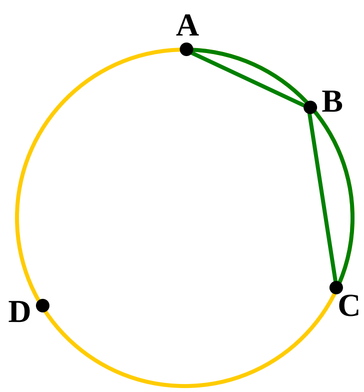
INSCRIBED ANGLES, TANGENTS, AND SECANTS

In this unit you will learn about inscribed angles, tangents, and secants. You will explore the relationship between inscribed angles and their intercepted arcs. You will investigate polygons inscribed in circles and polygons circumscribed about circles. You will learn about a point of tangency and examine lines that are tangent to circles and how they relate to radii. You will learn about secants and their connection with arcs, arc measure, and tangents.

Inscribed Angles

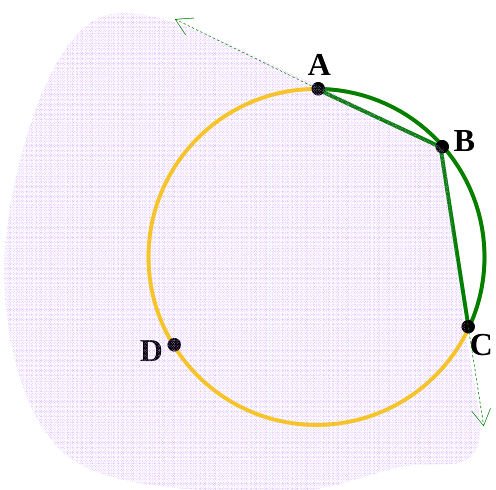
inscribed angle – An inscribed angle in a circle is an angle that has its vertex located on the circle and its rays are chords.

intercepted arc – An intercepted arc is an arc that lies in the interior of an inscribed angle and is formed by the intersection of the rays of an inscribed angle with the circle.



$\angle ABC$ is an inscribed angle.

$\overset{\frown}{ADC}$ is an intercepted arc of $\angle ABC$.



*Note: $\overset{\frown}{ADC}$ lies in the interior of $\angle ABC$.

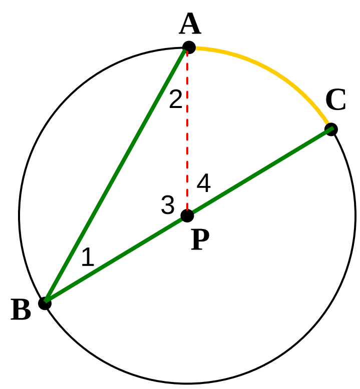
Theorem 23-A

If an angle is inscribed in a circle, then the measure of the angle is one-half the measure of the intercepted arc.

There are three cases to this proof.

Let's take a look at the case where the center of the circle lies on one of the rays of the inscribed angle.

Case 1: The center of the circle lies on one of the rays of the inscribed angle. (In the figure below, center point P lies on inscribed $\angle ABC$.)



Given: $\odot$P, Inscribed $\angle ABC$

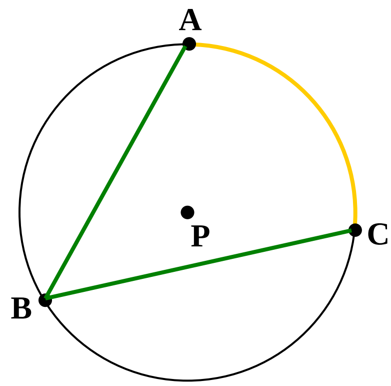
Prove: $m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$

*Numbers have been used to make easy reference to the angles.

Statement	**Reason**
Draw radius *PA*.	Radius *PA* is added as an auxilary line.
$m\angle 4 = m\angle 1 + m\angle 2$	Exterior Angle Theorem
$\overline{PB} \cong \overline{PA}$	Radii of the same circle are congruent.
$m\angle 2 = m\angle 1$	Angles that are opposite congruent sides in a triangle are congruent.
$m\angle 4 = m\angle 1 + m\angle 1$	Substitution
$m\angle 4 = 2(m\angle 1)$	Simplify
$\frac{1}{2}(m\angle 4) = m\angle 1$	Division
$m\angle 1 = \frac{1}{2}(m\angle 4)$	Symmetric Property of Equality
$m\angle 4 = m\overset{\frown}{AC}$	Definition of Arc Measure
$m\angle 1 = \frac{1}{2} m\overset{\frown}{AC}$	Substitution
$\therefore m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$	

In Case 2, the center of the circle lies within the inscribed angle.

Case 2: The center of the circle lies in the interior of the inscribed angle. (In the figure below, center point P lies in the interior of inscribed $\angle ABC$.)



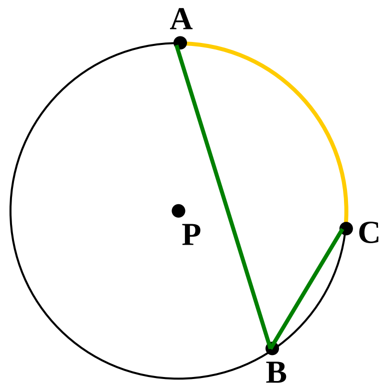
A similar proof could be developed as illustrated for Case 1.

Example 1: If the measurement of $\angle ABC = 56°$, what is the measure of $\overset{\frown}{AC}$?

$m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$	Theorem 23-A
Let $x = m\overset{\frown}{AC}$.	
$56° = \frac{1}{2} x$	Substitution
$112° = x$	Multiplication Property
$m\overset{\frown}{AC} = 112°$	

In Case 3 the center of the circle lies outside of the inscribed angle.

Case 3: The center of the circle lies in the exterior of the inscribed angle. (In the figure below, center point P lies in the exterior of inscribed $\angle ABC$.)



A similar proof could be developed as illustrated for Case 1.

Example 2: If the measurement of $\overset{\frown}{AC} = 139°$, what is the measure of $\angle ABC$?

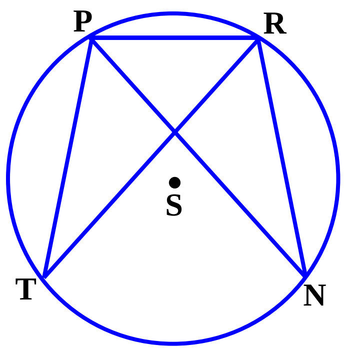
$m\angle ABC = \frac{1}{2} m\overset{\frown}{AC}$	Theorem 23-A
$m\angle ABC = \frac{1}{2}(139)$	Substitution
$m\angle ABC = 69.5°$	Simplify

Theorem 23-A holds true for all three of these cases; that is, the measurement of an inscribed angle is 1/2 the measure of its intercepted arc.

Theorem 23-B

If two inscribed angles intercept the same arc, then the angles are congruent.

Example 3: In the figure below, what angle is congruent to $\angle PTR$?



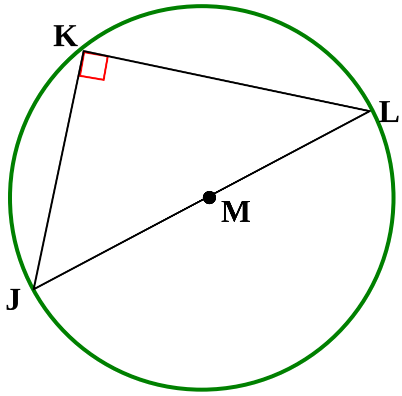
$\angle PTR$ intercepts $\overset{\frown}{PR}$.

$\angle RNP$ intercepts $\overset{\frown}{PR}$.

$\therefore\ \angle RNP \cong \angle PTR$ — Theorem 23-B The inscribed angles are congruent because they intercept the same arc.

Theorem 23-C

An angle that is inscribed in a circle is a right angle if and only if its intercepted arc is a semicircle.

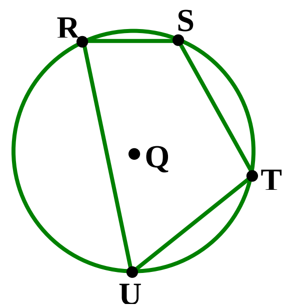


$\overset{\frown}{JKL}$ is a semicircle.	Given
$\angle JKL$ is an inscribed angle.	Given
$m\overset{\frown}{JKL} = 180°$	Definition of arc measure
$m\angle JKL = 90°$	Theorem 23-A

inscribed polygon – An inscribed polygon within a circle is a polygon whose vertices lie on the circle.

Theorem 23-D

If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.



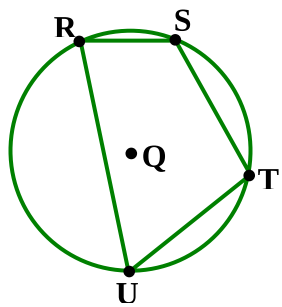
Example 4: Take any pair of opposite angles in the quadrilateral RSTU and explain Theorem 23-D.

We will organize the answer into statements and reasons.

Statement	**Reason**
$\angle RST$ and $\angle RUT$ are opposite angles in quadrilateral $RSTU$.	Given
$\angle RST$ is an inscribed angle of $\overset{\frown}{RUT}$.	Definition of Inscribed Angle
$m\angle RST = \frac{1}{2} m\overset{\frown}{RUT}$	Theorem 23-A
$\angle RUT$ is an inscribed angle of $\overset{\frown}{RST}$.	Definition of Inscribed Angle
$m\angle RUT = \frac{1}{2} m\overset{\frown}{RST}$	Theorem 23-A
$m\angle RST + m\angle RUT = \frac{1}{2} m\overset{\frown}{RUT} + \frac{1}{2} m\overset{\frown}{RST}$	Addition Property
$m\angle RST + m\angle RUT = \frac{1}{2}(m\overset{\frown}{RUT} + m\overset{\frown}{RST})$	Distributive Property
$m\overset{\frown}{RUT} + m\overset{\frown}{RST} = 360°$	Definition of Arc Measure
$m\angle RST + m\angle RUT = \frac{1}{2}(360°)$	Substitution
$m\angle RST + m\angle RUT = 180°$	Simplify
$m\angle RST$ and $m\angle RUT$ are supplementary angles.	Defintion of Supplementary Angles

Therefore, opposite angles $\angle RST$ and $\angle RUT$ of quadrilateral $RSTU$ are supplementary angles.

Example 5: For inscribed quadrilateral $RSTU$, if $\angle RST$ measures $125°$ and $\angle STU$ measures $95°$, then find the measures of $\angle TUR$ and $\angle URS$.



$\angle RST$ and $\angle TUR$ are opposite angles and are supplementary by Theorem 23-D.

$m\angle RST + m\angle TUR = 180°$	Definition of Supplementary Angles
$125° + m\angle TUR = 180°$	Substitution
$m\angle TUR = 55°$	Subtraction

$\angle STU$ and $\angle URS$ are opposite angles and are supplementary by Theorem 23-D.

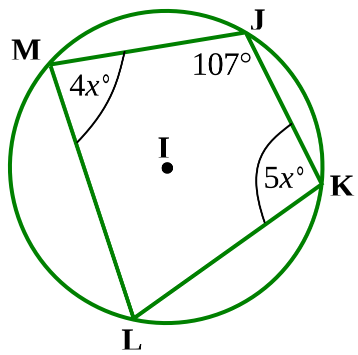
$m\angle STU + m\angle URS = 180°$	Definition of Supplementary Angles
$95° + m\angle URS = 180°$	Substitution
$m\angle URS = 85°$	Subtraction

To check the answers:

$m\angle RST + m\angle STU + m\angle TUR + m\angle URS = 360°$	The four angles of a quadrilateral total 360°.
$125° + 95° + 55° + 85° = 360°$	Substitution
$360° = 360°$	Simplify

Angles TUR and URS measure 55 degrees and 85 degrees, respectively.

Example 6: For inscribed quadrilateral *JKLM*, find the size of angles *K*, *L*, and *M*.



$m\angle M + m\angle K = 180$	Theorem 23-D and Definition of Supplementary Angles
$4x + 5x = 180$	Substitution
$9x = 180$	Simplify
$x = 20$	Division Property
$\angle M = 4x = 4(20) = 80°$	Substitution
$\angle K = 5x = 5(20) = 100°$	Substitution
$\angle J + \angle L = 180°$	Theorem 23-D and Definition of Supplementary Angles
$107° + \angle L = 180°$	Substitution
$\angle L = 73°$	Subtraction

To check the answers:

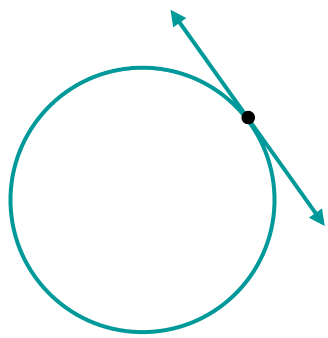
$m\angle J + m\angle K + m\angle L + m\angle M = 360°$	The four angles of a quadrilateral total 360°.
$107° + 100° + 73° + 80° = 360°$	Substitution
$360° = 360°$	Simplify

Angles M, K, and L measure 80, 100, and 73 degrees, respectively.

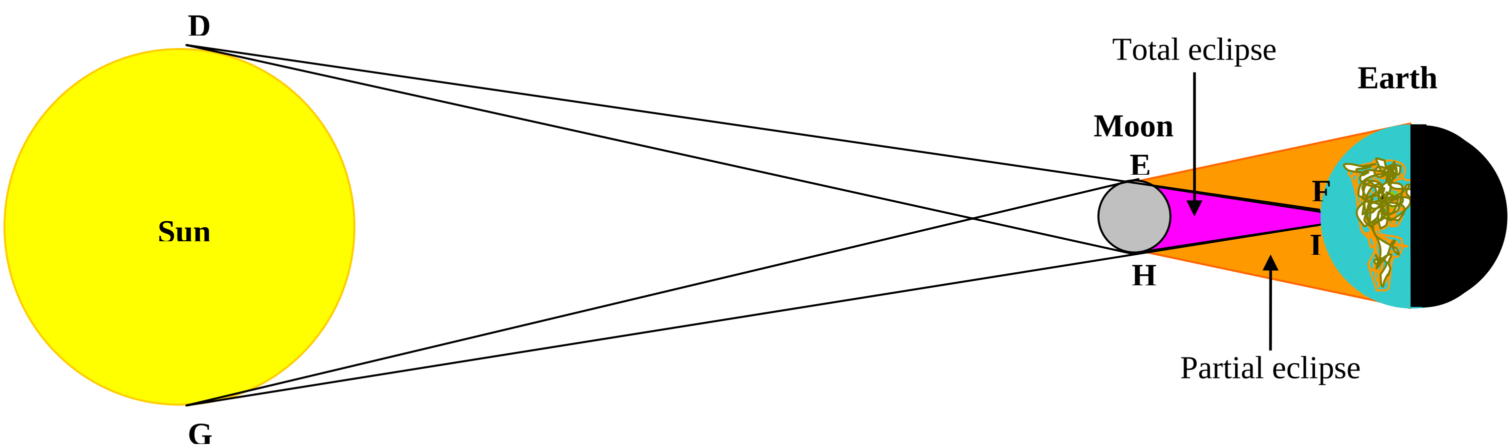
Tangents

tangent line – A tangent line to a circle is a line that intersects the circle at *exactly* one point. (It appears to brush the edge of a circle.)

point of tangency – A point of tangency is the point where a tangent line intersects with a circle.



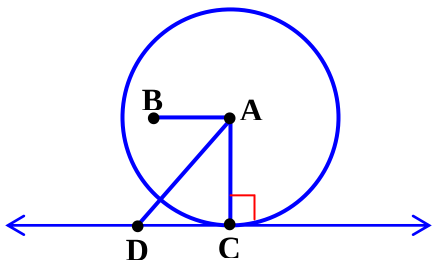
Example 1: The drawing below shows how the sun, moon, and earth are aligned for a solar eclipse. Identify the tangents lines which partition an area on the earth that experiences a total solar eclipse.



The pink area, the area between $\overline{EF}$ and $\overline{HI}$, is the area that experiences the total solar eclipse. One tangent line, $\overline{DE}$, that creates this area, is drawn from the upper most point on the sun, point D, through the uppermost point on the moon, point E, to the earth, point F. The second tangent line, $\overline{GH}$, that creates the area that experiences the total solar eclipse, is drawn from the lowest point on the sun, point G, through the lowest point on the moon, Point H, to the earth, point I.

Theorem 23-E

If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency.



Study the diagram to the left. $\overline{AB}$ is shorter than $\overline{AC}$. $\overline{AD}$ is longer than $\overline{AC}$. Any segment other than $\overline{AC}$ drawn to tangent $\overleftrightarrow{DC}$ will be longer than $\overline{AC}$. The shortest segment from a point to a line is a perpendicular segment. Thus, radius $\overline{AC} \perp \overleftrightarrow{DC}$.

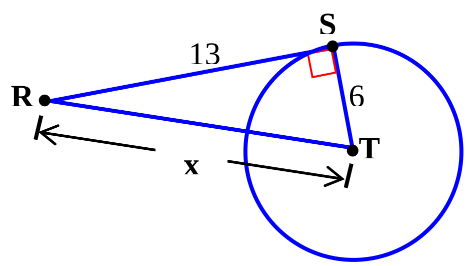
Example 2: Find x for the given information.

Given:

$\odot T$

Tangent RS = 13

Radius ST = 6



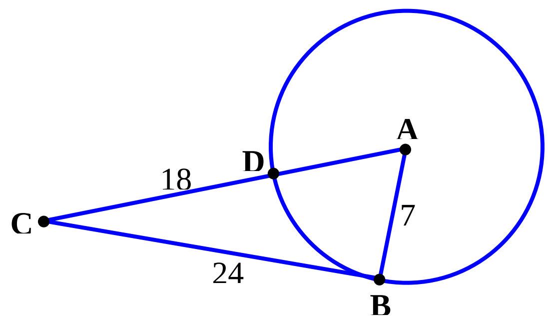
By theorem 23-E, $\overline{ST} \perp \overline{RS}$; thus, $\triangle RST$ is a right triangle.

$(13)^2 + (6)^2 = x^2$	Pythagorean Theorem
$169 + 36 = x^2$	Simplify
$205 = x^2$	Simplfiy
$x = \sqrt{205}$	Take the square root of both sides of the equation.
$x \approx 14.3$	Simplify

Theorem 23-F (Converse of 23-E)

If a radius is perpendicular to a line at the point at which the line intersects the circle, then the line is a tangent.

Example 3: Determine if $\overline{BC}$ is a tangent line to circle A.



Step 1: First determine the length of *AC*.

$AD = 7$	$\overline{AD}$ and $\overline{AB}$ are radii; therefore they are congruent.
$CD + DA = AC$	Segment Addition
$18 + 7 = AC$	Substitution
$AC = 25$	Simplify

Step 2: Prove that $\overline{AB} \perp \overline{BC}$ (Theorem 23-F) by showing that $\triangle$ABC is a right triangle and that $\overline{\text{AB}}$ and $\overline{\text{BC}}$ are the legs.

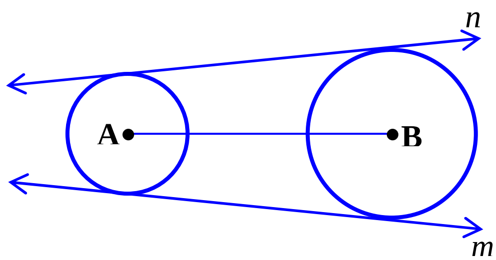
$(7)^2 + (24)^2 = (25)^2$	Pythagorean Theorem ($AB = 7,\ BC = 24,\ AC = 25$)
49 + 576 = 625	Simplify
625 = 625	Simplfiy

Thus, $\triangle ABC$ is a right triangle and segments $\overline{AB}$ and $\overline{BC}$ are the perpendicular legs.

common tangent – A common tangent is a line or line segment that is tangent to two circles in the same plane.

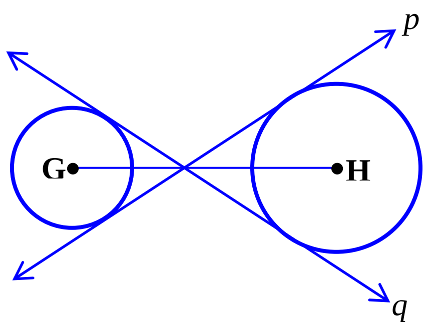
There are two types of common tangents: common external tangents and common internal tangents.

Common External Tangent



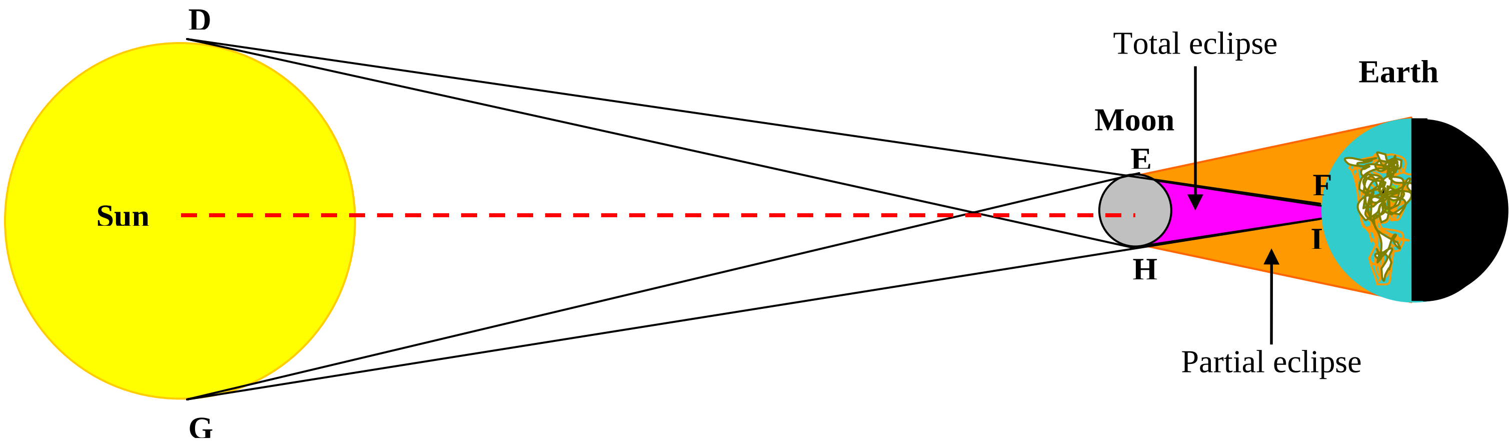
Common external tangents do not intersect the segment that has its endpoints on the centers of the two circles. Lines *n* and *m* are common external tangents for $\odot A$ and $\odot B$.

Common Internal Tangent



Common internal tangents intersect the segment that has its endpoints on the centers of the two circles. Lines *p* and *q* are common internal tangents for $\odot G$ and $\odot H$.

Example 4: Let's revisit our diagram of the solar eclipse in example 1. A red-dotted line has been added to represent the line segment that has its endpoints on the centers of the sun and the moon. Identify the common external tangents between the sun and the moon, and then identify the common internal tangents between the sun and the moon.



The common external tangents are $\overline{DE}$ and $\overline{GH}$.

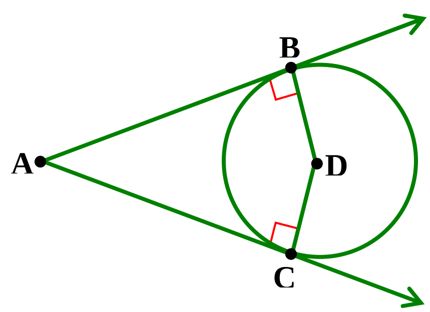
They DO NOT intersect the red dotted line.

The common internal tangents are $\overline{DH}$ and $\overline{GE}$.

They DO intersect the red dotted line.

Theorem 23-G

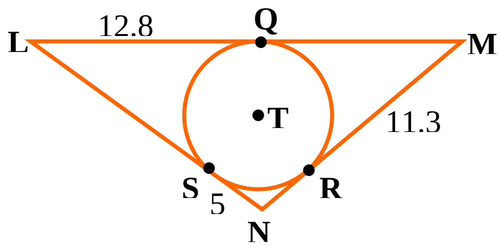
If two segments from the same exterior point are tangent to a circle, then the two segments are congruent.



$\overline{AB} \cong \overline{AC}$

circumscribed polygon – A circumscribed polygon about a circle is a polygon in which all of its sides are tangents to the circle.

Example 5: Triangle *LMN* is circumscribed around circle T. Segments LQ, MR, and SN measure 12.8, 11.3, and 5 centimeters, respectively. Find the perimeter of triangle LMN.



We will apply Theorem 23-G to solve this problem.

$NR = NS = 5$

$MQ = MR = 11.3$

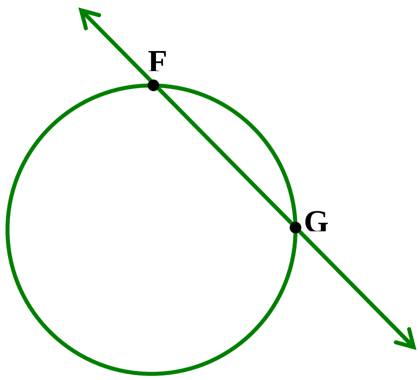
$LS = LQ = 12.8$

Perimeter $= LQ + LS + NS + NR + MR + MQ$	Definition of Perimeter
Perimeter $= 12.8 + 12.8 + 5 + 5 + 11.3 + 11.3$	Substitution
Perimeter $= 2(12.8) + 2(5) + 2(11.3)$	Simplify
Perimeter $= 58.2$	

The perimeter of $\triangle LMN$ is 58.2 centimeters.

Secants

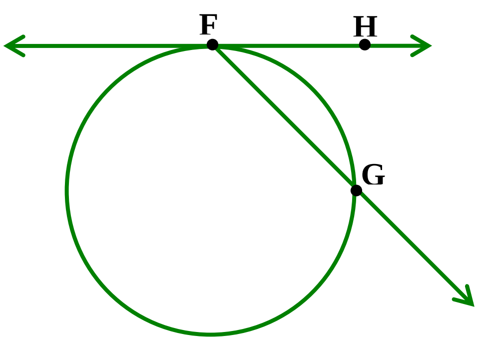
secant – A secant is a line that intersects a circle in exactly two points. A secant of a circle contains a chord of the circle.



$\overleftrightarrow{FG}$ is a secant.

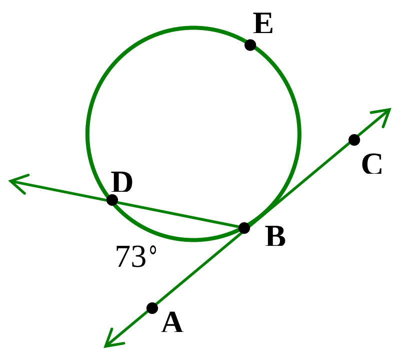
Theorem 23-H

If a secant and a tangent intersect at the point of tangency, then the measure of each angle formed is one-half the measure of its intercepted arc.



$$m\angle HFG = \frac{1}{2}m\overset{\frown}{FG}$$

Example 1: In the given diagram, arc DB measures 73 degrees. Point B is a point of tangency for line AC. Find the measure of angles ABD and CBD.



$m\angle ABD = \frac{1}{2} m\overset{\frown}{DB}$	Theorem 23-H
$m\angle ABD = \frac{1}{2}(73)$	Substitution
$m\angle ABD = 36.5°$	Simplify

$\angle ABD$ and $\angle CBD$ are a linear pair; therefore, they are supplementary.

$m\angle ABD + m\angle CBD = 180$	Definition of Supplementary Angles
$36.5 + m\angle CBD = 180$	Substitution
$m\angle CBD = 143.5°$	

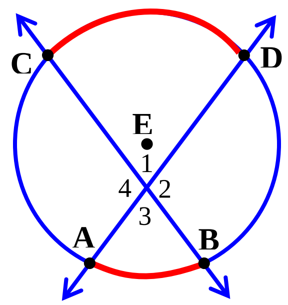
Now check the measure of $\angle CBD$ by applying Theorem 23-H.

$m\overset{\frown}{BED} = 360 - m\overset{\frown}{DB}$	Arc Measure
$m\overset{\frown}{BED} = 360 - 73$	Substitution
$m\overset{\frown}{BED} = 287°$	Simplify
$m\angle CBD = \frac{1}{2} m\overset{\frown}{BED}$	Theorem 23-H
$m\angle CBD = \frac{1}{2}(287)$	Substitution
$m\angle CBD = 143.5°$	Simplify

Angle ABD measures 36.5 degrees and angle CBD measures 143.5 degrees.

Theorem 23-I

If two secants intersect in the interior of a circle, then the measure of an angle formed is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.



$$m\angle 1 = \frac{1}{2}(m\overset{\frown}{CD} + m\overset{\frown}{AB})$$

$$m\angle 3 = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$$

$$m\angle 2 = \frac{1}{2}(m\overset{\frown}{BD} + m\overset{\frown}{AC})$$

$$m\angle 4 = \frac{1}{2}(m\overset{\frown}{AC} + m\overset{\frown}{BD})$$

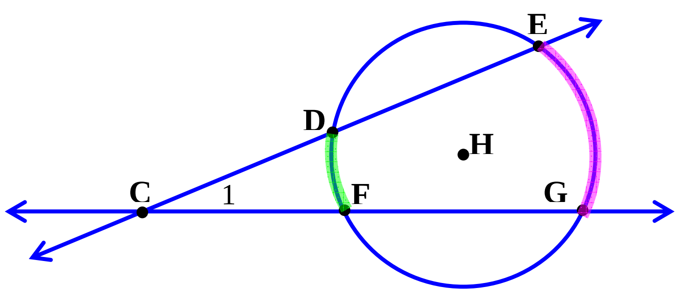
Example 2: In circle E above, determine the measure of angle 3 if $\overset{\frown}{CD}$ measures 78 degrees and $\overset{\frown}{AB}$ measures 40 degrees.

$m\angle 3 = \frac{1}{2}(m\overset{\frown}{AB} + m\overset{\frown}{CD})$	Theorem 23-I
$m\angle 3 = \frac{1}{2}(40 + 78)$	Substitution
$m\angle 3 = 59°$	Simplify

The measurement of angle 3 is 59 degrees.

Theorem 23-J	**The measure of an angle formed by two secants, a secant and a tangent, or two tangents intersecting in the exterior of a circle is equal to one-half the positive difference of the measures of the intercepted arcs.**

Theorem 23-J (Case 1): Secant-Secant



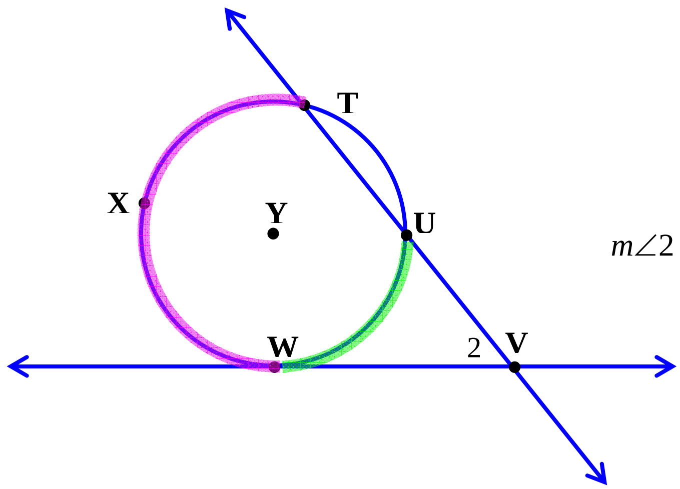
$$m\angle 1 = \frac{1}{2}(m\overset{\frown}{EG} - m\overset{\frown}{DF})$$

Example 3: In circle H above, find the measure of $\overset{\frown}{DF}$ if angle 1 measures 23 degrees and $\overset{\frown}{EG}$ measures 100 degrees.

$m\angle 1 = \frac{1}{2}(m\overset{\frown}{EG} - m\overset{\frown}{DF})$	Theorem 23-J
Let $x = m\overset{\frown}{DF}$.	
$23 = \frac{1}{2}(100 - x)$	Substitution
$46 = 100 - x$	Multiply both sides of the equation by 2.
$-54 = -x$	Subtract 100 from both sides of the equation.
$54 = x$	Muliply both sides of the equation by -1.
$m\overset{\frown}{DF} = 54°$	

The measurement of $\overset{\frown}{DF}$ is 54 degrees.

Theorem 23-J (Case 2): Secant-Tangent



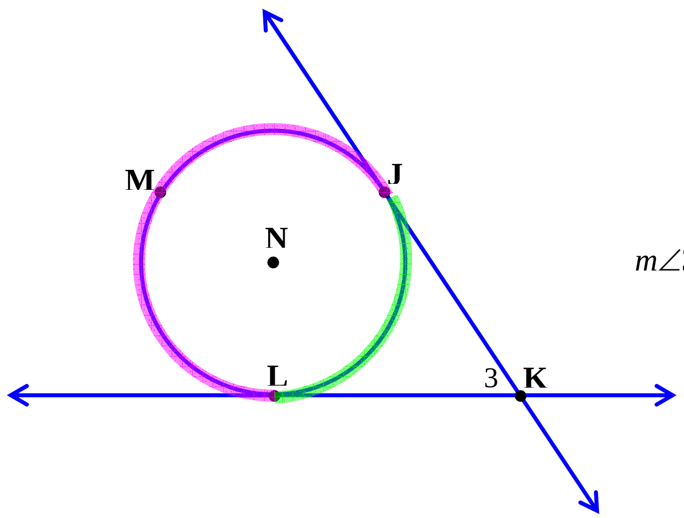
$$m\angle 2 = \frac{1}{2}(m\overset{\frown}{WXT} - m\overset{\frown}{WU})$$

Example 4: In circle Y above, find the measure of $\overset{\frown}{WXT}$ if angle 2 measures 49 degrees and $\overset{\frown}{WU}$ measures 93 degrees.

$m\angle 2 = \frac{1}{2}(m\overset{\frown}{WXT} - m\overset{\frown}{WU})$	Theorem 23-J
Let $x = m\overset{\frown}{WXT}$.	
$49 = \frac{1}{2}(x - 93)$	Substitution
$98 = x - 93$	Multiply both sides of the equation by 2.
$191 = x$	Add 93 to both sides of the equation.
$m\overset{\frown}{WXT} = 191°$	

The measurement of $\overset{\frown}{WXT}$ is 191 degrees.

Theorem 23-J (Case 3): Tangent-Tangent



$$m\angle 3 = \frac{1}{2}(m\overset{\frown}{LMJ} - m\overset{\frown}{LJ})$$

Example 5: In circle N above, find the measure of angle 3 if arc LMJ measures 240 degrees.

Step 1: First, find the measure of $\overset{\frown}{LJ}$.

$\overset{\frown}{LMJ} + \overset{\frown}{LJ} = 360$	Arc Measure
$240 + \overset{\frown}{LJ} = 360$	Substitution
$\overset{\frown}{LJ} = 120°$	Subtraction

Step 2: Apply Theorem 23-J to determine the measure of $\angle 3$.

$m\angle 3 = \frac{1}{2}(m\overset{\frown}{LMJ} - m\overset{\frown}{LJ})$	Theorem 23-J
$m\angle 3 = \frac{1}{2}(240 - 120)$	Substitution
$m\angle 3 = 60°$	Simplify

The measurement of angle 3 is 60 degrees.

Construction of an Inscribed Hexagon

The illustration below shows the steps for inscribing a regular hexagon within a circle.

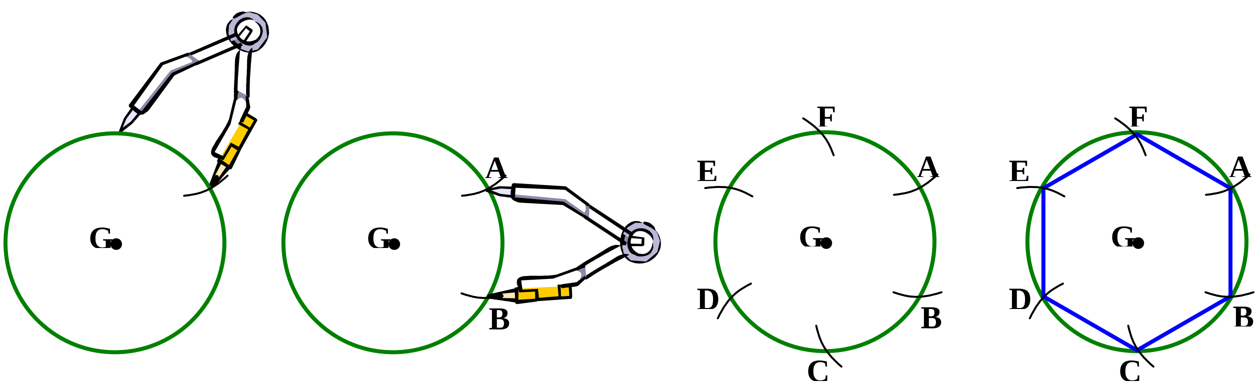
Step 1: Start by drawing a circle G with a compass.

Step 2: Without changing the setting on the compass, start anywhere on the circle and name the point A. Draw an arc that intersects the circle and name the point of intersection, B.

Step 3: Move the compass metal point to the point of intersection of the arc just drawn and draw another arc. Continue around the circle and draw six arcs. Make sure the setting of the compass does not change.

Step 4: Connect the consecutive points of intersection with line segments.

The polygon created is a regular hexagon.



Factoring Trinomials

Polynomials in the form of $x^2 + bx + c$ are factored in a different way because there are no common factors between the terms. In this section we are going to focus on quadratics whose leading coefficient is 1 (the coefficient of x^2). Remember that a polynomial whose degree is two (2) is called a quadratic.

When factoring a polynomial of the form $x^2 + bx + c,$ there is a pattern to discover.

* Look for factors of the constant term "*c*" whose sum is the coefficient of the linear term "*b*".

Example 1: Factor $x^2 + 5x + 6.$

a.) List the factors of the constant term (6).

$1 \cdot 6$

$2 \cdot 3$

b.) Find the sum of each set of factors.

1 + 6 = 7
2 + 3 = 5

c.) The sum of 2 and 3 is the coefficient of the linear term (5*x*); so, we are going to use the factors of 2 and 3 to factor the trinomial.

d.) The trinomial $x^2 + 5x + 6$ is factored as follows:

$(x + 2)(x + 3)$

factors of x^2

factors of 6 whose sum is 5

Example 2: Factor $x^2 + 11x + 18$.

factors of 18	sum
1, 18	19
2, 9	11
3, 6	9

$(x + 2)(x + 9)$ or $(x + 9)(x + 2)$

Example 3: Factor $x^2 + 13x + 30$

factors of 30	sum
1, 30	31
2, 15	17
3, 10	13
5, 6	11

$(x + 3)(x + 10)$ or $(x + 10)(x + 3)$

***Not all trinomials will contain (+) signs, so you must be very careful when finding factors of the constant term and choosing your sum.**

Example 4: Factor $x^2 - 10x + 16$.

*Notice in this trinomial that you are trying to find factors of a (+16) whose sum is (–10). This means that both factors will have to be negative.

factors of 16	sum
–1, –16	–17
–2, –8	–10
–4, –4	–8

(x – 2)(x – 8) or (x – 8)(x – 2)

***Sometimes it will be necessary to find factors of a negative constant. In this case you will have to list all possibilities in order to determine which sum is correct.**

Example 5: Factor $x^2 - 2x - 8$.

factors of –8	sum
–1, 8	7
1, –8	–7
–2, 4	2
2, –4	–2

***In this case you have to be very careful as to which factor is (+) and which factor is (–) because you are looking for the sum of two factors to equal the linear coefficient.**

(x + 2)(x – 4) or (x – 4)(x + 2)

**Just make sure in this case that the negative sign stays with the 4.*

The examples above now lead us into a discussion of signs within the trinomial and how to determine what signs will be used with the factors.

a.) If both signs are (+), then both factors will be (+).

$$x^2 + bx + c$$

$(x + \_\_\_)(x + \_\_\_)$

b.) If the last sign is (+) and the middle sign is (–), then both factors will be (–).

$$x^2 - bx + c$$

$(x - \_\_\_)(x - \_\_\_)$

c.) If the last sign is (–), then one factor is (+) and one factor is (–). You have to determine which is which depending on the sum you are trying to find.

$x^2 + bx - c$ or $x^2 - bx - c$

$(x + \_\_\_)(x - \_\_\_)$ $(x + \_\_\_)(x - \_\_\_)$

Solving Quadratic Equations

The process of factoring trinomials is used in solving quadratic equations. Quadratic equations are solved in order to determine what the zeros of a quadratic graph are. The zeros are the places at which the graph crosses the x-axis and will be discussed in more detail in later units. **The Zero Product Property** is used when solving quadratic equations.

> **Zero Product Property**
>
> If a and b are two real numbers and $ab = 0$, then $a = 0$ or $b = 0$.

***Make sure that before you factor, the equation is set equal to zero.**

Example 1: Solve $x^2 - 11x - 12 = 0$.

a.) Factor the trinomial.

$(x - 12)(x + 1) = 0$

b.) Set each factor equal to zero and solve the equations.

$x - 12 = 0$ *or* $x + 1 = 0$

$x = 12$ *or* $x = -1$

c.) The solution to this quadratic equation is $x = 12$ or $x = -1$.

Example 2: Solve $3x^2 + 11x = 4$.

a.) Set the equation equal to zero by subtracting 4 from each side.

$$3x^2 + 11x - 4 = 0$$

b.) Factor the trinomial using trial and error.

$$(3x - 1)(x + 4) = 0$$

c.) Set each factor equal to zero and solve the equations.

$3x - 1 = 0$ *or* $x + 4 = 0$

$3x = 1$ *or* $x = -4$

$x = \frac{1}{3}$