bh$$

$$T = 2B + Ph$$



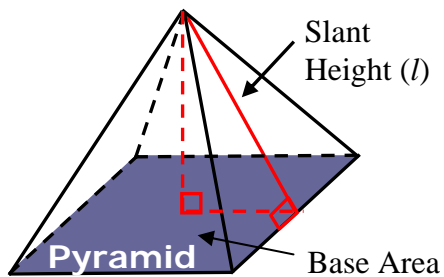
$$V = \pi r^2 h$$

$$T = 2\pi r^2 + 2\pi rh$$



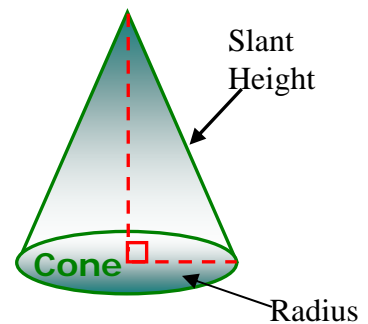
$$V = \frac{4}{3} \pi r^3$$

$$T = 4\pi r^2$$



$$V = \frac{1}{3} Bh$$

$$T = B + \frac{1}{2} Pl$$



$$V = \frac{1}{3} \pi r^2 h$$

$$T = \pi r^2 + \pi rl$$

Key for Variables:

r - radius

h - height (perpendicular to base)

l - slant height

B - area of base

P - perimeter of base

T - total surface area

V - volume